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Understanding Chao (Biometrika, 1982) [Paper on ΠPS Sampling Schemes]

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Abstract

Chao's (1982) sampling scheme offers a systematic approach to select samples based on probability proportional to size (PPS) sampling without replacement but it might be difficult to grasp, particularly for entry-level researchers. In response, this study revisits Chao's method with the aim of providing a simplified and more intuitive understanding. Drawing inspiration from efforts by Dr. Tommy Wright and subsequent group discussions with BKS, we present a step-by-step breakdown of Chao's scheme with illustrative examples, emphasizing its elegance and practicality. This study showcases the value of making complex statistical methods accessible for broader engagement in research and practice.

Keywords and Phrases: Inclusion probabilities, Πps sampling scheme, Chao's Sampling scheme, fixed sized design, algorithm.

AMS Classification: 62D05.

1. Introduction

Chao (1982) provided a general-purpose sampling scheme for the selection of a sample of n units from a finite population of N units by adopting a probability proportional to size sampling without replacement. The scheme keeps the sample size fixed and the population units are added one at a time.

In a way, the population size is increased from n to N, one unit at a time while keeping the sample size n fixed. When the population size is increased by one unit, a decision is made whether to sample the new unit or not. If sampled, then one of the units from the pre-existing sample is replaced by the new unit. As such, the sample size does not change while the population size increases by one. This sampling plan's construction via induction ensures that the first-order inclusion probabilities are proportional to the sizes, and it facilitates the computation of all high-order inclusion probabilities, providing deeper insights into the sampling outcomes and their implications. Its flexibility allows for application to different types of populations and sample sizes. The simplicity and elegance of Chao's method contribute to the practicality and effectiveness of this method in achieving a representative sample. Overall, the paper provides a valuable contribution to the field of unequal probability sampling.

Our purpose is to re-visit this scheme and bring out its elegance in our own way of understanding. We believe, our presentation, based on our re-visit, will enlighten entry-level researchers in this area of constructions of Fixed Size Πps sampling designs. While doing so, we provide an illustrative example to understand the process along with theoretical motivation. We intend to show the fundamental development of the method and related formulas.

In this context, it may be pertinent to mention one incident. Chao's paper is difficult to read and at first sight, it may look too obscure. In an attempt to decipher this paper, Dr. Tommy Wright, Head, Statistics Division, US BUREAU OF CENSUS, had prepared a 22-page 'made easy'! He invited the senior-most author of this manuscript [BKS] for a discussion on his understanding of Chao's paper. During the course of discussions, BKS felt that there should a simpler and logical way of understanding Chao's approach. He then prepared his own 'notes' towards "Chao – Made Easy"!!! That was sent to this group of researchers and our detailed study followed. We believe, entry-level researchers in this fascinating area of search, will get motivated after following the steps elaborated in this manuscript.

2. Scheme Parameters

$$[N, n, k, X = (0 < x_1 \leq x_2 \leq x_3 \leq \cdots \leq x_n \leq \cdots \leq x_N < \infty)]$$

Given the scheme parameters: Purpose is to suggest an algorithm for construction of ΠPS Sampling Schemes for fixed sample size n and for population sizes $k = n, n+1, n+2, \ldots$..upto a N $[> n]$ – not necessarily specified beforehand.

Note 1: For $k = n$, trivial unique solution is given by $s(n|k = n) = (U_1, U_2, \ldots, U_n)$ irrespective of the x- values $0 < x_1 \leq x_2 \leq x_3 \leq \cdots < x_n < \infty$.

We consider an illustrative example with $N = 6$ and $n = 3$. Set x-values as

$$X = [0 < a \leq b \leq c \leq d \leq e \leq f < \infty]$$

(with strict inequality in at least one place) and let the partial sums be denoted by

$$\sum a = a, \quad \sum b = a + b, \quad \sum c = a + b + c, \quad \sum d = a + b + c + d$$
$$\sum e = a + b + c + d + e, \quad \sum f = a + b + c + d + e + f$$

As is mentioned in Note 1, for $k = 3$, the trivial solution is $s(3|k = 3) = (U_1, U_2, U_3)$ with $P(s) = 1$, irrespective of the values assumed by (a, b, c).
For $k = 3$

