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Optimal Covariate Allocation in Asymmetrical Factorial Designs: A Comparative Analysis of Design Choices

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Abstract

Factorial designs play a crucial role in experimental research, enabling the examination of multiple factors and their influence on an outcome variable. This study delves into asymmetrical factorial designs, which are particularly useful when factors have differing numbers of levels. Expanding upon the work of Sinha *et al.* (2014, 2019), we examine a specific asymmetrical factorial design featuring two factors: one with two levels and the other with three, resulting in six distinct treatment combinations. We consider three design choices for assigning covariate values to these treatment pairs, aiming to optimize the estimation of treatment contrasts and two regression parameters. Each design offers a distinct covariate configuration, and we assess their efficiency through the information matrix and the average variance of treatment contrasts. The paper includes a comprehensive explanation of the statistical model, an evaluation of the information matrix and variance efficiency for each design, and a comparison of the average variance of treatment contrasts. Our results provide valuable insights into optimizing experimental designs in asymmetrical factorial contexts, emphasizing the trade-offs between various design strategies.

Keywords: Asymmetrical Factorial Design, Covariate Allocation, Treatment Contrasts, Regression Parameters, Variance Efficiency.

AMS Classification: 62K25.

1. Introduction

Factorial designs are a cornerstone of experimental research, allowing for the exploration of multiple factors and their effects on an outcome variable. Asymmetrical factorial designs, in particular, offer flexibility when the factors do not have a balanced number of levels. This paper extends the work of Sinha *et al.* (2014, 2019), which addressed various design set-ups and 2^n-factorial experiments, focusing on the optimal allocation of covariate values. For readers unfamiliar with this area of research, we refer to the Monograph by Das et al. (2015).

We investigate an asymmetrical factorial design with two factors: the first factor has two levels (0 and 1), while the second factor has three levels (0, 1 and 2). This configuration results in six treatment combinations: [(0,0), (0,1), (0,2), (1,0), (1,1), (1,2)]. We also consider two controllable non-stochastic covariates, X with values x_1 through x_5, and Y with values y_1 through y_5, in the closed interval [-1,1] - each of X and Y being associated with a pair of the treatment combinations. From these six treatment combinations, we derive five pairs of treatment combinations, as shown in Table 1.

Table 1: Paired combinations with their covariate values: Choice I

Set	Treatment pairs	x-values	y-values
1	(00)(01)	x_1	y_1
2	(00)(02)	x_2	y_2
3	(00)(10)	x_3	y_3
4	(10)(11)	x_4	y_4
5	(10)(12)	x_5	y_5

Under Choice I, the frequencies of the treatments are: $(00-3),(01-1),(02-1);(10-3),(11-1),(12-1)$.

Next, we explore second alternative non-transitive solution involving pairs of 5 treatment combinations across 5 sets.

Table 2: Paired combinations with their covariate values: Choice II

Set	Treatment pairs	x-values	y-values
1	(00)(01)	x_1	y_1
2	(00)(02)	x_2	y_2
3	(01)(11)	x_3	y_3
4	(10)(11)	x_4	y_4
5	(10)(12)	x_5	y_5

Under Choice II, the frequencies of the treatments are: $(00-2),(01-2),(02-1);(10-2),(11-2),(12-1)$.

Next, we explore third alternative non-transitive solution involving pairs of 5 treatment combinations across 5 sets.

Table 3: Paired combinations with their covariate values : Choice III

Set	Treatment pairs	x-values	y-values
1	(00)(01)	x_1	y_1
2	(00)(02)	x_2	y_2
3	(00)(10)	x_3	y_3
4	(00)(11)	x_4	y_4
5	(00)(12)	x_5	y_5

Under Choice III, the frequencies of the treatments are: $(00-5), (01-1), (02-1), (10-1), (11-1), (12-1)$.

No additional non-transitive solutions involving 5 sets of paired treatments are feasible from the six treatments.

Thus, we consider three different design choices for allocating covariate values to these treatment pairs, aiming to optimize the estimation of treatment contrasts and regression parameters. Each design choice provides a unique configuration of covariate values, and we evaluate their efficiency based on the information matrix and the average variance of treatment contrasts.

This paper proceeds as follows: Section 2 describes the standard model used for analysis. Section 3 details the information matrix and variance efficiency for each design choice. Section 4 compares the average variance of treatment contrasts across different choices. Finally, Section 5 discusses the findings and their implications for optimizing experimental designs.

2. Statistical Model

The standard model takes the form:

$$z_{ij} = \mu + \tau_j + \beta_1 x_i + \beta_2 y_i + e_{ij} = \tau_j^* + \beta_1 x_i + \beta_2 y_i + e_{ij} \tag{1}$$

where z_{ij} represents the observation in the experimental unit corresponding to the j-th treatment of i-th set; μ denotes the general mean effect; τ_j denotes the j-th treatment effect; β_1 and β_2 stand for the linear regression parameters, $\tau_j^* = \mu + \tau_j$ and x_i $[y_i]$ represents the covariate value to the i-th set associated with the linear effects parameter β_1 $[\beta_2]$.

The two mean responses for the two outputs in block 1: $z[(00); x_1, y_1]$ and $z[(01); x_1, y_1]$ are modelled as $\mu + \tau_{00} + \beta_1 x_1 + \beta_2 y_1$ and $\mu + \tau_{01} + \beta_1 x_1 + \beta_2 y_1$ respectively. Therefore, the treatment contrast $\tau_{00} - \tau_{01}$ can be readily estimated. It is readily verified that for each of the design choices, all the treatment contrasts are estimable.

Under the model (2.1) above, the standard representation can be given by [$\mathbf{Z}$, $\mathbf{T}\boldsymbol{\theta}$, $\sigma^2\mathbf{I}$], where $\mathbf{Z}$ is the 10×1 output vector, $\mathbf{T}$ is the 10×8 co-efficient matrix, and $\boldsymbol{\theta}$ is the parameter vector such that $\boldsymbol{\theta}' = (\tau^*_{00}, \tau^*_{01}, \tau^*_{02}, \tau^*_{10}, \tau^*_{11}, \tau^*_{12}, \beta_1, \beta_2)$.

Basically, we carry out routine data analysis - keeping in view information accrued for the β parameters from different choices of the design [in terms of the covariates' values x's and y's].

3. Data Analysis under different Choices

In this section, we analyze the information matrix and its implications for three distinct design choices. Each choice presents a unique configuration of covariate allocation, influencing the efficiency of variance estimation. We evaluate the performance of these choices to determine the optimal design based on the information provided.

3.1. Choice I

The information matrix $\mathbf{I}(\boldsymbol{\theta})$ of $\boldsymbol{\theta}$ under Choice I is given by

$$\mathbf{I}(\boldsymbol{\theta}) = \begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 & x_1+x_2+x_3 & y_1+y_2+y_3 \\ 0 & 1 & 0 & 0 & 0 & 0 & x_1 & y_1 \\ 0 & 0 & 1 & 0 & 0 & 0 & x_2 & y_2 \\ 0 & 0 & 0 & 3 & 0 & 0 & x_3+x_4+x_5 & y_3+y_4+y_5 \\ 0 & 0 & 0 & 0 & 1 & 0 & x_4 & y_4 \\ 0 & 0 & 0 & 0 & 0 & 1 & x_5 & y_5 \\ x_1+x_2+x_3 & x_1 & x_2 & x_3+x_4+x_5 & x_4 & x_5 & 2\sum_{i=1}^5 {x_i}^2 & 2\sum_{i=1}^5 x_i y_i \\ y_1+y_2+y_3 & y_1 & y_2 & y_3+y_4+y_5 & y_4 & y_5 & 2\sum_{i=1}^5 x_i y_i & 2\sum_{i=1}^5 {y_i}^2 \end{pmatrix}.$$

Then the information matrix of $\boldsymbol{\beta} = (\beta_1, \beta_2)' = I(\boldsymbol{\beta}) = \frac{1}{3}\begin{pmatrix} \phi(\mathbf{x},\mathbf{x}) & \phi(\mathbf{x},\mathbf{y}) \\ \phi(\mathbf{x},\mathbf{y}) & \phi(\mathbf{y},\mathbf{y}) \end{pmatrix}$

where

$\phi(\mathbf{x},\mathbf{y}) = ((x_1-x_2)(y_1-y_2) + (x_1-x_3)(y_1-y_3) + (x_2-x_3)(y_2-y_3)$
$+(x_3-x_4)(y_3-y_4) + (x_3-x_5)(y_3-y_5) + (x_4-x_5)(y_4-y_5))$.

