International Journal of Statistical Sciences *ISSN 1683-5603*
Vol. 25(1), March, 2025, pp 1-10
DOI: https://doi.org/10.3329/ijss.v25i1.81040

On Some Aspects of a Class of Discrete Probability Distributions

Kishore Kumar Das[1*] and Durba Purkayastha[2]

[1]Department of Statistics, Gauhati University, Guwahati, Assam, India
[2]Department of Statistics, B. N. College, Dhubri, Assam, India
[*]Correspondence should be addressed to Kishore Kumar Das
(Email: daskishore@gmail.com)

[Received November 17, 2024; Accepted February 20, 2025]

Abstract

A class of functions has been introduced, leading to the introduction of corresponding class of discrete probability distributions. This includes several known as well as newly identified distributions as specific cases. Various distributional properties of these proposed distributions have been explored. Finally, the proposed distributions are fitted with a zero-truncated Poisson distribution to life data, and we also test their goodness of fit.

Keywords: Weighted distributions, Geometric distribution, Probability generating function.

AMS Classification: 05A19, 62, 62R01.

1. Introduction

The geometric distribution, also known as the Furry distribution (Furry 1937), has been illustrated in Johnson et al. (1969, 2005). This distribution is a special case of the negative binomial distribution and possesses the non-aging or Markovian property, similar to the exponential distribution (Johnson et al. 1995, 2005). It is often considered an exponential distribution and is a specific case of the grouped exponential distribution (Spinelli 2001). The geometric distribution and its properties have been extensively studied by various researchers, as reported in Johnson et al. (1969, 2005), Marco (2021), among others. Much of this work requires updating in the book by Johnson et al. (2005). Parameter estimation for the geometric distribution is straightforward due to its classification as a power series distribution, with the first-moment equation also serving as the maximum likelihood equation (Johnson et al. 2005).

The logarithmic distribution has been documented in Johnson et al. (1969, 2005). It is a one-parameter generalized power series distribution with infinite support on the positive integers (Johnson et al. 2005). Comprehensive studies on the logarithmic distribution can be found in the works of Jonson et al. (1969, 1995, and 2005). The polylogarithm function finds its application in quantum statistics, where it appears as the closed form of integrals of the Fermi–Dirac and Bose-Einstein distributions, also known as the Fermi–Dirac integral or the Bose-Einstein integral. The contributions of the polylogarithmic distribution are documented in Kemp (2004), Gallardo et al. (2018), and Valero et al. (2022).

The concept of weighted distributions was developed by Rao (1965), with a comprehensive review of their applications found in Rao (1985) and Patil et al. (1986). These distributions are modified either through the method of ascertainment or the recording process when an event occurs (Johnson et al. 2005). A detailed description of weighted distribution has been illustrated in Johnson et. al. (1969, 1995, 2005), highlighting the contributions of Patil and Rao (1978), and Patil et al. (1986). Patil et al. (1986) suggested 10 types of useful weight functions in scientific work. The weighted distribution is also known as the moment distribution (Berg, 1978).

This study introduces and examines a class of discrete probability distributions (CDPD), proposing a new class of useful functions. This encompasses several familiar as well as newly identified distributions as specific cases. The distributional properties of the proposed distributions have been thoroughly examined. R language has been utilized for mathematical calculations and chart preparation. Finally, these proposed distributions are fitted with a zero-truncated Poisson distribution to life data and the test their goodness of fit.

2. Class of Useful Functions

A class of useful functions has been proposed for non-negative integer values of x as

$$L_{\nu}(\alpha,\beta,\theta)=\sum_{x=0}^{\infty}(\alpha+\beta x^{\nu})\theta^{x},\ \upsilon\geq 0,\ \alpha\geq 0,\beta>0 \text{ and } (0<\theta<1) \quad (2.1)$$

A list of functions can be obtained for different integer values of the parameter ν from (2.1).

