International Journal of Statistical Sciences *ISSN 1683-5603*
Vol. 26(1), March, 2026, pp 107-118
DOI: https://doi.org/10.3329/ijss.v26i1.88849

Hook Method of Integration by Parts and its Applications to Probability Theory

Anwar H. Joarder

Department of Computer Science and Engineering,
Khulna Khan Bahadur Ahsanullah University,
Gollamari 9100, Khulna City, Bangladesh
ORCID: https://orcid.org/my-orcid?orcid=0000-0002-0607-1729
WOS: https://www.webofscience.com/wos/author/record/N-1628-2019
SCOPUS: https://www.scopus.com/authid/detail.uri?authorId=35565161100
(Email: ajstat@gmail.com)

[Received May 6, 2025; Accepted August 30, 2025]

Abstract

Integration of the product of two functions involves steps that are often messy. Tabular methods available in online or in the literature are not that transparent. We present it in a way that the repeated integrals yield a series of parallel lines unless you stop at any step, and the last step yields a hook, and hence the name hook method. We apply it to calculate higher order moments and cumulative probability or survival functions of an exponential random variable related to service times.

Keywords: Integration by parts; exponential distribution; gamma distribution; cumulative distribution function; survival function.

AMS Classification: 44-10, 60-02, 60E05.

1. Introduction

Manning (1940) discussed the method of repeated integration by parts. It occurs in many examples in probability theory and mathematical statistics. Folley (1947) presented the integration of the product of two functions in an excellent manner. Murty (1980), Horowitz (1990), Nelson (1991), Gillman (1991), Kilmer (2008) and Rock (2017) used the method but Rock (2022) have simplified the integration and named it tabular method or row method of integration. We recommend going through Rock (2017) for the fundamentals and pedagogical issue of it. We still feel the method is not that transparent to a learner and we present it in a way that the repeated integrals yield a series of parallel lines unless you stop at any step, and the last step yields a hook, and hence the name hook method. We apply it to calculate higher order moments and cumulative probability or survival functions of an exponential random variable related to service times.

In Section 2, we discuss the method, and provide examples in Section 3. The examples include a case that suggests us to stop the repetition of integrals. In Section 4, we calculate moments of exponential probability model by using Hook Method. In Sections 5 and 6, we present examples of service times, and present general contribution to probability theory that provide compact results to well-known incomplete gamma function representations of the Cumulative Distribution Function or Survival Function.

2. The Hook Method

Consider differentiating the function $y = uv$ where both u and v are functions of x. We write $(uv)' = u'v + uv'$ which can be arranged as

$uv' = (uv)' - u'v,$ or,

$\int uv'dx = \int (uv)'dx - \int u'vdx,$ or,

$$\int u\left(\frac{dv}{dx}\right)dx = \int \frac{d}{dx}(uv)dx - \int\left(\frac{d}{dx}u\right)vdx,$$

$$\int u\ dv = uv - \int v\ du.$$

Let $u = f(x)$ and $v = g(x),$ then

$$\int f(x)g(x)dx = u\int g(x)dx - \int \frac{du}{dx}\left[\int g(x)dx\right]dx,$$

$$\int f(x)g(x)dx = u\int g(x)dx + (-1)\int \frac{du}{dx}\left[\int g(x)dx\right]dx,$$

$$I_1 = \int f(x)g(x)dx = f(x)g_1(x) + (-1)\int f^{(1)}(x)\left[g_1(x)\right]dx,$$

$$I_2 = \int f(x)g(x)dx = f(x)g_1(x) + (-1)f^{(1)}(x)\left[g_2(x)\right] + (+1)\int f^{(2)}(x)\left[g_2(x)\right]dx,$$

$$I_3 = \int f(x)g(x)dx = f(x)g_1(x) + (-1)f^{(1)}(x)\left[g_2(x)\right] + (+1)f^{(2)}(x)\left[g_3(x)\right] + (-1)\int f^{(3)}(x)\left[g_3(x)\right]dx,$$

$$I_4 = \int f(x)g(x)dx = +(+1)f(x)g_1(x) + (-1)f^{(1)}(x)\left[g_2(x)\right] + (+1)f^{(2)}(x)\left[g_3(x)\right] + (-1)f^{(3)}(x)\left[g_4(x)\right] + (+1)\int f^{(4)}(x)\left[g_4(x)\right]dx.$$