Now our aim is to maximize $\phi(\mathbf{x},\mathbf{x})$ and $\phi(\mathbf{y},\mathbf{y})$ and minimize $\phi(\mathbf{x},\mathbf{y})$ simultaneously.

When x's and y's are in the closed interval $[-1,\ 1]$, it is argued that $\phi(\boldsymbol{x},\boldsymbol{x}) \leq 16$ as also $\phi(\boldsymbol{y},\boldsymbol{y}) \leq 16$. Further to this, equality is achieved in both along with $\phi(\boldsymbol{x},\boldsymbol{y}) = 0$ when the allocation design assumes one of the 64 possible forms of $\mathbf{x}$ and $\mathbf{y}$, as outlined below.

Remark 1. Since $\phi(\mathbf{x},\mathbf{x})$ is a quadratic function of $x_1,\ x_2,\ x_3,\ x_4$, and x_5, the maximum value of $\phi(\mathbf{x},\mathbf{x})$ is attained only when all x_i values are either +1 or -1. Furthermore, from the expression of $\phi(\mathbf{x},\mathbf{x})$, we observe that when x_3 is fixed, we can maximize the

Table 4:

Sl.	x_1	x_2	x_3	x_4	x_5	y_1	y_2	y_3	y_4	y_5	Sl.	x_1	x_2	x_3	x_4	x_5	y_1	y_2	y_3	y_4	y_5
1	-1	1	-1	-1	1	1	-1	-1	1	1	33	-1	-1	1	-1	1	1	-1	-1	-1	1
2	-1	1	-1	-1	1	1	1	-1	1	-1	34	-1	-1	1	-1	1	-1	1	-1	-1	1
3	-1	1	-1	1	-1	1	-1	-1	1	1	35	-1	-1	1	1	-1	-1	1	-1	1	-1
4	-1	1	-1	1	-1	1	1	-1	-1	1	36	-1	-1	1	1	-1	1	-1	-1	1	-1
5	-1	1	-1	1	1	1	-1	-1	-1	1	37	-1	1	1	-1	-1	-1	1	-1	-1	1
6	-1	1	-1	1	1	1	-1	-1	1	-1	38	-1	1	1	-1	-1	-1	1	-1	1	-1
7	1	-1	-1	-1	1	-1	1	-1	1	1	39	-1	1	1	-1	1	-1	1	-1	1	1
8	1	-1	-1	-1	1	1	1	-1	1	-1	40	-1	1	1	-1	1	1	1	-1	-1	1
9	1	-1	-1	1	-1	-1	1	-1	1	1	41	-1	1	1	1	-1	-1	1	-1	1	1
10	1	-1	-1	1	-1	1	1	-1	-1	1	42	-1	1	1	1	-1	1	1	-1	1	-1
11	1	-1	-1	1	1	-1	1	-1	-1	1	43	1	-1	1	-1	-1	1	-1	-1	1	-1
12	1	-1	-1	1	1	-1	1	-1	1	-1	44	1	-1	1	-1	-1	1	-1	-1	-1	1
13	1	1	-1	1	-1	-1	1	-1	-1	1	45	1	-1	1	-1	1	1	-1	-1	1	1
14	1	1	-1	1	-1	1	-1	-1	-1	1	46	1	-1	1	-1	1	1	1	-1	-1	1
15	1	1	-1	-1	1	-1	1	-1	1	-1	47	1	-1	1	1	-1	1	-1	-1	1	1
16	1	1	-1	-1	1	1	-1	-1	1	-1	48	1	-1	1	1	-1	1	1	-1	1	-1
17	-1	1	-1	-1	1	-1	-1	1	-1	1	49	-1	-1	1	-1	1	-1	1	1	1	-1
18	-1	1	-1	-1	1	-1	1	1	-1	-1	50	-1	-1	1	-1	1	1	-1	1	1	-1
19	-1	1	-1	1	-1	-1	-1	1	1	-1	51	-1	-1	1	1	-1	-1	1	1	-1	1
20	-1	1	-1	1	-1	-1	1	1	-1	-1	52	-1	-1	1	1	-1	1	-1	1	-1	1
21	-1	1	-1	1	1	-1	1	1	-1	1	53	-1	1	1	-1	-1	1	-1	1	-1	1
22	-1	1	-1	1	1	-1	1	1	1	-1	54	-1	1	1	-1	-1	1	-1	1	1	-1
23	1	-1	-1	-1	1	-1	-1	1	-1	1	55	-1	1	1	1	-1	-1	-1	1	-1	1
24	1	-1	-1	-1	1	1	-1	1	-1	-1	56	-1	1	1	1	-1	1	-1	1	-1	-1
25	1	-1	-1	1	-1	-1	-1	1	1	-1	57	-1	1	1	-1	1	-1	-1	1	1	-1
26	1	-1	-1	1	-1	1	-1	1	-1	-1	58	-1	1	1	-1	1	1	-1	1	-1	-1
27	1	-1	-1	1	1	1	-1	1	-1	1	59	1	-1	1	-1	-1	-1	1	1	-1	1
28	1	-1	-1	1	1	1	-1	1	1	-1	60	1	-1	1	-1	-1	-1	1	1	1	-1
29	1	1	-1	-1	1	-1	1	1	-1	1	61	1	-1	1	-1	1	-1	-1	1	1	-1
30	1	1	-1	-1	1	1	-1	1	-1	1	62	1	-1	1	-1	1	-1	1	1	-1	-1
31	1	1	-1	1	-1	-1	1	1	1	-1	63	1	-1	1	1	-1	-1	-1	1	-1	1
32	1	1	-1	1	-1	1	-1	1	1	-1	64	1	-1	1	1	-1	-1	1	1	-1	-1

two parts $(x_1 - x_2)^2 + (x_1 - x_3)^2 + (x_2 - x_3)^2$ and $(x_3 - x_4)^2 + (x_3 - x_5)^2 + (x_4 - x_5)^2$ separately. Both parts reach their maximum for the same set of solutions due to their symmetric nature. Therefore, by exploring all four choices for each part (independent of the other) with a given x_3 [say, +1], we obtain, out of 16, a total of 9 solutions. Likewise we also have another 9 solutions when x_3 assumes a different value, viz., -1. On the whole we end up with a total of 18 solutions, as shown in the following table, where $\phi(\mathbf{x}, \mathbf{x})$ attains its maximum value of 16.

Sl.no.	x_1	x_2	x_3	x_4	x_5	Sl.no.	x_1	x_2	x_3	x_4	x_5
1	1	-1	1	1	-1	10	1	-1	-1	1	-1
2	1	-1	1	-1	1	11	1	-1	-1	-1	1
3	1	-1	1	-1	-1	12	1	-1	-1	1	1
4	-1	1	1	1	-1	13	-1	1	-1	1	-1
5	-1	1	1	-1	1	14	-1	1	-1	-1	1
6	-1	1	1	-1	-1	15	-1	1	-1	1	1
7	-1	-1	1	1	-1	16	1	1	-1	1	-1
8	-1	-1	1	-1	1	17	1	1	-1	-1	1
9	-1	1	1	-1	-1	18	1	1	-1	1	1

Similarly, we can argue that $\phi(\mathbf{y}, \mathbf{y}) \leq 16$. Out of the 18×18=324 possible solutions for $(\mathbf{x}, \mathbf{y})$, where both $\phi(\mathbf{x}, \mathbf{x}) \leq 16$ and $\phi(\mathbf{y}, \mathbf{y}) \leq 16$ are satisfied, we identified 64 solutions, as shown in Table 4, that meet the additional condition $\phi(\mathbf{x}, \mathbf{y}) = 0$. These solutions were derived using the R code provided in Appendix 1.2. In fine, we find that for Choice I, $Det(I(\boldsymbol{\beta})) = 256/9 = 28.44$.