Table 1

s	$P(s)$
(123)	1

For $k = 4$ the solution is as given in Table 2,

Table 2

s	$P(s)$
(123)	$1 - \frac{3d}{\sum d}$
(124)	$1 - \frac{3c}{\sum d}$
(134)	$1 - \frac{3b}{\sum d}$
(234)	$1 - \frac{3a}{\sum d}$

Proof: For k=4, n=3, we start with the solution for $k = n = 3$ i.e., $s(3) = (1, 2, 3)$ with probability 1. Now we introduce U_4 and consider its inclusion/exclusion probabilities. Its exclusion leads to the unique triplet (1, 2, 3). On the other hand, being a ΠPS design of size 3, $P(U_4 \; is \; included) = \pi_4 = \frac{3d}{\sum d}$ and hence

$$P(U_4 \; is \; excluded) = 1 - \frac{3d}{\sum d} = P[s = (1, 2, 3) | k = 4, n = 3].$$

Likewise,

$$P[s = (1, 2, 4) | k = 4, n = 3] = Prob.[U_3 \; is \; excluded \; while \; k = 4] = 1 - \frac{3c}{\sum d}.$$

Similarly, it transpires that

$$Q_1 = 1 - \frac{3a}{\sum d} \; and \; Q_2 = 1 - \frac{3b}{\sum d}.$$

Hence the solution for k = 4 is attained.

We will designate these 4 samples as PARENTAL SAMPLES when we extend the sampling operation from k = 4 to k=5.

3. Case of k = 5 and n = 3:

When U_5 is introduced with x-value 'e', we use $\sum e$ as reference value hence forth. Further,

$$P[U_5 \; is \; included | k = 5, n = 3] = \pi_5 = \frac{3e}{\sum e}$$

[ΠPS property for U_5]

We now expand the Table: 2 for [n=3, k = 4] to one for [$n = 3, k = 5$]. Each Parental Sample will generate 4 samples with/without inclusion of U_5.

Therefore, to start with, there are 4x4=16 triplets of samples – each of size 3. A careful analysis suggests that there are altogether 10 different samples, each being a triplet on its own.

[Watch: $\binom{5}{3}$=10].

For $k = 5$

Table 3

Parental Sample	s	$P(s)$
(123)	(123)	$\left(1 - \frac{3d}{\sum d}\right)\left(1 - \frac{3e}{\sum e}\right)$
	(125)	$\left(1 - \frac{3d}{\sum d}\right)\left(\frac{e}{\sum e}\right)$
	(135)	$\left(1 - \frac{3d}{\sum d}\right)\left(\frac{e}{\sum e}\right)$
	(235)	$\left(1 - \frac{3d}{\sum d}\right)\left(\frac{e}{\sum e}\right)$
(124)	(124)	$\left(1 - \frac{3c}{\sum d}\right)\left(1 - \frac{3e}{\sum e}\right)$
	(125)	$\left(1 - \frac{3c}{\sum d}\right)\left(\frac{e}{\sum e}\right)$
	(145)	$\left(1 - \frac{3c}{\sum d}\right)\left(\frac{e}{\sum e}\right)$

	(245)	$\left(1-\frac{3c}{\sum d}\right)\left(\frac{e}{\sum e}\right)$
(134)	(134)	$\left(1-\frac{3b}{\sum d}\right)\left(1-\frac{3e}{\sum e}\right)$
	(135)	$\left(1-\frac{3b}{\sum d}\right)\left(\frac{e}{\sum e}\right)$
	(145)	$\left(1-\frac{3b}{\sum d}\right)\left(\frac{e}{\sum e}\right)$
	(345)	$\left(1-\frac{3b}{\sum d}\right)\left(\frac{e}{\sum e}\right)$
(234)	(234)	$\left(1-\frac{3a}{\sum d}\right)\left(1-\frac{3e}{\sum e}\right)$
	(235)	$\left(1-\frac{3a}{\sum d}\right)\left(\frac{e}{\sum e}\right)$
	(245)	$\left(1-\frac{3a}{\sum d}\right)\left(\frac{e}{\sum e}\right)$
	(345)	$\left(1-\frac{3a}{\sum d}\right)\left(\frac{e}{\sum e}\right)$

Note 2: In the above we have all the 4 Parental Samples [$wrt\ k = 4$] regenerated when k=5 is considered. This happens when U_5 is excluded. Besides, there are 3x4=12 new samples generated in the process – each with inclusion of U_5.