Special Case: For $\alpha=0,\beta=1$ the function (2.1) is denoted by $L_{\upsilon}(\theta)\equiv L_{\upsilon}(0,1,\theta)$, and we term it a L-function of order υ. It is expressed as

$$L_{\upsilon}(\theta)=\sum_{x=0}^{\infty}x^{\upsilon}\theta^{x},\ \upsilon\geq 0,\ 0<\theta<1 \quad (2.2)$$

It is also defined as a power series in θ.

The other form of the above function may be proposed for strictly positive integer value of x as

$$L_{-\nu}(\alpha,\beta,\theta)=\sum_{x=1}^{\infty}(\alpha+\beta x^{-\nu})\theta^{x},\ \upsilon\geq 0,\ \alpha\geq 0,\beta>0 \text{ and } (0<\theta<1) \quad (2.3)$$

Special Case: For $\alpha=0,\beta=1$, the function (2.3) is known as a polylogarithm (Valero et al. 2022). It is defined by a power series in θ also denoted by $Li_{\upsilon}(\theta)$ and is defined for positive integer values of x as

$$L_{-\upsilon}(0,1,\theta)=Li_{\upsilon}(\theta)=\sum_{x=1}^{\infty}\frac{\theta^{x}}{x^{\upsilon}},\quad \upsilon\geq 0,\ 0<\theta<1 \quad (2.4)$$

It is also valid for a negative value of υ, then $Li_{-\upsilon}(\theta)=L_{\upsilon}(\theta)$.

3. Class of Discrete Probability Distributions

Two classes of discrete probability distributions are defined for given integer values of ν and $\alpha\geq 0$ and $\beta>0$ as

Definition 3.1. A random variable X is said to follow a class of discrete probability distributions of type I (CDPDI), if it has the probability mass function (pmf)

$$P(X=x)=\begin{cases}\frac{(\alpha+\beta x^{\nu})\theta^{x}}{L_{\nu}(\alpha,\beta,\theta)}, \alpha\geq 0,\ \beta>0,\ \upsilon\geq 0, 0<\theta<1, x=0,1,2,\ldots \\ 0,\ otherwise\end{cases} \quad (3.1)$$

where $L_{\nu}(\alpha,\beta,\theta)$ is defined in (2.1).

Special Case: For $\alpha=0,\beta=1$, the pmf (3.1) yields the distribution of L- function of order υ and is defined as follows.

Definition 3.2 A random variable X is said to follow a distribution of L- function of order υ, if it has the probability mass function

$$p(x) = \begin{cases} \frac{1}{L_\upsilon(\theta)} x^\upsilon \theta^x, \ 0 < \theta < 1, \upsilon > 0, \ x \in \{1,2,3,\dots\} \\ 0, \ Otherwise \end{cases}, \tag{3.2}$$

where $L_\upsilon(0,1,\theta) = L_\upsilon(\theta) = \sum_{x=1}^{\infty} x^\upsilon \theta^\upsilon$.

Definition 3.3 A random variable X is said to follow a class of discrete probability distributions of type II (CDPDII), if it has the probability mass function

$$P(X = x) = \begin{cases} \frac{\left(\alpha + \frac{\beta}{x^\upsilon}\right)\theta^x}{L_{-\nu}(\alpha,\beta,\theta)}, \alpha \ge 0, \ \beta > 0, 0 < \theta < 1, \upsilon \ge 0, \ x = 1,2,\dots \\ 0, \ otherwise \end{cases} \tag{3.3}$$

where $L_{-\upsilon}(\alpha,\beta,\theta)$ is defined in (2.3).

Special Case: The pmf (3.3) is said to be a polylogarithmic distributions for $\alpha = 0, \beta = 1$. The pmf of polylogarithmic distributions is given by

$$P(X = x) = \frac{\theta^x}{[Li_\upsilon(\theta)]x^\nu}, \ 0 < \theta < 1, \ \nu \ge 0, x = 1,2,\dots \tag{3.4}$$

where $Li_\nu(\theta)$ is defined in (2.4).