3.2. Choice II

The information matrix $\mathbf{I}(\boldsymbol{\theta})$ of $\boldsymbol{\theta}$ under Choice II is given by

$$\mathbf{I}(\boldsymbol{\theta}) = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & x_1+x_2 & y_1+y_2 \\ 0 & 2 & 0 & 0 & 0 & 0 & x_1+x_3 & y_1+y_3 \\ 0 & 0 & 1 & 0 & 0 & 0 & x_2 & y_2 \\ 0 & 0 & 0 & 2 & 0 & 0 & x_4+x_5 & y_4+y_5 \\ 0 & 0 & 0 & 0 & 2 & 0 & x_3+x_4 & y_3+y_4 \\ 0 & 0 & 0 & 0 & 0 & 1 & x_5 & y_5 \\ x_1+x_2 & x_1+x_3 & x_2 & x_4+x_5 & x_3+x_4 & x_5 & 2\sum_{i=1}^5 {x_i}^2 & 2\sum_{i=1}^5 x_i y_i \\ y_1+y_2 & y_1+y_3 & y_2 & y_4+y_5 & y_3+y_4 & y_5 & 2\sum_{i=1}^5 x_i y_i & \sum_{i=1}^5 {y_i}^2 \end{pmatrix}.$$

Then the information matrix of $\boldsymbol{\beta} = (\beta_1, \beta_2)'$=$I(\boldsymbol{\beta}) = \frac{1}{2}\begin{pmatrix} \psi(\mathbf{x},\mathbf{x}) & \psi(\mathbf{x},\mathbf{y}) \\ \psi(\mathbf{x},\mathbf{y}) & \psi(\mathbf{y},\mathbf{y}) \end{pmatrix}$

where

$\psi(\mathbf{x},\mathbf{y}) = (x_1-x_2)(y_1-y_2)+(x_1-x_3)(y_1-y_3)+(x_3-x_4)(y_3-y_4)+(x_4-x_5)(y_4-y_5)$.

$Det(I(\boldsymbol{\beta})) = \frac{1}{4}\left(\psi(\mathbf{x},\mathbf{x})\psi(\mathbf{y},\mathbf{y}) - (\psi(\mathbf{x},\mathbf{y}))^2\right) \leq 44$.

Equality is attained when the allocation design takes on one of the 16 possible configurations of $\mathbf{x}$ and $\mathbf{y}$, as described below:

Table 5: Solutions (x, y) where the maximum value of $Det(I(\boldsymbol{\beta}))$ is attained under Choice II

Sl. No.	x_1	x_2	x_3	x_4	x_5	y_1	y_2	y_3	y_4	y_5	$\psi(\mathbf{x},\mathbf{x})$	$\psi(\mathbf{y},\mathbf{y})$	$\psi(\mathbf{x},\mathbf{y})$	$Det(I(\boldsymbol{\beta}))$
1	1	-1	-1	1	-1	-1	1	-1	1	-1	16	12	4	44
2	1	-1	-1	1	-1	-1	1	1	1	-1	16	12	-4	44
3	1	-1	-1	1	-1	1	-1	-1	-1	1	16	12	4	44
4	1	-1	-1	1	-1	1	-1	1	-1	1	16	12	-4	44
5	-1	1	-1	1	-1	1	-1	-1	1	-1	12	16	4	44
6	-1	1	-1	1	-1	-1	1	1	-1	1	12	16	-4	44
7	-1	1	1	1	-1	1	-1	-1	1	-1	12	16	-4	44
8	-1	1	1	1	-1	-1	1	1	-1	1	12	16	4	44
9	1	-1	-1	-1	1	1	-1	-1	1	-1	12	16	4	44
10	1	-1	-1	-1	1	-1	1	1	-1	1	12	16	-4	44
11	1	-1	1	-1	1	1	-1	-1	1	-1	12	16	-4	44
12	1	-1	1	-1	1	-1	1	1	-1	1	12	16	4	44
13	-1	1	1	-1	1	-1	1	-1	1	-1	16	12	-4	44
14	-1	1	1	-1	1	-1	1	1	1	-1	16	12	4	44
15	-1	1	1	-1	1	1	-1	-1	-1	1	16	12	-4	44
16	-1	1	1	-1	1	1	-1	1	-1	1	16	12	4	44

Remark 2. We use the R code provided in Appendix 1.3 to generate Table 5.

3.3. Choice III

The information matrix $\mathbf{I}(\boldsymbol{\theta})$ of $\boldsymbol{\theta}$ under Choice III is given by

$$\mathbf{I}(\boldsymbol{\theta}) = \begin{pmatrix} 5 & 0 & 0 & 0 & 0 & 0 & \sum_{i=1}^{5} x_i & \sum_{i=1}^{5} y_i \\ 0 & 1 & 0 & 0 & 0 & 0 & x_1 & y_1 \\ 0 & 0 & 1 & 0 & 0 & 0 & x_2 & y_2 \\ 0 & 0 & 0 & 1 & 0 & 0 & x_3 & y_3 \\ 0 & 0 & 0 & 0 & 1 & 0 & x_4 & y_4 \\ 0 & 0 & 0 & 0 & 0 & 1 & x_5 & y_5 \\ \sum_{i=1}^{5} x_i & x_1 & x_2 & x_3 & x_4 & x_5 & 2\sum_{i=1}^{5} {x_i}^2 & 2\sum_{i=1}^{5} x_i y_i \\ \sum_{i=1}^{5} y_i & y_1 & y_2 & y_3 & y_4 & y_5 & 2\sum_{i=1}^{5} x_i y_i & 2\sum_{i=1}^{5} {y_i}^2 \end{pmatrix}.$$

Then the information matrix of $\boldsymbol{\beta} = (\beta_1, \beta_2)' = I(\boldsymbol{\beta}) = \begin{pmatrix} \chi(\mathbf{x}, \mathbf{x}) & \chi(\mathbf{x}, \mathbf{y}) \\ \chi(\mathbf{x}, \mathbf{y}) & \chi(\mathbf{y}, \mathbf{y}) \end{pmatrix}$
where $\chi(\mathbf{x}, \mathbf{y}) = \sum x_i y_i - \frac{1}{5}\left(\sum_{i=1}^{5} x_i\right)\left(\sum_{i=1}^{5} y_i\right)$.
$Det(I(\boldsymbol{\beta})) = \chi(\mathbf{x}, \mathbf{x})\chi(\mathbf{y}, \mathbf{y}) - (\chi(\mathbf{x}, \mathbf{y}))^2 \leq 22.4$.

Equality is attained when the allocation design takes on one of the 240 possible configurations of $\mathbf{x}$ and $\mathbf{y}$, as described below in Table 10.

Remark 3. From Table 10 given in Appendix 1.1, the maximum value of 22.4 is attained for $Det(I(\boldsymbol{\beta}))$ when possible combinations of the x's and y's have 2 (or 3) values being '1' each, while the rest are '-1' each. It may be noted that to start with, there are altogether $4 \times 10 \times 10 = 400$ possible paired choices for x and y. Out of these, we have 240 choices each of which provides the above maximum value under Choice III. We use R code to identify these 240 solutions, which is provided in Appendix 1.4.

Remark 4. Globally, however, it transpires that Choice III is least preferred and Choice II is most preferred for joint estimation of the beta-parameters.

Remark 5. Since the designs are all treatment connected, we may as well compare them with respect to minimum average variance of treatment contrasts.

4. Average Variance of five treatment contrasts under different Choices

We choose a full set of 5 treatment contrasts as $\boldsymbol{\eta} = (\tau_{00} - \tau_{01}, \tau_{00} - \tau_{02}, \tau_{00} - \tau_{10}, \tau_{00} - \tau_{11}, \tau_{00} - \tau_{12})'$.

We work out variance of each of the treatment contrasts and compare the aggregates corresponding to different design choices. Before we do the computations, we will introduce self-explanatory notations for Y-values.

Table 6: Paired combinations with their Y values : Choice I

Set	Treatment pairs	Y-values
1	(00) (01)	$Y_{00.1}, Y_{01.1}$
2	(00)(02)	$Y_{00.2}, Y_{02.1}$
3	(00)(10)	$Y_{00.3}, Y_{10.1}$
4	(10)(11)	$Y_{10.2}, Y_{11.1}$
5	(10)(12)	$Y_{10.3}, Y_{12.1}$

4.1. Design Choice I

With reference to Choice I, we note that

$\hat{\boldsymbol{\eta}}\prime = (Y_{00.1} - Y_{01.1}, Y_{00.2} - Y_{02.1}, Y_{00.3} - Y_{10.1}, Y_{00.3} - Y_{10.1} + Y_{10.2} - Y_{11.1}, Y_{00.3} - Y_{10.1} + Y_{10.3} - Y_{12.1})$

Note that $Y_{00.1}$ corresponds to the position of (00) in the first set of Choice I. Likewise, all such repeat allocations are marked.