It is interesting to note that these 12 samples are, in fact, six (6) distinct triplets – each with frequency 2. These naturally correspond to inclusion of U_5 with prob. $\frac{3e}{\sum e}$.

We show the 6 triplets below.

Table 4

s	$P(s)$
(125)	$\left(1-\frac{3c}{\sum d}\right)\left(\frac{e}{\sum e}\right)+\left(1-\frac{3d}{\sum d}\right)\left(\frac{e}{\sum e}\right)$
(135)	$\left(1-\frac{3b}{\sum d}\right)\left(\frac{e}{\sum e}\right)+\left(1-\frac{3d}{\sum d}\right)\left(\frac{e}{\sum e}\right)$
(145)	$\left(1-\frac{3b}{\sum d}\right)\left(\frac{e}{\sum e}\right)+\left(1-\frac{3c}{\sum d}\right)\left(\frac{e}{\sum e}\right)$
(235)	$\left(1-\frac{3a}{\sum d}\right)\left(\frac{e}{\sum e}\right)+\left(1-\frac{3d}{\sum d}\right)\left(\frac{e}{\sum e}\right)$
(245)	$\left(1-\frac{3a}{\sum d}\right)\left(\frac{e}{\sum e}\right)+\left(1-\frac{3c}{\sum d}\right)\left(\frac{e}{\sum e}\right)$
(345)	$\left(1-\frac{3a}{\sum d}\right)\left(\frac{e}{\sum e}\right)+\left(1-\frac{3b}{\sum d}\right)\left(\frac{e}{\sum e}\right)$

This subtotal of $P(\ldots)$'s is given by

$$\left(\frac{3e}{\sum e}\right)\left[\left(1-\frac{3a}{\sum d}\right)+\left(1-\frac{3b}{\sum d}\right)+\left(1-\frac{3c}{\sum d}\right)+\left(1-\frac{3d}{\sum d}\right)\right]$$

$= \frac{3e}{\sum e}$.

Verification of ΠPS Sampling Scheme for N=k=5, n=3:

$\pi_1 = P(123) + P(124) + P(134) + P(125) + P(135) + P(145)$

$= \left(\frac{\left(\frac{3a}{\sum d}\right) (\sum e - e)}{\sum e} \right) = \frac{3a}{\sum e}$, *since* $\sum e - e = \sum d$.

Concentrate on π_2:

$\pi_2 = P(123) + P(124) + P(234) + P(125) + P(235) + P(245)$

$= \frac{3b}{\sum e}$, *as expected*.

Now comes π_3:

$\pi_3 = P(123) + P(134) + P(234) + P(135) + P(235) + P(345)$

$= \frac{3c}{\sum e}$, *hence verified*.

Now we focus on π_4:

$\pi_4 = P(124) + P(134) + P(234) + P(145) + P(245) + P(345)$

$= \frac{3d}{\sum e}$, *as it should be*.

Finally, we focus on π_5:

$\pi_5 = P(125) + P(135) + P(145) + P(235) + P(245) + P(345)$

$= \frac{3e}{\sum e}$, *as expected*.

ROUTINE COMPUTATIONS OF SECOND ORDER INCLUSION PROBABILITIES FOLLOW FROM ANALYSIS OF TABLE 4.

1) $\pi_{12} = P(123) + P(124) + P(125)$

$= \left[\left(1 - \frac{3c}{\sum d}\right) + \left(1 - \frac{3d}{\sum d}\right)\right]\left(1 - \frac{2e}{\sum e}\right)$

2) $\pi_{13} = P(123) + P(134) + P(135)$

$= \left[\left(1 - \frac{3b}{\sum d}\right) + \left(1 - \frac{3d}{\sum d}\right)\right]\left(1 - \frac{2e}{\sum e}\right)$

3) $\pi_{14} = P(124) + P(134) + P(145)$

$= \left[\left(1 - \frac{3b}{\sum d}\right) + \left(1 - \frac{3c}{\sum d}\right)\right]\left(1 - \frac{3e}{\sum e}\right)$

4) $\pi_{15} = P(125) + P(135) + P(145)$

$= \frac{6ae}{(\sum d)(\sum e)}$

5) $\pi_{23} = P(123) + P(234) + P(235)$

$= \left[\left(1 - \frac{3a}{\sum d}\right) + \left(1 - \frac{3d}{\sum d}\right)\right]\left(1 - \frac{2e}{\sum e}\right)$