The pdf (3.3) can also be expressed as

$$P(X = x) = \begin{cases} \frac{\left(\alpha + \frac{\beta}{x^\upsilon}\right)\theta^x}{\alpha\theta(1-\theta)^{-1} + \beta Li_\nu(\theta)}, \quad \alpha \ge 0, \ \beta > 0, \ \upsilon \ge 0, 0 < \theta < 1, \ x = 1,2,\dots \\ 0, \ otherwise \end{cases} \tag{3.5}$$

Remarks:

1. For $\alpha = 0,$ and $\beta = 1$, the pmf's (3.3) and (3.5) both becomes the pmf (3.4).
2. Suppose $\alpha = 0, \beta = 1$ and $\theta = 1$, then the pmf's (3.3) and (3.5) becomes the zeta distribution for positive integer $x \ge 1$ with pmf (Johnson, 2005, Valero et al. 2022)

$$P(X = x) = \frac{x^{-\nu}}{\zeta(\nu)}, \ \nu > 1, x \in \{1,2,\dots\}, \tag{3.6}$$

where $\zeta(\upsilon) = \sum_{x=1}^{\infty} x^{-\upsilon}$ is the Riemann zeta function. The probability generating function (pgf) of a Zipf distributed random variable is

$$G_X(t) = E(t^X) = \sum_{x=1}^{\infty} \frac{t^x x^{-\upsilon}}{\zeta(\upsilon)} = \frac{Li_\upsilon(t)}{Li_\upsilon(1)}, \ |t| < 1, \text{and } \upsilon > 1 \tag{3.7}$$

where the $Li_\upsilon(t)$ is known as the polylogarithm function or Li function of order υ and is given by

$$Li_\upsilon(t) = \sum_{x=1}^{\infty} \frac{t^x}{x^\upsilon} \tag{3.8}$$

The Li function of any specified order is defined for any arbitrary complex number and any complex number t, for $|t| < 1$. Through analytic continuation, this function extends across the entire complex plane. For Re(t) > 0, and all t except those that are real and greater than or equal to one, the polylogarithm function can be expressed using the integral of Bose-Einstein distribution as (Valero et al. 2022)

$$Li_{\upsilon}(t) = \frac{1}{\Gamma(\upsilon)} \int_0^{\infty} \frac{u^{\upsilon-1}}{\frac{\exp(u)}{t} - 1}\, du \tag{3.9}$$

1. Suppose $\alpha = 0$, and $\theta = 1$, then the pmf (3.3) and (3.5) yields a new distribution call it a modified zeta distribution for positive integer $x \geq 1$ with pmf

$$P(X = x) = \frac{\beta x^{-\nu}}{\zeta(\beta,\nu)}, \quad \nu > 1,\ \beta > 1 \tag{3.10}$$

where $\zeta(\beta, \nu) = \sum_{x=1}^{\infty} \beta x^{-\nu}$.

4. Some Distributional Properties of CDPDs

4.1 Probability Generating Function of CDPDI

The probability generating function (pgf) of the CDPDI is

$$G_X(t) = \frac{L_{\nu}(\alpha,\beta,t\theta)}{L_{\nu}(\alpha,\beta,\theta)} \tag{4.1}$$

Remark 4: For $\alpha = 0, \beta = 1$, the pgf of the distribution of L- function of order υ is given by

$$G_X(t) = \frac{L_{\nu}(0,1,t\theta)}{L_{\nu}(0,1,\theta)} = \frac{L_{\nu}(t\theta)}{L_{\nu}(\theta)} \tag{4.1a}$$

The probability distributions of different CDPDIs are obtained from the probability generating functions using the following relation

$$P(X = x) = p(x) = \frac{1}{x!} \frac{d^x}{dt^x} \frac{L_{\nu}(\alpha,\beta,t\theta)}{L_{\nu}(\alpha,\beta,\theta)} \Big|_{t=0} \tag{4.2}$$

The descending factorial moments of the probability distributions of different CDPDI's are obtained from the pgf using the following relation

$$\mu'_{(r)} = \frac{d^r}{dt^r} \frac{L_{\nu}(\alpha,\beta,t\theta)}{L_{\nu}(\alpha,\beta,\theta)} \Big|_{t=1} \tag{4.3}$$

For different values of α, β and υ, one may get the descending factorial moments of the distributions putting $r = 1,2,3,4, ...$ in (4.3).