This yields : Variance components as multiples of σ^2 and the multipliers are respectively : 2, 2, 2, 4, 4. Therefore average variance amounts to 14/5 times σ^2.

4.2. Design Choice II

Table 7: Paired combinations with their Y-values : Choice II

Set	Treatment pairs	Y-values
1	(00)(01)	$Y_{00.1}, Y_{01.1}$
2	(00)(02)	$Y_{00.2}, Y_{02.1}$
3	(01)(11)	$Y_{01.2}, Y_{11.1}$
4	(10)(11)	$Y_{10.1}, Y_{11.2}$
5	(10)(12)	$Y_{10.2}, Y_{12.1}$

We find that

$$\hat{\boldsymbol{\eta}}' = (Y_{00.1} - Y_{01.1}, Y_{00.2} - Y_{02.1}, Y_{00.1} - Y_{01.1} + Y_{01.2} - Y_{11.1} + Y_{11.2} - Y_{10.1}, Y_{00.1} - Y_{01.1} + Y_{01.2} - Y_{11.1}, Y_{00.1} - Y_{01.1} + Y_{01.2} - Y_{11.1} + Y_{11.2} - Y_{10.1} + Y_{10.2} - Y_{12.1}).$$

This yields the variances of estimates of the components of $\boldsymbol{\eta}$ as multiples of σ^2 with the respective multipliers as : 2, 2, 6, 4, 8 and hence the average variance as $(22/5)\times\sigma^2$.

4.3. Design Choice III

We find that

$\hat{\boldsymbol{\eta}}' = (Y_{00.1} - Y_{01.1}, Y_{00.2} - Y_{02.1}, Y_{00.3} - Y_{10.1}, Y_{00.4} - Y_{11.1}, Y_{00.5} - Y_{12.1})$

This yields the variances of estimates of the components of $\boldsymbol{\eta}$ each one as multiple of σ^2, again each with the multiplier as 2. The average variance is $2\times\sigma^2$.

Table 8: Paired combinations with their Y- values : Choice III

Set	Treatment pairs	Y-values
1	(00)(01)	$Y_{00.1}, Y_{01.1}$
2	(00)(02)	$Y_{00.2}, Y_{02.1}$
3	(00)(10)	$Y_{00.3}, Y_{10.1}$
4	(00)(11)	$Y_{00.4}, Y_{11.1}$
5	(00)(12)	$Y_{00.5}, Y_{12.1}$

On the whole, therefore, we conclude that variance comparisons suggest the ranking : Choice III better than Choice I better than Choice II.

5. Discussion

From Section 3, it is evident that Choice III is the least preferred, while Choice II is the most preferred when the primary objective is to estimate the regression parameters with maximum efficiency. However, as noted in Section 4, when the focus shifts to the five treatment contrasts $\boldsymbol{\eta}$, Choice III is superior to Choice I, which in turn is superior to Choice II in terms of average variance. Thus, the preferences in these two situations are directly opposite.
When we are interested in both $\boldsymbol{\eta}$ and $\boldsymbol{\beta}$ parameters, the following table presents the average variance of $(\hat{\boldsymbol{\eta}}, \hat{\boldsymbol{\beta}})$ under the three different choices.

Table 9: Average Variance for $(\boldsymbol{\eta}, \boldsymbol{\beta})$

Choice	Average Variance of estimates
I	$\left(\frac{\frac{6}{16}+14}{7}\right)\sigma^2 = 2.05\sigma^2$
II	$\left(\frac{\frac{16/2+12/2}{44}+22}{7}\right)\sigma^2 = 3.19\sigma^2$
III	$\left(\frac{\frac{4.8+4.8}{22.4}+10}{7}\right)\sigma^2 = 1.49\sigma^2$

Therefore, in terms of average variance, when considering the joint estimation of $(\boldsymbol{\eta}, \boldsymbol{\beta})$, Choice III is superior to Choice I, which in turn is superior to Choice II.
Finally, the case of a single controllable quantitative covariate can be directly inferred from our analysis of two covariates, and thus, it does not warrant further elaboration here.

References

[1] Das, P., Dutta, G., Mandal, N.K., Sinha, B.K. (2015). Optimal Covariate Designs (OCDs). Springer, New Delhi. $https://doi.org/10.1007/978-81-322-2461-7_1$

[2] Sinha, Bikas K., Rao, P.S.S.N.V.P., Mathew, T., and Rao, S. B. (2014). A new class of optimal designs in the presence of a quantitative covariate. *International Journal of Statistical Sciences*, **14**, 1-15.

[3] Sinha, Bikas K. and Rao, P.S.S.N.V.P. (2019). Some aspects of optimal covariate designs in factorial experiments. *Statistics and Applications*, **17**, 77-86.

Appendix 1.1: Solutions (x, y) where the maximum value of $Det(I(\boldsymbol{\beta}))$ is attained

Table 10:

Sl. No.	x_1	x_2	x_3	x_4	x_5	y_1	y_2	y_3	y_4	y_5	$\chi(\mathbf{x},\mathbf{x})$	$\chi(\mathbf{y},\mathbf{y})$	$\chi(\mathbf{x},\mathbf{y})$	$Det(I(\boldsymbol{\beta}))$
1	1	1	-1	-1	-1	1	-1	1	-1	-1	4.8	4.8	0.8	22.4
2	1	1	-1	-1	-1	-1	1	1	-1	-1	4.8	4.8	0.8	22.4
3	1	1	-1	-1	-1	1	-1	-1	1	-1	4.8	4.8	0.8	22.4
4	1	1	-1	-1	-1	-1	1	-1	1	-1	4.8	4.8	0.8	22.4
5	1	1	-1	-1	-1	1	-1	1	1	-1	4.8	4.8	-0.8	22.4
6	1	1	-1	-1	-1	-1	1	1	1	-1	4.8	4.8	-0.8	22.4
7	1	1	-1	-1	-1	1	-1	-1	-1	1	4.8	4.8	0.8	22.4
8	1	1	-1	-1	-1	-1	1	-1	-1	1	4.8	4.8	0.8	22.4
9	1	1	-1	-1	-1	1	-1	1	-1	1	4.8	4.8	-0.8	22.4
10	1	1	-1	-1	-1	-1	1	1	-1	1	4.8	4.8	-0.8	22.4
11	1	1	-1	-1	-1	1	-1	-1	1	1	4.8	4.8	-0.8	22.4
12	1	1	-1	-1	-1	-1	1	-1	1	1	4.8	4.8	-0.8	22.4
13	1	-1	1	-1	-1	1	1	-1	-1	-1	4.8	4.8	0.8	22.4
14	1	-1	1	-1	-1	-1	1	1	-1	-1	4.8	4.8	0.8	22.4
15	1	-1	1	-1	-1	1	-1	-1	1	-1	4.8	4.8	0.8	22.4
16	1	-1	1	-1	-1	1	1	-1	1	-1	4.8	4.8	-0.8	22.4
17	1	-1	1	-1	-1	-1	-1	1	1	-1	4.8	4.8	0.8	22.4
18	1	-1	1	-1	-1	-1	1	1	1	-1	4.8	4.8	-0.8	22.4
19	1	-1	1	-1	-1	1	-1	-1	-1	1	4.8	4.8	0.8	22.4
20	1	-1	1	-1	-1	1	1	-1	-1	1	4.8	4.8	-0.8	22.4
21	1	-1	1	-1	-1	-1	-1	1	-1	1	4.8	4.8	0.8	22.4
22	1	-1	1	-1	-1	-1	1	1	-1	1	4.8	4.8	-0.8	22.4
23	1	-1	1	-1	-1	1	-1	-1	1	1	4.8	4.8	-0.8	22.4
24	1	-1	1	-1	-1	-1	-1	1	1	1	4.8	4.8	-0.8	22.4
25	-1	1	1	-1	-1	1	1	-1	-1	-1	4.8	4.8	0.8	22.4
26	-1	1	1	-1	-1	1	-1	1	-1	-1	4.8	4.8	0.8	22.4
27	-1	1	1	-1	-1	-1	1	-1	1	-1	4.8	4.8	0.8	22.4
28	-1	1	1	-1	-1	1	1	-1	1	-1	4.8	4.8	-0.8	22.4
29	-1	1	1	-1	-1	-1	-1	1	1	-1	4.8	4.8	0.8	22.4
30	-1	1	1	-1	-1	1	-1	1	1	-1	4.8	4.8	-0.8	22.4
31	-1	1	1	-1	-1	-1	1	-1	-1	1	4.8	4.8	0.8	22.4
32	-1	1	1	-1	-1	1	1	-1	-1	1	4.8	4.8	-0.8	22.4
33	-1	1	1	-1	-1	-1	-1	1	-1	1	4.8	4.8	0.8	22.4
34	-1	1	1	-1	-1	1	-1	1	-1	1	4.8	4.8	-0.8	22.4
35	-1	1	1	-1	-1	-1	1	-1	1	1	4.8	4.8	-0.8	22.4
36	-1	1	1	-1	-1	-1	-1	1	1	1	4.8	4.8	-0.8	22.4
37	1	1	1	-1	-1	1	-1	-1	1	-1	4.8	4.8	-0.8	22.4
38	1	1	1	-1	-1	-1	1	-1	1	-1	4.8	4.8	-0.8	22.4
39	1	1	1	-1	-1	1	1	-1	1	-1	4.8	4.8	0.8	22.4
40	1	1	1	-1	-1	-1	-1	1	1	-1	4.8	4.8	-0.8	22.4
41	1	1	1	-1	-1	1	-1	1	1	-1	4.8	4.8	0.8	22.4
42	1	1	1	-1	-1	-1	1	1	1	-1	4.8	4.8	0.8	22.4
43	1	1	1	-1	-1	1	-1	-1	-1	1	4.8	4.8	-0.8	22.4
44	1	1	1	-1	-1	-1	1	-1	-1	1	4.8	4.8	-0.8	22.4
45	1	1	1	-1	-1	1	1	-1	-1	1	4.8	4.8	0.8	22.4
46	1	1	1	-1	-1	-1	-1	1	-1	1	4.8	4.8	-0.8	22.4
47	1	1	1	-1	-1	1	-1	1	-1	1	4.8	4.8	0.8	22.4
48	1	1	1	-1	-1	-1	1	1	-1	1	4.8	4.8	0.8	22.4