6) $\pi_{24} = P(124) + P(234) + P(245)$
$= \left[\left(1-\frac{3a}{\sum d}\right)+\left(1-\frac{3c}{\sum d}\right)\right]\left(1-\frac{2e}{\sum e}\right)$
7) $\pi_{25} = P(125) + P(235) + P(245)$
$= \frac{6be}{(\sum d)(\sum e)}$
8) $\pi_{34} = P(134) + P(234) + P(345)$
$= \left[\left(1-\frac{3a}{\sum d}\right)+\left(1-\frac{3b}{\sum d}\right)\right]\left(1-\frac{2e}{\sum e}\right)$
9) $\pi_{35} = P(135) + P(235) + P(345)$
$= \frac{6ce}{(\sum d)(\sum e)}$
10) $\pi_{45} = P(145) + P(245) + P(345)$
$= \frac{6de}{(\sum d)(\sum e)}$

Hence, by summing over 1), 2), 3) and 4), we deduce
$\pi_{12} + \pi_{13} + \pi_{14} + \pi_{15}$
$= \frac{6a}{\sum e} = 2 \times \frac{3a}{\sum e} = 2\pi_1, \textit{as expected}.$

Likewise,
$\pi_{21} + \pi_{23} + \pi_{24} + \pi_{25}$
$= \frac{6b}{\sum e} = 2 \times \frac{3b}{\sum e} = 2\pi_2$

Similarly
$\pi_{31} + \pi_{32} + \pi_{34} + \pi_{35}$
$= \frac{6c}{\sum e} = 2 \times \frac{3c}{\sum e} = 2\pi_3$

Finally,
$\pi_{41} + \pi_{42} + \pi_{43} + \pi_{45}$
$= \frac{6d}{\sum e} = 2 \times \frac{3d}{\sum e} = 2\pi_4$

Further, with reference to unit U_5, we deduce
$\pi_{51} + \pi_{52} + \pi_{53} + \pi_{54}$
$= \frac{6e}{(\sum d)(\sum e)}(\sum d) = 2 \times \frac{3e}{\sum e} = 2\pi_5$.

Therefore,
$$\pi_i = np_i \textit{ for all } i, \;\; \textit{where } p_i = \frac{X_i}{\sum X_i} \textit{ and } \sum_{j(\neq i)} \pi_{ij} = (n-1)\pi_i \quad \textit{for all } i.$$

The sampling design is a fixed size ($n = 3, k = 5$) sampling design. Therefore, the third property namely,
$\sum\sum_{i\neq j} \pi_{ij} = Var(\nu(s)) + E[\nu(s)(\nu(s)-1)]$
is trivially satisfied.

We may as well examine this property directly.

$$\sum\sum_{i\neq j}\pi_{ij} = (\pi_{12}+\pi_{13}+\pi_{14}+\pi_{15}) + (\pi_{21}+\pi_{23}+\pi_{24}+\pi_{25}) + (\pi_{31}+\pi_{32}+\pi_{34}+\pi_{35})$$
$$+ (\pi_{41}+\pi_{42}+\pi_{43}+\pi_{45}) + (\pi_{51}+\pi_{52}+\pi_{53}+\pi_{54})$$
$$= 2\pi_1 + 2\pi_2 + 2\pi_3 + 2\pi_4 + 2\pi_5$$
$$= 2(\pi_1+\pi_2+\pi_3+\pi_4+\pi_5)$$
$$= 2\times 3$$
$$= n(n-1)$$

Alternatively,
We know that $\sum_{j(\neq i)}\pi_{ij} = (n-1)\pi_i \quad for\ all\ i.$

$$\sum\sum_{i\neq j}\pi_{ij} = \sum_i (n-1)\pi_i$$
$$= (n-1)\sum_i \pi_i$$
$$= (n-1)\left(\frac{3a}{\sum e}+\frac{3b}{\sum e}+\frac{3c}{\sum e}+\frac{3d}{\sum e}+\frac{3e}{\sum e}\right)$$
$$= (n-1)\times 3$$
$$= n(n-1)$$

As the sampling design is a fixed size sampling design, $v(s) = n.$

$$So, \sum\sum_{i\neq j}\pi_{ij} = Var\big(v(s)\big) + v(s)(v(s)-1) = n(n-1)$$

where v = no of distinct unts in sample, is trivially satisfied. It may be noted that in the whole analysis we end up with a fixed size sampling design. As such the third property is trivially satisfied for all the resulting designs.