Proceeding thus, we have

$$I_a = \int f(x)g(x)dx = (+1)f(x)g_1(x) + (-1)f^{(1)}(x)[g_2(x)] + (+1)f^{(2)}(x)[g_3(x)] +(-1)f^{(3)}(x)\left[g_4(x)\right] + \ldots + (-1)f^{(a-1)}(x)\left[g_a(x)\right] + (-1)^a\int f^{(a)}(x)\left[g_a(x)\right]dx.$$

The above can be written in a compact way by using summation notation as follows:

$$I_a = \int f(x)g(x)dx = \sum_{j=0}^{a-1}(-1)^j f^{(j)}(x)\left[g_{j+1}(x)\right] + (-1)^a\int f^{(a)}(x)\left[g_a(x)\right]dx.$$

The following method provides a picturesque view of the formula:

0	1	2	3	...	$a-2$	$a-1$	a
$f(x)$	$f^{(1)}(x)$	$f^{(2)}(x)$	$f^{(3)}(x)$	...	$f^{(a-2)}(x)$	$f^{(a-1)}(x)$	$f^{(a)}(x)$
$g(x)$	$g_1(x)$	$g_2(x)$	$g_3(x)$	...	$g_{a-2}(x)$	$g_{a-1}(x)$	$g_a(x)$

We observe that the integral at every step, say, at I_1, I_2, I_3, …, the summands are telescopic products in the following table but the last summand is a product of two functions. Thus the last two summand forms a hook in the above table, and hence we name the method “Hook Method”.

Throughout the next section we insert constant of additivity at the final answer.

3. Some Examples of Hook Method

Example 3.1 Evaluate $I = \int x^3 e^{ax} dx$ by the hook method.

Solution:

	0	1	2	3	4
Diff	x^3	$3x^2$	$6x$	6	0
Inte	e^{ax}	$\frac{e^{ax}}{a}$	$\frac{e^{ax}}{a^2}$	$\frac{e^{ax}}{a^3}$	$\frac{e^{ax}}{a^4}$

$$I_4 = (+1)(x^3)\left[\frac{e^{ax}}{a}\right] + (-1)(3x^2)\left[\frac{e^{ax}}{a^2}\right] + (+1)(6x)\left[\frac{e^{ax}}{a^3}\right] + (-1)(6)\left[\frac{e^{ax}}{a^4}\right] + 1(+1)\int (0)\left[\frac{e^{-x}}{a^4}\right] dx,$$

$$I_4 = (x^3)\left[\frac{e^{ax}}{a}\right] - (3x^2)\left[\frac{e^{ax}}{a^2}\right] + (6x)\left[\frac{e^{ax}}{a^3}\right] - (6)\left[\frac{e^{ax}}{a^4}\right].$$

$$J = \int_0^\infty x^3 e^{ax} dx = \left[(x^3)\frac{e^{ax}}{a}\right]_0^\infty - \left[(3x^2)\frac{e^{ax}}{a^2}\right]_0^\infty + \left[(6x)\frac{e^{ax}}{a^3}\right]_0^\infty - \left[(6)\frac{e^{ax}}{a^4}\right]_0^\infty .$$

Note that if $a = -1$, then

$$J = \int_0^\infty x^3 e^{ax} dx = 0 - 0 + 0 + 6 = 6 \text{ which is known to be } \Gamma(4).$$

An example is presented below where we stop at an Easily Integrable Column.