Sl. No.	x_1	x_2	x_3	x_4	x_5	y_1	y_2	y_3	y_4	y_5	$\chi(\mathbf{x},\mathbf{x})$	$\chi(\mathbf{y},\mathbf{y})$	$\chi(\mathbf{x},\mathbf{y})$	$Det(I(\boldsymbol{\beta}))$
49	1	-1	-1	1	-1	1	1	-1	-1	-1	4.8	4.8	0.8	22.4
50	1	-1	-1	1	-1	1	-1	1	-1	-1	4.8	4.8	0.8	22.4
51	1	-1	-1	1	-1	1	1	1	-1	-1	4.8	4.8	-0.8	22.4
52	1	-1	-1	1	-1	-1	1	-1	1	-1	4.8	4.8	0.8	22.4
53	1	-1	-1	1	-1	-1	-1	1	1	-1	4.8	4.8	0.8	22.4
54	1	-1	-1	1	-1	-1	1	1	1	-1	4.8	4.8	-0.8	22.4
55	1	-1	-1	1	-1	1	-1	-1	-1	1	4.8	4.8	0.8	22.4
56	1	-1	-1	1	-1	1	1	-1	-1	1	4.8	4.8	-0.8	22.4
57	1	-1	-1	1	-1	1	-1	1	-1	1	4.8	4.8	-0.8	22.4
58	1	-1	-1	1	-1	-1	-1	-1	1	1	4.8	4.8	0.8	22.4
59	1	-1	-1	1	-1	-1	1	-1	1	1	4.8	4.8	-0.8	22.4
60	1	-1	-1	1	-1	-1	-1	1	1	1	4.8	4.8	-0.8	22.4
61	-1	1	-1	1	-1	1	1	-1	-1	-1	4.8	4.8	0.8	22.4
62	-1	1	-1	1	-1	-1	1	1	-1	-1	4.8	4.8	0.8	22.4
63	-1	1	-1	1	-1	1	1	1	-1	-1	4.8	4.8	-0.8	22.4
64	-1	1	-1	1	-1	1	-1	-1	1	-1	4.8	4.8	0.8	22.4
65	-1	1	-1	1	-1	-1	-1	1	1	-1	4.8	4.8	0.8	22.4
66	-1	1	-1	1	-1	1	-1	1	1	-1	4.8	4.8	-0.8	22.4
67	-1	1	-1	1	-1	-1	1	-1	-1	1	4.8	4.8	0.8	22.4
68	-1	1	-1	1	-1	1	1	-1	-1	1	4.8	4.8	-0.8	22.4
69	-1	1	-1	1	-1	-1	1	1	-1	1	4.8	4.8	-0.8	22.4
70	-1	1	-1	1	-1	-1	-1	-1	1	1	4.8	4.8	0.8	22.4
71	-1	1	-1	1	-1	1	-1	-1	1	1	4.8	4.8	-0.8	22.4
72	-1	1	-1	1	-1	-1	-1	1	1	1	4.8	4.8	-0.8	22.4
73	1	1	-1	1	-1	1	-1	1	-1	-1	4.8	4.8	-0.8	22.4
74	1	1	-1	1	-1	-1	1	1	-1	-1	4.8	4.8	-0.8	22.4
75	1	1	-1	1	-1	1	1	1	-1	-1	4.8	4.8	0.8	22.4
76	1	1	-1	1	-1	-1	-1	1	1	-1	4.8	4.8	-0.8	22.4
77	1	1	-1	1	-1	-1	1	-1	1	1	4.8	4.8	0.8	22.4
78	1	1	-1	1	-1	-1	1	1	1	-1	4.8	4.8	0.8	22.4
79	1	1	-1	1	-1	-1	1	-1	-1	1	4.8	4.8	-0.8	22.4
80	1	1	-1	1	-1	1	-1	-1	1	1	4.8	4.8	0.8	22.4
81	1	1	-1	1	-1	1	1	-1	-1	1	4.8	4.8	0.8	22.4
82	1	1	-1	1	-1	-1	-1	1	1	1	4.8	4.8	0.8	22.4
83	1	1	-1	1	-1	-1	1	-1	-1	1	4.8	4.8	0.8	22.4
84	-1	-1	1	1	-1	1	-1	1	-1	-1	4.8	4.8	0.8	22.4
85	-1	-1	1	1	-1	-1	1	1	-1	-1	4.8	4.8	0.8	22.4
86	-1	-1	1	1	-1	1	1	1	-1	-1	4.8	4.8	-0.8	22.4
87	-1	-1	1	1	-1	1	-1	-1	1	-1	4.8	4.8	0.8	22.4
88	-1	-1	1	1	-1	-1	1	-1	1	-1	4.8	4.8	0.8	22.4
89	-1	-1	1	1	-1	1	1	-1	1	-1	4.8	4.8	-0.8	22.4
90	-1	-1	1	1	-1	-1	-1	1	-1	1	4.8	4.8	0.8	22.4
91	-1	-1	1	1	-1	1	-1	1	-1	1	4.8	4.8	-0.8	22.4
92	-1	-1	1	1	-1	-1	1	1	-1	1	4.8	4.8	-0.8	22.4
93	-1	-1	1	1	-1	-1	-1	-1	1	1	4.8	4.8	0.8	22.4
94	-1	-1	1	1	-1	1	-1	-1	1	1	4.8	4.8	-0.8	22.4
95	-1	-1	1	1	-1	-1	1	-1	1	1	4.8	4.8	-0.8	22.4
96	1	-1	1	1	-1	1	1	-1	-1	-1	4.8	4.8	-0.8	22.4
97	1	-1	1	1	-1	-1	1	1	-1	-1	4.8	4.8	-0.8	22.4
98	1	-1	1	1	-1	1	1	1	-1	-1	4.8	4.8	0.8	22.4
99	1	-1	1	1	-1	-1	1	-1	1	-1	4.8	4.8	-0.8	22.4
100	1	-1	1	1	-1	1	1	-1	1	-1	4.8	4.8	0.8	22.4
101	1	-1	1	1	-1	1	1	-1	1	-1	4.8	4.8	0.8	22.4