4. Case for k=6:

We now continue to work out the nature of sampling scheme for k=6, n=3. This time we consider $X_6 = f$ and introduce $\sum f$. Naturally each of the 16 samples will, in its turn, result into 4 samples with/without inclusion of U_6.

Thus, in principle, 64 samples will be generated and of these, there will be 16 PARENTAL LEVEL SAMPLES OF SECONDARY TYPE i.e., OF STAGE II--- INCLUDING THE FOUR PRIMARY LEVEL SAMPLES OF STAGE I.

Note 3. There are altogether 64 triplets of samples and these are divided into 20 distinct triplets – with varying frequencies like 1, 2, 4 and 6.
For k=6, n=3

Table 5

Stage I sample	Stage 2 sample	Stage 3 sample (s)	P(s)
123	123	123	$\left(1-\frac{3d}{\sum d}\right)\left(1-\frac{3e}{\sum e}\right)\left(1-\frac{3f}{\sum f}\right)$
		126	$\left(1-\frac{3d}{\sum d}\right)\left(1-\frac{3e}{\sum e}\right)\left(\frac{f}{\sum f}\right)$
		136	$\left(1-\frac{3d}{\sum d}\right)\left(1-\frac{3e}{\sum e}\right)\left(\frac{f}{\sum f}\right)$

		236	$\left(1-\frac{3d}{\sum d}\right)\left(1-\frac{3e}{\sum e}\right)\left(\frac{f}{\sum f}\right)$
	125	125	$\left(1-\frac{3d}{\sum d}\right)\left(\frac{e}{\sum e}\right)\left(1-\frac{3f}{\sum f}\right)$
		126	$\left(1-\frac{3d}{\sum d}\right)\left(\frac{e}{\sum e}\right)\left(\frac{f}{\sum f}\right)$
		156	$\left(1-\frac{3d}{\sum d}\right)\left(\frac{e}{\sum e}\right)\left(\frac{f}{\sum f}\right)$
		256	$\left(1-\frac{3d}{\sum d}\right)\left(\frac{e}{\sum e}\right)\left(\frac{f}{\sum f}\right)$
	135	135	$\left(1-\frac{3d}{\sum d}\right)\left(\frac{e}{\sum e}\right)\left(1-\frac{3f}{\sum f}\right)$
		136	$\left(1-\frac{3d}{\sum d}\right)\left(\frac{e}{\sum e}\right)\left(\frac{f}{\sum f}\right)$
		156	$\left(1-\frac{3d}{\sum d}\right)\left(\frac{e}{\sum e}\right)\left(\frac{f}{\sum f}\right)$
		356	$\left(1-\frac{3d}{\sum d}\right)\left(\frac{e}{\sum e}\right)\left(\frac{f}{\sum f}\right)$
	235	235	$\left(1-\frac{3d}{\sum d}\right)\left(\frac{e}{\sum e}\right)\left(1-\frac{3f}{\sum f}\right)$
		236	$\left(1-\frac{3d}{\sum d}\right)\left(\frac{e}{\sum e}\right)\left(\frac{f}{\sum f}\right)$
		256	$\left(1-\frac{3d}{\sum d}\right)\left(\frac{e}{\sum e}\right)\left(\frac{f}{\sum f}\right)$
		356	$\left(1-\frac{3d}{\sum d}\right)\left(\frac{e}{\sum e}\right)\left(\frac{f}{\sum f}\right)$
124	124	124	$\left(1-\frac{3c}{\sum d}\right)\left(1-\frac{3e}{\sum e}\right)\left(1-\frac{3f}{\sum f}\right)$
		126	$\left(1-\frac{3c}{\sum d}\right)\left(1-\frac{3e}{\sum e}\right)\left(\frac{f}{\sum f}\right)$
		146	$\left(1-\frac{3c}{\sum d}\right)\left(1-\frac{3e}{\sum e}\right)\left(\frac{f}{\sum f}\right)$
		246	$\left(1-\frac{3c}{\sum d}\right)\left(1-\frac{3e}{\sum e}\right)\left(\frac{f}{\sum f}\right)$
	125	125	$\left(1-\frac{3c}{\sum d}\right)\left(\frac{e}{\sum e}\right)\left(1-\frac{3f}{\sum f}\right)$
		126	$\left(1-\frac{3c}{\sum d}\right)\left(\frac{e}{\sum e}\right)\left(\frac{f}{\sum f}\right)$
		156	$\left(1-\frac{3c}{\sum d}\right)\left(\frac{e}{\sum e}\right)\left(\frac{f}{\sum f}\right)$
		256	$\left(1-\frac{3c}{\sum d}\right)\left(\frac{e}{\sum e}\right)\left(\frac{f}{\sum f}\right)$
	145	145	$\left(1-\frac{3c}{\sum d}\right)\left(\frac{e}{\sum e}\right)\left(1-\frac{3f}{\sum f}\right)$
		146	$\left(1-\frac{3c}{\sum d}\right)\left(\frac{e}{\sum e}\right)\left(\frac{f}{\sum f}\right)$