Example 3.2 Evaluate the integral $I = \int (\ln x) x^4 dx.$

Solution:

	0	1	2	3
Diff	$\ln x$	$\frac{1}{x}$	$\frac{-1}{x^2}$	$\frac{2}{x^3}$
Inte	x^4	$\frac{1}{5}x^5$	$\frac{1}{5\times 6}x^6$	$\frac{1}{5\times 6\times 7}x^7$

$$I_0 = \int (\ln x) x^4 dx.$$

$$I_1 = (+1)(\ln x)\left(\frac{1}{5}x^5\right) + (-1)\int \left(\frac{1}{x}\right)\left(\frac{1}{5}x^5\right) dx$$

Since the last integral is easily integrable, we stop.

$$I_2 = \frac{1}{5}x^5 \ln x - \int \frac{1}{5}x^4 dx + c_2,$$

$$I_2 = \frac{1}{5}x^5 \ln x - \frac{1}{25}x^5 + C.$$

Note that if we do not stop at the second step, the integral may continue at the third summand or the fourth, and so on.

An example is presented below where we stop at a column **reproducing the original integral.**

Example 3.3 Evaluate the integral $J = \int e^x \sin x dx.$

Solution:

	0	1	2	3
Diff	e^x	e^x	e^x	e^x
Inte	$\sin x$	$-\cos x$	$-\sin x$	$\cos x$

$$J = \int e^x \sin x dx,$$

$$J_1 = (+1)e^x(-\cos x) + \int (-1)e^x(-\cos x) dx,$$

$$J_2 = (+1)e^x(-\cos x) + (-1)e^x(-\sin x) + \int (+1)e^x(-\sin x)dx,$$

Since the integral is reproducing the original integral, we better stop as the following:

$$J_2 = -e^x \cos x + e^x \sin x - J + C.$$

Since $J_2 = J$, we have

$$2J = e^x(\sin x - \cos x) + C,$$

$$J = \frac{1}{2}e^x(\sin x - \cos x) + C,$$

where C is the constant of integration.

4. Higher Order Moments by Hook Method

Theorem 4.1 Let $X \sim G(1, \beta)$ with pdf $\frac{1}{\beta}e^{-x/\beta}$ where $0 < x < \infty,\ 0 < \beta < \infty.$ Then

$$E(X^a) = a!\beta^a,\ a = 1, 2, \ldots. \text{ and } \mu_a = a!\beta^a \sum_{k=0}^{a} \frac{(-1)^k}{k!}.$$

Proof. By definition,

$E(X^a) = \int_0^\infty x^a \frac{1}{\beta} e^{-x/\beta} dx = \frac{1}{\beta} I,$ where $I = \int_0^\infty x^a e^{-x/\beta} dx$ can be evaluated by the Hook Method as follows:

	0	1	2		$a-1$	a
Diff	x^a	ax^{a-1}	$a(a-1)x^{a-2}$	$\cdots$	$(a-1)!x$	$a!$
Inte	$e^{-x/\beta}$	$(-\beta)e^{-x/\beta}$	$(-\beta)^2 e^{-x/\beta}$	$\cdots$	$(-\beta)^{a-1} e^{-x/\beta}$	$(-\beta)^a e^{-x/\beta}$

$$I = 0 + 0 + \ldots + 0 + (-1)^a a!(-\beta)^a \int_0^\infty e^{-x/\beta} dx,$$

$$I = (-1)^a a!(-\beta)^{a+1} \left[e^{-x/\beta} \right]_0^\infty,$$

$$I = a!\beta^{a+1}(0-1) = a!\beta^{a+1}.$$

Then we have

$$E(X^a) = \frac{1}{\beta}\left(a!\beta^{a+1}\right) = a!\beta^a,\ a = 1, 2, \ldots$$

By definition, $\mu_a = E\left[X - E(X)\right]^a$,

$\mu_a = E(X-\beta)^a$.