Sl. No.	x_1	x_2	x_3	x_4	x_5	y_1	y_2	y_3	y_4	y_5	$\chi(\mathbf{x},\mathbf{x})$	$\chi(\mathbf{y},\mathbf{y})$	$\chi(\mathbf{x},\mathbf{y})$	$Det(I(\boldsymbol{\beta}))$
102	1	-1	1	1	-1	-1	1	1	1	-1	4.8	4.8	0.8	22.4
103	1	-1	1	1	-1	1	-1	-1	-1	1	4.8	4.8	-0.8	22.4
104	1	-1	1	1	-1	-1	-1	1	-1	1	4.8	4.8	-0.8	22.4
105	1	-1	1	1	-1	1	-1	1	-1	1	4.8	4.8	0.8	22.4
106	1	-1	1	1	-1	-1	-1	-1	1	1	4.8	4.8	-0.8	22.4
107	1	-1	1	1	-1	1	-1	-1	1	1	4.8	4.8	0.8	22.4
108	1	-1	1	1	-1	-1	-1	1	1	1	4.8	4.8	0.8	22.4
109	-1	1	1	1	-1	1	1	-1	-1	-1	4.8	4.8	-0.8	22.4
110	-1	1	1	1	-1	1	-1	1	-1	-1	4.8	4.8	-0.8	22.4
111	-1	1	1	1	-1	1	1	1	-1	-1	4.8	4.8	0.8	22.4
112	-1	1	1	1	-1	1	-1	-1	1	-1	4.8	4.8	-0.8	22.4
113	-1	1	1	1	-1	1	1	-1	1	-1	4.8	4.8	0.8	22.4
114	-1	1	1	1	-1	1	-1	1	1	-1	4.8	4.8	0.8	22.4
115	-1	1	1	1	-1	-1	1	-1	-1	1	4.8	4.8	-0.8	22.4
116	-1	1	1	1	-1	-1	-1	1	-1	1	4.8	4.8	-0.8	22.4
117	-1	1	1	1	-1	-1	1	1	-1	1	4.8	4.8	0.8	22.4
118	-1	1	1	1	-1	-1	-1	-1	1	1	4.8	4.8	-0.8	22.4
119	-1	1	1	1	-1	-1	1	-1	1	1	4.8	4.8	0.8	22.4
120	-1	1	1	1	-1	-1	-1	-1	1	1	4.8	4.8	-0.8	22.4
121	1	-1	1	1	-1	1	1	-1	-1	-1	4.8	4.8	0.8	22.4
122	1	-1	1	1	-1	-1	1	1	1	-1	4.8	4.8	0.8	22.4
123	1	-1	1	1	-1	1	-1	-1	-1	1	4.8	4.8	-0.8	22.4
124	1	-1	1	1	-1	-1	-1	1	-1	1	4.8	4.8	-0.8	22.4
125	1	-1	1	1	-1	1	-1	1	-1	1	4.8	4.8	0.8	22.4
126	1	-1	1	1	-1	-1	-1	-1	1	1	4.8	4.8	-0.8	22.4
127	1	-1	1	1	-1	-1	1	-1	1	1	4.8	4.8	0.8	22.4
128	1	-1	1	1	-1	-1	1	1	-1	1	4.8	4.8	-0.8	22.4
129	1	-1	1	1	-1	-1	-1	-1	1	1	4.8	4.8	0.8	22.4
130	1	-1	1	1	-1	-1	1	-1	1	1	4.8	4.8	-0.8	22.4
131	1	-1	-1	-1	1	-1	1	-1	1	1	4.8	4.8	-0.8	22.4
132	1	-1	-1	-1	1	-1	-1	1	1	1	4.8	4.8	-0.8	22.4
133	-1	1	-1	-1	1	1	1	-1	-1	-1	4.8	4.8	0.8	22.4
134	-1	1	-1	-1	1	-1	1	1	-1	-1	4.8	4.8	0.8	22.4
135	-1	1	-1	-1	1	1	1	1	-1	-1	4.8	4.8	-0.8	22.4
136	-1	1	-1	-1	1	-1	1	-1	1	-1	4.8	4.8	0.8	22.4
137	-1	1	-1	-1	1	1	1	-1	1	-1	4.8	4.8	-0.8	22.4
138	-1	1	-1	-1	1	-1	1	1	1	-1	4.8	4.8	-0.8	22.4
139	-1	1	-1	-1	1	1	-1	-1	-1	1	4.8	4.8	0.8	22.4
140	-1	1	-1	-1	1	-1	-1	1	-1	1	4.8	4.8	0.8	22.4
141	-1	1	-1	-1	1	1	-1	1	-1	1	4.8	4.8	-0.8	22.4
142	-1	1	-1	-1	1	-1	-1	-1	1	1	4.8	4.8	0.8	22.4
143	-1	1	-1	-1	1	1	-1	-1	1	1	4.8	4.8	-0.8	22.4
144	1	1	-1	-1	1	1	-1	1	-1	-1	4.8	4.8	-0.8	22.4
145	1	1	-1	-1	1	-1	1	1	-1	-1	4.8	4.8	-0.8	22.4
146	1	1	-1	-1	1	1	1	1	-1	-1	4.8	4.8	0.8	22.4
147	1	1	-1	-1	1	1	-1	-1	1	-1	4.8	4.8	-0.8	22.4
148	1	1	-1	-1	1	-1	1	-1	1	-1	4.8	4.8	-0.8	22.4
149	1	1	-1	-1	1	1	1	-1	1	-1	4.8	4.8	0.8	22.4
150	1	1	-1	-1	1	-1	-1	1	-1	1	4.8	4.8	-0.8	22.4
151	1	1	-1	-1	1	1	-1	1	-1	1	4.8	4.8	0.8	22.4
152	1	1	-1	-1	1	-1	1	1	-1	1	4.8	4.8	0.8	22.4
153	1	1	-1	-1	1	-1	-1	-1	1	1	4.8	4.8	-0.8	22.4
154	1	1	-1	-1	1	1	-1	-1	1	1	4.8	4.8	0.8	22.4
155	1	1	-1	-1	1	-1	1	-1	1	1	4.8	4.8	0.8	22.4
156	-1	-1	1	-1	1	1	-1	1	-1	-1	4.8	4.8	0.8	22.4