The pgf of the CDPDII is

$$G_X(t) = \frac{L_{-\nu}(\alpha,\beta,t\theta)}{L_{-\nu}(\alpha,\beta,\theta)} \tag{4.4}$$

Remark 5: For $\alpha = 0, \beta = 1$, the pgf of the polylogarithmic distributions is given by

$$G_X(t) = \frac{L_{-\nu}(0,1,t\theta)}{L_{-\nu}(0,1,\theta)} = \frac{Li_{\nu}(t\theta)}{Li_{\nu}(\theta)} \tag{4.4a}$$

The probability distributions of different CDPDIIs are obtained from the probability generating functions using the following relation

$$P(X = x) = p(x) = \frac{1}{x!} \frac{d^x}{dt^x} \frac{L_{-\nu}(\alpha,\beta,t\theta)}{L_{-\nu}(\alpha,\beta,\theta)} \Big|_{t=0} \tag{4.5}$$

The descending factorial moments of the probability distributions of different CDPDII's are obtained from the pgf using the following relation

$$\mu'_{(r)} = \frac{d^r}{dt^r} \frac{L_{-\nu}(\alpha,\beta,t\theta)}{L_{-\nu}(\alpha,\beta,\theta)} \Big|_{t=1} \tag{4.6}$$

For different values of α, β and υ, one may get the descending factorial moments of the distributions putting $r = 1,2,3,4, ...$ in (4.6).

4.2 Moments of the CDPDs

The rth raw moment of the CDPDI is given by

$$\mu_r' = \frac{\alpha L_r(\theta)}{L_\upsilon(\alpha,\beta,\theta)} + \frac{\beta L_{\upsilon+r}(\theta)}{\theta^r L_\upsilon(\alpha,\beta,\theta)} \tag{4.7}$$

Remark 6: For $\alpha = 0, \beta = 1$, the rth raw moment of the distribution of L- function of order υ is given by

$$\mu_r' = \frac{L_{\upsilon+r}(\theta)}{\theta^r L_\upsilon(0,1,\theta)} = \frac{L_{\upsilon+r}(\theta)}{\theta^r L_\upsilon(\theta)} \tag{4.8}$$

One can obtain different raw moments by putting $r = 1,2,3,4, \ldots$ in (4.7) and (4.8).

The rth raw moment of the CDPDII is given by

$$\mu_r' = \frac{\alpha L_r(\theta)}{L_{-\upsilon}(\alpha,\beta,\theta)} + \frac{\beta \theta^r Li_{\upsilon-r}(\theta)}{L_{-\upsilon}(\alpha,\beta,\theta)} \tag{4.9}$$

Remark 7: For $\alpha = 0, \beta = 1$, the rth raw moment of the polylogarithmic distribution is given by

$$\mu_r' = \frac{\theta^r Li_{\upsilon-r}(\theta)}{L_{-\upsilon}(0,1,\theta)} = \frac{\theta^r Li_{\upsilon-r}(\theta)}{Li_\upsilon(\theta)} \tag{4.10}$$

One can obtain different raw moments by putting $r = 1,2,3,4, \ldots$ in (4.9) and (4.10).

4.3 A Few Distributions Derived from the CDPDI

In this subsection, a few known as well as new distributions will be derived from (3.2) for different values of ν.