By binomial expansion, we have

$$\mu_a = E\sum_{k=0}^{a}\binom{a}{k}(-\beta)^k X^{a-k}.$$

By linearity of expectation operator, we have

$$\mu_a = \sum_{k=0}^{a}\binom{a}{k}(-\beta)^k E\left(X^{a-k}\right).$$

By the raw moments of exponential probability distribution, we have

$$\mu_a = \sum_{k=0}^{a}\frac{a!}{k!(a-k)!}(-\beta)^k (a-k)\beta^{a-k},$$

which completes the proof.

5. The Exponential Probability Model

The pdf of an exponential random variable X is $f(x) = \lambda\, e^{-\lambda x},\ 0 < x < \infty,\ 0 < \lambda < \infty$. Its mgf (moment generating function) is given by

$$M_X(t) = \frac{\lambda}{\lambda - t} = \left(1 - \frac{t}{\lambda}\right)^{-1},\ t < \lambda.$$

We will denote the exponential distribution by $X \sim G(1, 1/\lambda)$.

Example 5.1 If X is the amount of time a customer is served and it has an has exponential distribution with mean time $1/\lambda$. If two customer are randomly selected, then the probability that both are served in $w = 5$ minutes is given by

$$P(X+Y \le w) = \iint\limits_{x+y\le w} (\lambda e^{-\lambda x})(\lambda e^{-\lambda y})dydx.$$

$$P(X+Y \le w) = \int_{x=0}^{w}\int_{y=0}^{w-x} \lambda e^{-\lambda x}\lambda e^{-\lambda y}dydx,$$

$$P(X+Y \le w) = \int_{x=0}^{w} \lambda e^{-\lambda x} \left(\int_{y=0}^{w-x} \lambda e^{-\lambda y} dy \right) dx,$$

$$P(X+Y \le w) = 1 - e^{-\lambda w} - \lambda e^{-\lambda w} w,$$

$$P(X+Y \le w) = 1 - (1+\lambda w) e^{-\lambda w}.$$

If we denote $W = X+Y,$ then

$$F_W(w) = 1 - (1+\lambda w) e^{-\lambda w}.$$

The above is the CDF (Cumulative Distribution Function) of the random variable $W = X+Y.$ The pdf of it is

$$f_W(w) = 0 - e^{-\lambda w}(-\lambda) - \lambda \left[\{1\} e^{-\lambda w} + w\{ e^{-\lambda w}(-\lambda)\} \right],$$

$$f_W(w) = \lambda^2 w e^{-\lambda w}, \quad 0 < w < \infty, \; 0 < \lambda < \infty.$$

Observe that the above is the pdf of a gamma random variable $G(2, 1/\lambda).$

If we denote $w = 5,$ then the required probability is

$$F_W(5) = \int_0^5 \lambda^2 w e^{-\lambda w} dw$$

is evaluated in the following question.

Example 5.2 If X is the amount of time a customer is served and it has an has exponential distribution with mean time $1/\lambda.$ If n customer are randomly selected, then the probability that all are served in w minutes is given by

$$P(X_1 + X_2 + ... + X_n \le w) = \int_{x_1 + x_2 + ... + x_n \le w} \lambda^n e^{-\lambda(x_1 + x_2 + ... + x_n} dx_1 dx_2 ... dx_n.$$

The solution to the above integral is the expression in (6.2).

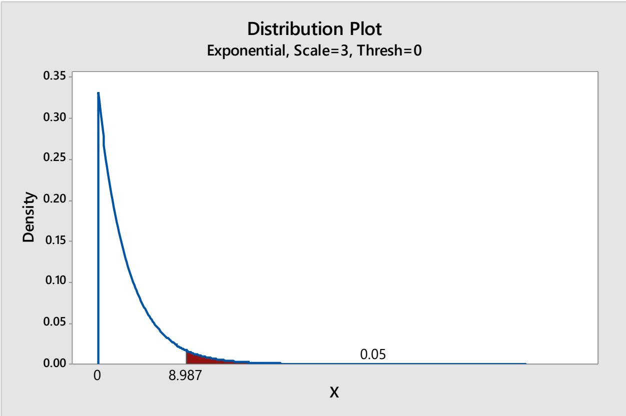
6. The Gamma Probability Model: The Distribution of Total Time