Sl. No.	x_1	x_2	x_3	x_4	x_5	y_1	y_2	y_3	y_4	y_5	$\chi(\mathbf{x},\mathbf{x})$	$\chi(\mathbf{y},\mathbf{y})$	$\chi(\mathbf{x},\mathbf{y})$	$Det(I(\boldsymbol{\beta}))$
157	-1	-1	1	-1	1	-1	1	1	-1	-1	4.8	4.8	0.8	22.4
158	-1	-1	1	-1	1	1	1	1	-1	-1	4.8	4.8	-0.8	22.4
159	-1	-1	1	-1	1	-1	-1	1	1	-1	4.8	4.8	0.8	22.4
160	-1	-1	1	-1	1	-1	-1	1	1	-1	4.8	4.8	0.8	22.4
161	-1	-1	1	-1	1	1	-1	1	1	-1	4.8	4.8	-0.8	22.4
162	-1	-1	1	-1	1	-1	1	1	1	-1	4.8	4.8	-0.8	22.4
163	-1	-1	1	-1	1	1	-1	-1	-1	1	4.8	4.8	0.8	22.4
164	-1	-1	1	-1	1	-1	1	-1	-1	1	4.8	4.8	0.8	22.4
165	-1	-1	1	-1	1	1	1	-1	-1	1	4.8	4.8	-0.8	22.4
166	-1	-1	1	-1	1	-1	-1	-1	1	1	4.8	4.8	0.8	22.4
167	-1	-1	1	-1	1	1	-1	-1	1	1	4.8	4.8	-0.8	22.4
168	-1	-1	1	-1	1	-1	1	-1	1	1	4.8	4.8	-0.8	22.4
169	1	-1	1	-1	1	1	1	-1	-1	-1	4.8	4.8	-0.8	22.4
170	1	-1	1	-1	1	-1	1	1	-1	-1	4.8	4.8	-0.8	22.4
171	1	-1	1	-1	1	1	1	1	-1	-1	4.8	4.8	0.8	22.4
172	1	-1	1	-1	1	1	-1	-1	1	-1	4.8	4.8	-0.8	22.4
173	1	-1	1	-1	1	-1	-1	1	1	-1	4.8	4.8	-0.8	22.4
174	1	-1	1	-1	1	1	-1	1	1	-1	4.8	4.8	0.8	22.4
175	1	-1	1	-1	1	-1	1	-1	-1	1	4.8	4.8	-0.8	22.4
176	1	-1	1	-1	1	1	1	-1	-1	1	4.8	4.8	0.8	22.4
177	1	-1	1	-1	1	-1	1	1	-1	1	4.8	4.8	0.8	22.4
178	1	-1	1	-1	1	-1	-1	-1	1	1	4.8	4.8	-0.8	22.4
179	1	-1	1	-1	1	1	-1	-1	1	1	4.8	4.8	0.8	22.4
180	1	-1	1	-1	1	-1	-1	1	1	1	4.8	4.8	0.8	22.4
181	-1	1	1	-1	1	1	1	-1	-1	-1	4.8	4.8	-0.8	22.4
182	-1	1	1	-1	1	1	-1	1	-1	-1	4.8	4.8	-0.8	22.4
183	-1	1	1	-1	1	1	1	1	-1	-1	4.8	4.8	0.8	22.4
184	-1	1	1	-1	1	-1	1	-1	1	-1	4.8	4.8	-0.8	22.4
185	-1	1	1	-1	1	-1	-1	1	1	-1	4.8	4.8	-0.8	22.4
186	-1	1	1	-1	1	-1	1	1	1	-1	4.8	4.8	0.8	22.4
187	-1	1	1	-1	1	1	-1	-1	-1	1	4.8	4.8	-0.8	22.4
188	-1	1	1	-1	1	1	1	-1	-1	1	4.8	4.8	0.8	22.4
189	-1	1	1	-1	1	1	-1	1	-1	1	4.8	4.8	0.8	22.4
190	-1	1	1	-1	1	-1	-1	-1	1	1	4.8	4.8	-0.8	22.4
191	-1	1	1	-1	1	-1	1	-1	1	1	4.8	4.8	0.8	22.4
192	-1	1	1	-1	1	-1	1	1	1	1	4.8	4.8	0.8	22.4
193	-1	-1	-1	1	1	1	-1	-1	1	-1	4.8	4.8	0.8	22.4
194	-1	-1	-1	1	1	-1	1	-1	1	-1	4.8	4.8	0.8	22.4
195	-1	-1	-1	1	1	1	1	-1	1	-1	4.8	4.8	-0.8	22.4
196	-1	-1	-1	1	1	-1	-1	1	1	-1	4.8	4.8	0.8	22.4
197	-1	-1	-1	1	1	1	-1	1	1	-1	4.8	4.8	-0.8	22.4
198	-1	-1	-1	1	1	-1	1	1	1	-1	4.8	4.8	-0.8	22.4
199	-1	-1	-1	1	1	1	-1	-1	-1	1	4.8	4.8	0.8	22.4
200	-1	-1	-1	1	1	-1	1	-1	-1	1	4.8	4.8	0.8	22.4
201	-1	-1	-1	1	1	1	1	-1	-1	1	4.8	4.8	-0.8	22.4
202	-1	-1	-1	1	1	-1	-1	1	-1	1	4.8	4.8	0.8	22.4
203	-1	-1	-1	1	1	1	-1	1	-1	1	4.8	4.8	-0.8	22.4
204	-1	-1	-1	1	1	-1	1	1	-1	1	4.8	4.8	-0.8	22.4
205	1	-1	-1	1	1	1	1	-1	-1	-1	4.8	4.8	-0.8	22.4
206	1	-1	-1	1	1	1	-1	1	-1	-1	4.8	4.8	-0.8	22.4
207	1	-1	-1	1	1	-1	1	-1	1	-1	4.8	4.8	-0.8	22.4
208	1	-1	-1	1	1	1	1	-1	1	-1	4.8	4.8	0.8	22.4
209	1	-1	-1	1	1	-1	-1	1	1	-1	4.8	4.8	-0.8	22.4
210	1	-1	-1	1	1	1	-1	1	1	-1	4.8	4.8	0.8	22.4

Sl. No.	x_1	x_2	x_3	x_4	x_5	y_1	y_2	y_3	y_4	y_5	$\chi(\mathbf{x},\mathbf{x})$	$\chi(\mathbf{y},\mathbf{y})$	$\chi(\mathbf{x},\mathbf{y})$	$Det(I(\boldsymbol{\beta}))$
211	1	-1	-1	1	1	-1	1	-1	-1	1	4.8	4.8	-0.8	22.4
212	1	-1	-1	1	1	1	1	-1	-1	1	4.8	4.8	0.8	22.4
213	1	-1	-1	1	1	-1	-1	1	-1	1	4.8	4.8	-0.8	22.4
214	1	-1	-1	1	1	1	-1	1	-1	1	4.8	4.8	0.8	22.4
215	1	-1	-1	1	1	-1	1	-1	1	1	4.8	4.8	0.8	22.4
216	1	-1	-1	1	1	-1	-1	1	1	1	4.8	4.8	0.8	22.4
217	-1	1	-1	1	1	1	1	-1	-1	-1	4.8	4.8	-0.8	22.4
218	-1	1	-1	1	1	-1	1	1	-1	-1	4.8	4.8	-0.8	22.4
219	-1	1	-1	1	1	1	-1	-1	1	-1	4.8	4.8	-0.8	22.4
220	-1	1	-1	1	1	1	1	-1	1	-1	4.8	4.8	0.8	22.4
221	-1	1	-1	1	1	-1	-1	1	1	-1	4.8	4.8	-0.8	22.4
222	-1	1	-1	1	1	-1	1	1	1	-1	4.8	4.8	0.8	22.4
223	-1	1	-1	1	1	1	-1	-1	-1	1	4.8	4.8	-0.8	22.4
224	-1	1	-1	1	1	1	1	-1	-1	1	4.8	4.8	0.8	22.4
225	-1	1	-1	1	1	-1	-1	1	-1	1	4.8	4.8	-0.8	22.4
226	-1	1	-1	1	1	-1	1	1	-1	1	4.8	4.8	0.8	22.4
227	-1	1	-1	1	1	1	-1	-1	1	1	4.8	4.8	0.8	22.4
228	-1	1	-1	1	1	-1	-1	1	1	1	4.8	4.8	0.8	22.4
229	-1	-1	1	1	1	1	-1	1	-1	-1	4.8	4.8	-0.8	22.4
230	-1	-1	1	1	1	-1	1	1	-1	-1	4.8	4.8	-0.8	22.4
231	-1	-1	1	1	1	1	-1	-1	1	-1	4.8	4.8	-0.8	22.4
232	-1	-1	1	1	1	-1	1	-1	1	-1	4.8	4.8	-0.8	22.4
233	-1	-1	1	1	1	1	-1	1	1	-1	4.8	4.8	0.8	22.4
234	-1	-1	1	1	1	-1	1	1	1	-1	4.8	4.8	0.8	22.4
235	-1	-1	1	1	1	1	-1	-1	-1	1	4.8	4.8	-0.8	22.4
236	-1	-1	1	1	1	-1	1	-1	-1	1	4.8	4.8	-0.8	22.4
237	-1	-1	1	1	1	1	-1	1	-1	1	4.8	4.8	0.8	22.4
238	-1	-1	1	1	1	1	-1	-1	1	1	4.8	4.8	0.8	22.4
239	-1	-1	1	1	1	-1	1	-1	1	1	4.8	4.8	0.8	22.4
240	-1	-1	1	1	1	1	-1	1	1	-1	4.8	4.8	0.8	22.4