4.3.1 Geometric Distribution

The L- function of order $\upsilon = 0$ is represented by a geometric distribution with the probability mass function (pmf):

$$P(X = x) = (1-\theta)\theta^x, \quad x = 0,1,2,\ldots, \quad 0 < \theta \leq 1 \tag{4.11}$$

If $\theta = (1-p)$ is taken as the probability of failure, the geometric distribution (4.11) is used to model the number of failures before the first success. The pmf (4.11) indicates the probability that the first success occurs at $(x+1)$th independent trials, each trial having a failure probability of θ.

$$P(X = x) = p(1-p)^x, \quad x = 0,1,2,\ldots \tag{4.12}$$

Again, if X follows a geometric distribution with parameter p, then $Y = X + 1$ represent the shifted geometric distribution with the pmf

$$P(Y = y) = (1-p)^{y-1}p, \quad y = 1,2,\ldots \tag{4.13}$$

The pmfs' (4.11), (4.12) and (4.13) all describe the geometric distribution. Specifically, pmf (4.11) models the number of successes before the first failure, pmf (4.12) models the number of failures before the first success, and pmf (4.13) models the total number of trials up to and including the first success. There are numerous references discussing the geometric distribution (e.g., Johnson, 1969; Johnson et al., 2005; Marco, 2021).

4.3.2 A Few Distributions Derived from the L- function of order υ

The distribution of the L- function of order $\upsilon = 1$ becomes the size biased geometric distribution of order 1, which is defined in the following sub-section.

4.3.2.1 Size Biased Geometric Distribution (SBGD) of order 1

The size biased geometric distribution of order 1 is defined as

Definition 4.1 A random variable X is said to follow size biased geometric distribution of order 1 if it possesses the following probability mass function

$$P(X = x) = p(x) = \begin{cases} (1-\theta)^2 x\, \theta^{x-1}, \ 0 < \theta < 1, \ x = 1,2,\dots \\ 0, Otherwise \end{cases} \tag{4.14}$$

The distributional properties of the size biased geometric distribution of order 1 are described in the following sections.

4.3.2.2 Probability Generating Function

The pgf of the size biased geometric distribution of order 1 is given by

$$G_X(t) = E(t^X) = \frac{t(1-\theta)^2}{(1-t\theta)^2} \tag{4.15}$$

4.3.2.3 Moments of the Distribution

The mean and the variance of the size biased geometric distribution of order 1 are given below:

Mean of the distribution, $\mu_1' = \frac{(1+\theta)}{(1-\theta)}$

Variance of the distribution, $\mu_2 = \frac{2\theta}{(1-\theta)^2}$

4.3.2.4 Estimation

The parameter of the size biased geometric distribution of order 1 is estimated by the method of moments and the maximum likelihood estimation.

By the Method of Moments

Let the random sample $x_1, x, \dots, x_n$ be of n observations from the size biased geometric population of order 1 with parameter θ. Let m_1 be the mean obtained from the observations. Then the mean of the size biased geometric distribution of order 1 is $\hat{\theta} = \frac{m_1-1}{m_1+1}$, $m_1 > 1$.

By the Method of Maximum Likelihood Estimation

The maximum likelihood estimates of random sample $x_1, x, \dots, x_n$ of n observations from the size biased geometric population of order 1 is $\hat{\theta} = \frac{\bar{x}-1}{\bar{x}+1}$, $\bar{x} > 1$.

It is seen that the estimate of the parameter θ by the method of moments and the maximum likelihood estimator method is the same.

4.3.2.5 Graphical representation of the sized biased geometric distribution of order 1

The following plots represent the pattern of the size biased geometric distribution of order 1 for different values of θ.