Let $X_j \sim G(1, 1/\lambda), \;\; j = 1, 2, ..., n.$ If $W = X_1 + X_2 + ... + X_n,$ then

$$M_W(w) = M_{X_1}(t) M_{X_2}(t) ... M_{X_n}(t) = \left(1 - \frac{t}{\lambda}\right)^{-1} \left(1 - \frac{t}{\lambda}\right)^{-1} ... \left(1 - \frac{t}{\lambda}\right)^{-1} = \left(1 - \frac{t}{\lambda}\right)^{-n}, \; t < \lambda$$

which is the moment generating function of a gamma random variable $X \sim G(n, 1/\lambda).$

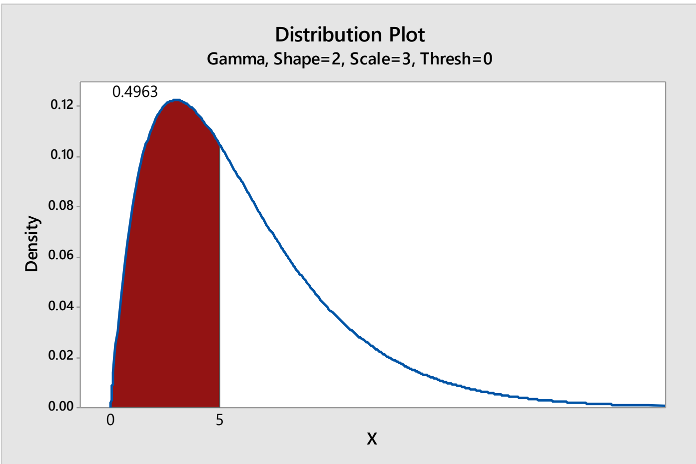
Example 6.1 Suppose that the service time of a customer is exponentially distributed with average $3(=1/\lambda)$. The probability that two $2(=n)$ customers together will be served in less than $5(=w)$ minutes can be derived by a gamma probability distribution.



If W is the total time to get served, then $W \sim G(2, 1/\lambda)$ with pdf

$$f(w) = (\lambda w)\ \lambda e^{-\lambda w} = \lambda^2 w\ \lambda e^{-\lambda w},\ 0 < w < \infty,\ 0 < \lambda < \infty.$$

We want $P(W \le 5) = \int_0^5 \lambda^2 w e^{-\lambda w} dw.$



The above is actually the CDF of W at $w = 5$ and is also written by

$$F_W(5)=\int_0^5 \lambda^2 we^{-\lambda w}dw,\ \text{where } W\sim G(2,1/\lambda).$$

Transforming $\lambda w = y$ we have

$$F_W(5)=\lambda^2\int_0^{5\lambda}\frac{y}{\lambda}\,e^{-y}\left(\frac{dy}{\lambda}\right),$$

$$F_W(5)=\int_0^{5\lambda} y\,e^{-y}dy$$

which is the CDF of $F_W(5)=F_Y(5\lambda)=\int_0^{5\lambda} y\,e^{-y}dy$ where $Y\sim G(2,1).$

The integral is evaluated below by Hook Method:

	0	1	2
Diff	y	1	0
Inte	e^{-y}	$\frac{e^{-y}}{-1}$	$\frac{e^{-y}}{(-1)^2}$

$$I=(+1)y\times\frac{e^{-y}}{-1}+(-1)\{1\}\times\frac{e^{-y}}{(-1)^2}+(+1)\int(+1)\{0\}\left[\frac{e^{-y}}{(-1)^2}\right],$$

$$F_W(5)=(+1)\left[y\times\frac{e^{-y}}{-1}\right]_0^{5\lambda}+(-1)\left[\{1\}\times\frac{e^{-y}}{(-1)^2}\right]_0^{5\lambda}+0,$$

$$F_W(5)=-5\lambda e^{-5\lambda}-e^{-5\lambda}+1=1-2.67e^{-1.67}\approx 0.50$$

The probability is calculated below by the LIG (Lower Incomplete Gamma) Function

$\gamma(a,b)=\int_0^b u^{a-1}\,e^{-u}du$ where U is a gamma random variable $U\sim G(a,1).$