Appendix 1.2: Construction of Table 4

```
# Define the 18x5 matrix based on your provided data
choices <- matrix(c(
  1, -1,  1,  1, -1,
  1, -1,  1, -1,  1,
  1, -1,  1, -1, -1,
 -1,  1,  1,  1, -1,
 -1,  1,  1, -1,  1,
 -1,  1,  1, -1, -1,
 -1, -1,  1,  1, -1,
 -1, -1,  1, -1,  1,
 -1, -1,  1, -1, -1,
  1, -1, -1,  1,  1,
  1, -1, -1,  1, -1,
  1, -1, -1, -1,  1,
 -1,  1, -1,  1,  1,
 -1,  1, -1,  1, -1,
 -1,  1, -1, -1,  1,
  1,  1, -1,  1,  1,
  1,  1, -1,  1, -1,
  1,  1, -1, -1,  1
```

```
), nrow = 18, ncol = 5, byrow = TRUE)

# Initialize a list to store valid combinations
valid_combinations <- list()

# Loop through all combinations of rows for x's and y's
for (i in 1:18) {
  for (j in 1:18) {
    # Extract the current choices for x's and y's
    x <- choices[i, ]
    y <- choices[j, ]

    # Calculate the expression
    expression_value <- (x[1] - x[2]) * (y[1] - y[2]) +
                        (x[1] - x[3]) * (y[1] - y[3]) +
                        (x[2] - x[3]) * (y[2] - y[3]) +
                        (x[3] - x[4]) * (y[3] - y[4]) +
                        (x[3] - x[5]) * (y[3] - y[5]) +
                        (x[4] - x[5]) * (y[4] - y[5])

    # Check if the expression equals zero
    if (expression_value == 0) {
      # Store the combination if valid
      valid_combinations[[length(valid_combinations) + 1]] <- c(x, y)
    }
  }
}

# Print the valid combinations in the form (x, y)
if (length(valid_combinations) > 0) {
  cat("Valid combinations (x, y) where the expression equals 0:\n")

  for (combo in valid_combinations) {
    cat("(x, y) =", combo, "\n")
  }

  # Also print as a matrix
  valid_combinations_matrix <- do.call(rbind, valid_combinations)
  cat("\nMatrix format:\n")
  print(valid_combinations_matrix)
} else {
  cat("No valid combinations found.\n")
}
```

Appendix 1.3: Construction of Table 5

```
# Function to evaluate the expression and output term1, term2, term3, and result
evaluate_function <- function(x, y) {
  term1 <- (x[1] - x[2])^2 + (x[1] - x[3])^2 + (x[3] - x[4])^2 + (x[4] - x[5])^2
```

```
  term2 <- (y[1] - y[2])^2 + (y[1] - y[3])^2 + (y[3] - y[4])^2 + (y[4] - y[5])^2
  term3 <- (x[1] - x[2]) * (y[1] - y[2]) + (x[1] - x[3]) * (y[1] - y[3]) +
           (x[3] - x[4]) * (y[3] - y[4]) + (x[4] - x[5]) * (y[4] - y[5])

  # Calculate the objective function
  result <- 1/4 * ((term1 * term2) - term3^2)

  # Return all terms and the result
  return(list(term1 = term1, term2 = term2, term3 = term3, result = result))
}

# Generate all combinations of x_i = 1 or -1, and y_i = 1 or -1
generate_combinations <- function() {
  grid <- expand.grid(rep(list(c(-1, 1)), 5))
  return(as.matrix(grid))
}

# Find all solutions where the maximum value is attained
find_all_max_solutions <- function() {
  x_combinations <- generate_combinations()
  y_combinations <- generate_combinations()

  max_value <- -Inf
  solutions <- data.frame() # Initialize an empty data frame

  # Loop over all combinations of x and y
  for (i in 1:nrow(x_combinations)) {
    for (j in 1:nrow(y_combinations)) {
      x <- x_combinations[i, ]
      y <- y_combinations[j, ]

      evaluation <- evaluate_function(x, y)
      value <- evaluation$result

      if (value > max_value) {
        # Found a new maximum value, reset the solutions data frame
        max_value <- value
        solutions <- data.frame() # Reset the data frame
        # Add the new solution
        solutions <- rbind(solutions, data.frame(
          x1 = x[1], x2 = x[2], x3 = x[3], x4 = x[4], x5 = x[5],
          y1 = y[1], y2 = y[2], y3 = y[3], y4 = y[4], y5 = y[5],
          term1 = evaluation$term1, term2 = evaluation$term2,
          term3 = evaluation$term3, result = value
        ))
      } else if (value == max_value) {
        # Found another solution with the same maximum value
        solutions <- rbind(solutions, data.frame(
          x1 = x[1], x2 = x[2], x3 = x[3], x4 = x[4], x5 = x[5],
          y1 = y[1], y2 = y[2], y3 = y[3], y4 = y[4], y5 = y[5],
```

```
          term1 = evaluation$term1, term2 = evaluation$term2,
          term3 = evaluation$term3, result = value
        ))
      }
    }
  }

  list(max_value = max_value, solutions = solutions)
}

# Run the search for all maximum solutions
result <- find_all_max_solutions()

# Output the result in tabular form
cat("Maximum Value:", result$max_value, "\n")
cat("Solutions (x, y) where maximum is attained:\n")
print(result$solutions)
```

Appendix 1.4: Construction of Table 10

```
# Function to evaluate the expression and output term1, term2, term3, and result
evaluate_function <- function(x, y) {
  term1 <- x[1]^2 + x[2]^2 + x[3]^2 + x[4]^2 + x[5]^2
  - (x[1] + x[2] + x[3] + x[4] + x[5])^2 / 5
  term2 <- y[1]^2 + y[2]^2 + y[3]^2 + y[4]^2 + y[5]^2
  - (y[1] + y[2] + y[3] + y[4] + y[5])^2 / 5
  term3 <- x[1]*y[1] + x[2]*y[2] + x[3]*y[3] + x[4]*y[4] + x[5]*y[5]
  - ((x[1] + x[2] + x[3] + x[4] + x[5]) * (y[1] + y[2] + y[3] + y[4] + y[5])) / 5

  # Calculate the objective function
  result <- (term1 * term2) - term3^2

  # Return all terms and the result
  return(list(term1 = term1, term2 = term2, term3 = term3, result = result))
}

# Generate all combinations of x_i = 1 or -1, and y_i = 1 or -1
generate_combinations <- function() {
  grid <- expand.grid(rep(list(c(-1, 1)), 5))
  return(as.matrix(grid))
}

# Find all solutions where the maximum value is attained
find_all_max_solutions <- function() {
  x_combinations <- generate_combinations()
  y_combinations <- generate_combinations()

  max_value <- -Inf
  solutions <- data.frame() # Initialize an empty data frame
```

```
  # Loop over all combinations of x and y
  for (i in 1:nrow(x_combinations)) {
    for (j in 1:nrow(y_combinations)) {
      x <- x_combinations[i, ]
      y <- y_combinations[j, ]

      evaluation <- evaluate_function(x, y)
      value <- evaluation$result

      if (value > max_value) {
        # Found a new maximum value, reset the solutions data frame
        max_value <- value
        solutions <- data.frame() # Reset the data frame
        # Add the new solution
        solutions <- rbind(solutions, data.frame(
          x1 = x[1], x2 = x[2], x3 = x[3], x4 = x[4], x5 = x[5],
          y1 = y[1], y2 = y[2], y3 = y[3], y4 = y[4], y5 = y[5],
          term1 = evaluation$term1, term2 = evaluation$term2,
          term3 = evaluation$term3, result = value
        ))
      } else if (value == max_value) {
        # Found another solution with the same maximum value
        solutions <- rbind(solutions, data.frame(
          x1 = x[1], x2 = x[2], x3 = x[3], x4 = x[4], x5 = x[5],
          y1 = y[1], y2 = y[2], y3 = y[3], y4 = y[4], y5 = y[5],
          term1 = evaluation$term1, term2 = evaluation$term2,
          term3 = evaluation$term3, result = value
        ))
      }
    }
  }

  list(max_value = max_value, solutions = solutions)
}

# Run the search for all maximum solutions
result <- find_all_max_solutions()

# Output the result in tabular form
cat("Maximum Value:", result$max_value, "\n")
cat("Solutions (x, y) where maximum is attained:\n")
print(result$solutions)
```