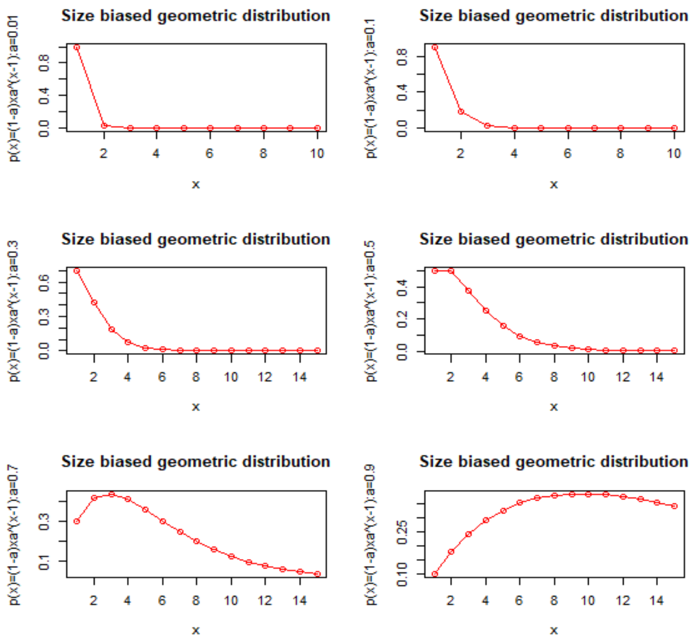

Figure 1: Plots of the size biased geometric distribution of order 1 for different values of θ

Proposition 4.1.1 Let X be a random variable following size biased geometric distribution of order 1. Then the size biased geometric distribution of order 1 does not possess the non-aging properties, i.e.,

$$P(X = x + j|X \geq j) \neq P(X = x) \quad (4.16)$$

unless $j = 0$.

4.3.3.1 Size Biased Geometric Distribution (SBGD) of Order 2

The distribution of L- function of order $\upsilon = 2$ gives the size biased geometric distribution of order 2 and is defined as

Definition 4.2 A random variable X is said to follow sized biased geometric distribution of order 2, if it has the probability mass function

$$P(X = x) = p(x) = \begin{cases} \frac{(1-\theta)^3}{(1+\theta)} x^2\, \theta^{x-1}, & 0 < \theta < 1, x = 1,2, \ldots \\ 0, Otherwise & \end{cases} \quad (4.17)$$

The distributional properties of the size biased geometric distribution of order 2 are described in the following sections.

4.3.3.2 Probability Generating Function

The pgf of the size biased geometric distribution of order 2 is given by

$$G_X(t) = E(t^X) = \frac{(1-\theta)^3}{(1-t\theta)^3} \frac{t(1+t\theta)}{(1+\theta)} \quad (4.18)$$

4.3.3.3 Graphical representation of the sized biased geometric distribution of order 2

The following plots represent the pattern of the sized biased geometric distribution of order 2 for different values of θ.

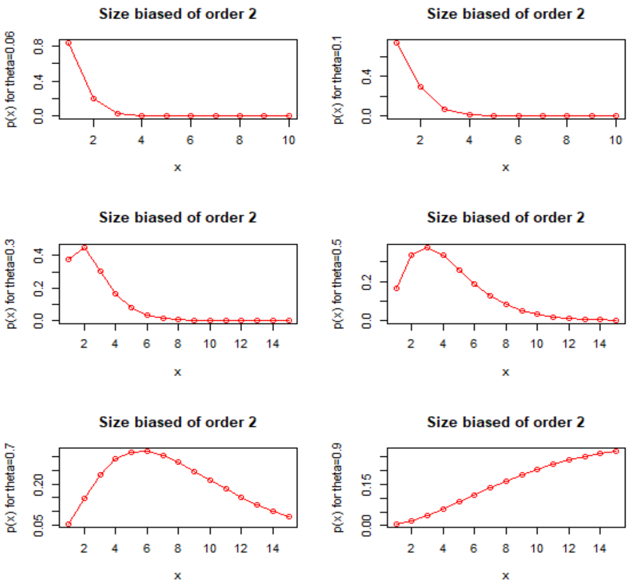

Figure 2: Plots of the size biased geometric distribution of order 2 for different values of θ

4.3.3.4 Estimation

By the Method of Moments

Let the random sample $x_1, x, \dots, x_n$ be of n observations from the size biased geometric population of order 2 with parameter θ. Let m_1 be the mean obtained from the observations. Then the mean of the size biased geometric distribution of order 2 is obtained by solving a quadratic equation and considering the positive value of θ as the estimate of θ. Thus, the estimate of θ by the method of the moment is $\hat{\theta} = \frac{\sqrt{m_1^2+3}-2}{m_1+1}$ and m_1 must be greater than 1.