The required probability is the CDF given by $F_W(5)=\int_0^5 \lambda^2 we^{-\lambda w}dw,$ where $W\sim G(2,1/\lambda).$

Transforming $\lambda w = u$ we have

$$F_W(5) = \lambda^2 \int_0^{5\lambda} \frac{y}{\lambda} e^{-y} \left(\frac{dy}{\lambda} \right),$$

$$F_W(5) = \int_0^{5\lambda} u\ e^{-u} du \text{ where } U \sim G(2,1).$$

$$F_W(5) = \int_0^{5\lambda} u\ e^{-u} du = \gamma(2, 5\lambda) = \gamma(2, 5/3) \approx 0.496332,$$

where the last step is by the following

https://www.danielsoper.com/statcalc/calculator.aspx?id=24

Theorem 6.1 Suppose that the service time X of a customer is exponentially distributed with average time $1/\lambda$ minutes so that $W = X_1 + X_2 + \ldots + X_n$ is the total time taken by n customers to get served. Then $S_W(w) = P(W > w)$, the probability that n customers together will be served in more than w minutes is

$$S_W(w) = \sum_{j=0}^{n-1} \frac{(\lambda w)^j}{j!} e^{-\lambda w}. \tag{6.1}$$

The above can be written explicitly as

$$S(w) = \left[1 + \lambda w + \frac{(\lambda w)^2}{2!} + \frac{(\lambda w)^3}{3!} + \ldots. + \frac{(\lambda w)^{n-1}}{(n-1)!} \right] e^{-\lambda w}.$$

Proof. Since $W = X_1 + X_2 + \ldots + X_n \sim G(n, \lambda)$, we have

$$F_W(w) = P(W \le w) = \int_0^w \frac{\lambda^n}{\Gamma(n)} z^{n-1} e^{-\lambda z} dz.$$

By substituting $t = \lambda z$, we have

$$1 - S_W(w) = \frac{1}{\Gamma(n)} \int_0^{\lambda w} t^{n-1} e^{-t} dt = \frac{\gamma(n, \lambda w)}{\Gamma(n)}. \tag{6.2}$$

If n is an integer, then the lower incomplete gamma function can be evaluated in a compact form by Hook Method of Integration, and the theorem is proved.

The integral $\gamma(a,b)$ is available in https://www.danielsoper.com/statcalc/calculator.aspx?id=24 but we consider evaluating $\gamma(a,b)=\int_0^b x^{a-1}\ e^{-x}dx$ by the Hook Method as follows:

	0	1	2	3		$a-2$	$a-1$
Diff	x^{a-1}	$(a-1)x^{a-2}$	$(a-1)(a-2)x^{a-3}$	$(a-1)(a-2)(a-3)x^{a-4}$	$\cdots$	$(a-1)!x$	$(a-1)!$
Inte	e^{-x}	$(-1)e^{-x}$	$(-1)^2e^{-x}$	$(-1)^3e^{-x}$	$\cdots$	$(-1)^{a-2}e^{-x}$	$(-1)^{a-1}e^{-x}$

$$\gamma(a,b)=\left[x^{a-1}(-1)e^{-x}\right]_0^b-\left[(a-1)x^{a-2}(-1)^2e^{-x}\right]_0^b+\left[(a-1)(a-2)x^{a-3}(-1)^3e^{-x}\right]_0^b-\ldots$$

$$+\left[(-1)^{a-2}(a-1)!x(-1)^{a-1}e^{-x}\right]_0^b+(-1)^{a-1}\int_0^b(a-1)!(-1)^{a-1}e^{-x}dx,$$

$$\gamma(a,b)=-\left[b^{a-1}+(a-1)b^{a-2}+(a-1)(a-2)b^{a-3}+\ldots+(a-1)!b+(a-1)!\right]e^{-b}+(a-1)!$$

Then we have

$$\gamma(n,\lambda w)=-\left[(\lambda w)^{n-1}+(n-1)(\lambda w)^{n-2}+(n-1)(n-2)(\lambda w)^{n-3}+\ldots+(n-1)!\lambda w+(n-1)!(1-e^{\lambda w})\right]e^{-\lambda w},$$

$$\frac{\gamma(n,\lambda w)}{\Gamma(n)}=-\left[\frac{(\lambda w)^{n-1}}{(n-1)!}+\frac{(n-1)}{(n-1)!}(\lambda w)^{n-2}+\frac{(n-1)(n-2)}{(n-1)!}(\lambda w)^{n-3}+\ldots+\frac{(n-1)!}{(n-1)!}\lambda w+\frac{(n-1)!}{(n-1)!}(1-e^{\lambda w})\right]e^{-\lambda w},$$

$$\frac{\gamma(n,\lambda w)}{\Gamma(n)}=-\left[\frac{(\lambda w)^{n-1}}{(n-1)!}+\frac{1}{(n-2)!}(\lambda w)^{n-2}+\frac{1}{(n-3)!}(\lambda w)^{n-3}+\ldots+\lambda w+1\right]e^{-\lambda w}+1,$$

$$1-\frac{\gamma(n,\lambda w)}{\Gamma(n)}=\left[\frac{(\lambda w)^{n-1}}{(n-1)!}+\frac{1}{(n-2)!}(\lambda w)^{n-2}+\frac{1}{(n-3)!}(\lambda w)^{n-3}+\ldots+\lambda w+1\right]e^{-\lambda w},$$

which can be written as (6.1).

We remark that $S_W(w)=F_Y(n)$ where $W\sim G(n,1/\lambda)$ and Y is a Poisson variable with mean $\lambda w.$

7. Conclusion

We believe the gentle proof of Hook Method, some examples and applications in probability theory would be beneficial to students and instructors of statistics. The method can be applied to any continuous probability model.

Acknowledgement: The author sincerely thanks the reviewer for his encouraging and detailed feedback, which was both inspiring and useful.

References

[1] Folley, K. W. (1947). Integration by parts. The American Mathematical Monthly, 54(9), 542–543. https://doi.org/10.2307/2304674

[2] Gillman, L. (1991). More on tabular integration by parts. The College Mathematics Journal, 22(5), 407–410. https://doi.org/10.1080/07468342.1991.11973416

[3] Horowitz, D. (1990). Tabular integration by parts. The College Mathematics Journal, 21(4), 307–311. https://doi.org/10.1080/07468342.1990.11973325

[4] Kilmer, S. J. (2008). Integration by parts and infinite series. Mathematics Magazine, 81, 51–55.

[5] Manning, F. L. (1940). A formula for repeated integration by parts. The American Mathematical Monthly, 47(9), 643–644. https://doi.org/10.2307/2304117

[6] Murty, V. N. (1980). Integration by parts. Two-Year College Mathematics Journal, 11, 90–94.

[7] Nelsen, R. B. (1991). Proof without words: integration by parts. Mathematics Magazine. 64, 130.

[8] Rock, J. A. (2017). A lecture on integration by parts. Mathematical Scientist, 42, 1–9. https://drive.google.com/file/d/1aadLkBgGh5iri_NHXFUmyYRvYMAfjLzu/view

[9] Rock, J.A. (2022). Row integration by parts. International Journal of Mathematical Education in Science and Technology, 53(4), 1037-1043.

[10] Spickerman, W. R. and Daniels, J. W. (1983). A note on integration by parts. Mathematics Teacher. 76, 337–338.