By the Method of Maximum Likelihood Estimation

The maximum likelihood estimates of random sample $x_1, x, \dots, x_n$ of n observations from the size biased geometric population of order 2 is $\hat{\theta} = \frac{\sqrt{\bar{x}^2+3}-2}{\bar{x}+1}$, $\bar{x} > 1$.

It is also seen that the estimate of the parameter θ of size biased geometric distribution of order 2 by the method of moments and the maximum likelihood estimator method are the same.

4.3.3.5 Moments of the Distribution

The mean and the variance of the size biased geometric distribution of order 2 are given below:

Mean of the distribution, $\mu_1' = \frac{\theta^2+4\theta+1}{(1-\theta^2)}$

Variance of the distribution, $\mu_2 = \frac{4\theta(\theta^2+\theta+1)}{(1-\theta^2)^2}$

5. Fitting of Some Distributions to Numerical Data

In the following, we present the fitting of the size-biased geometric distribution (SBGD) of order 1 and order 2 to numerical data, alongside the zero-truncated Poisson distribution. Tables 5.1 and 5.2 compare the expected frequencies with the corresponding observed frequencies. It is observed that in Table 5.1, the SBGD of order 2 fits the data better than the SBGD of order 1 and the truncated Poisson distribution. Conversely, in Table 5.2, the SBGD of order 1 fits the data better than the zero-truncated Poisson distribution and the SBGD of order 2.

Table 5.1: The following gives the distribution of the number of albino children in families of live children indicating at least one albino child (Pearson's data).

Albino children in families of five children

No. of albinos in family	No. of families	Zero truncated Poisson	SBGD of order 1	SBGD of order 2
1	25	30.03	29.90	27.50
2	23	18.84	17.59	20.27
3	10	7.88	7.76	8.40
4	1	2.47	3.04	2.75
5	1	0.78	1.71	1.08
Total	60	60	60	60
Estimated value of the parameter		1.255	0.294	0.184
Degrees of freedom		2	2	2
Chi-square		3.2683	4.78	2.07

(Source: J.B.S. Haldane, The estimation of the frequencies of recessive conditions in man, *Annals of Eugenics*, *Vol.8*.)

Table 5.2: gives the frequencies of eggs laid by gallflies in flower heads. The count of flower heads with no eggs is not available.

No. of eggs	No. of flower heads	Zero truncated Poisson	SBGD of order 1	SBGD of order 2
1	22	12.65	21.63	16.04
2	18	19.76	21.81	23.75
3	18	20.57	16.50	19.79
4	11	16.04	11.09	13.03
5	9	10.04	6.99	7.54
6	6	5.22	4.23	4.02
7	3	2.33	2.49	2.02
8	0	0.91	1.43	0.98
9	1	0.48	1.83	0.83
Total	88	88	88	88
Estimated value of the parameter		3.124	0.504	0.370
Degrees of freedom		5	5	5
Chi-square		11.79	3.71	7.44

(Source: Handbook of Methods of Applied Statistics, Vol. 1, I. M. Chakravarty, R. G. Laha, and J. Roy, Wiley.)

Acknowledgement: The authors express their gratitude to Professor Bikas K. Sinha, Former Professor at the Indian Statistical Institute, Kolkata, for his constructive comments and advice. They also extend their appreciation to one reviewer for his valuable suggestions and critical feedback, which significantly enhanced the presentation of this article.

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