		156	$\left(1-\frac{3c}{\sum d}\right)\left(\frac{e}{\sum e}\right)\left(\frac{f}{\sum f}\right)$
		456	$\left(1-\frac{3c}{\sum d}\right)\left(\frac{e}{\sum e}\right)\left(\frac{f}{\sum f}\right)$
	245	245	$\left(1-\frac{3c}{\sum d}\right)\left(\frac{e}{\sum e}\right)\left(1-\frac{3f}{\sum f}\right)$
		246	$\left(1-\frac{3c}{\sum d}\right)\left(\frac{e}{\sum e}\right)\left(\frac{f}{\sum f}\right)$
		256	$\left(1-\frac{3c}{\sum d}\right)\left(\frac{e}{\sum e}\right)\left(\frac{f}{\sum f}\right)$
		456	$\left(1-\frac{3c}{\sum d}\right)\left(\frac{e}{\sum e}\right)\left(\frac{f}{\sum f}\right)$
134	134	134	$\left(1-\frac{3b}{\sum d}\right)\left(1-\frac{3e}{\sum e}\right)\left(1-\frac{3f}{\sum f}\right)$
		136	$\left(1-\frac{3b}{\sum d}\right)\left(1-\frac{3e}{\sum e}\right)\left(\frac{f}{\sum f}\right)$
		146	$\left(1-\frac{3b}{\sum d}\right)\left(1-\frac{3e}{\sum e}\right)\left(\frac{f}{\sum f}\right)$
		346	$\left(1-\frac{3b}{\sum d}\right)\left(1-\frac{3e}{\sum e}\right)\left(\frac{f}{\sum f}\right)$
	135	135	$\left(1-\frac{3b}{\sum d}\right)\left(\frac{e}{\sum e}\right)\left(1-\frac{3f}{\sum f}\right)$
		136	$\left(1-\frac{3b}{\sum d}\right)\left(\frac{e}{\sum e}\right)\left(\frac{f}{\sum f}\right)$
		156	$\left(1-\frac{3b}{\sum d}\right)\left(\frac{e}{\sum e}\right)\left(\frac{f}{\sum f}\right)$
		356	$\left(1-\frac{3b}{\sum d}\right)\left(\frac{e}{\sum e}\right)\left(\frac{f}{\sum f}\right)$
	145	145	$\left(1-\frac{3b}{\sum d}\right)\left(\frac{e}{\sum e}\right)\left(1-\frac{3f}{\sum f}\right)$
		146	$\left(1-\frac{3b}{\sum d}\right)\left(\frac{e}{\sum e}\right)\left(\frac{f}{\sum f}\right)$
		156	$\left(1-\frac{3b}{\sum d}\right)\left(\frac{e}{\sum e}\right)\left(\frac{f}{\sum f}\right)$
		456	$\left(1-\frac{3b}{\sum d}\right)\left(\frac{e}{\sum e}\right)\left(\frac{f}{\sum f}\right)$
	345	345	$\left(1-\frac{3b}{\sum d}\right)\left(\frac{e}{\sum e}\right)\left(1-\frac{3f}{\sum f}\right)$
		346	$\left(1-\frac{3b}{\sum d}\right)\left(\frac{e}{\sum e}\right)\left(\frac{f}{\sum f}\right)$
		356	$\left(1-\frac{3b}{\sum d}\right)\left(\frac{e}{\sum e}\right)\left(\frac{f}{\sum f}\right)$
		456	$\left(1-\frac{3b}{\sum d}\right)\left(\frac{e}{\sum e}\right)\left(\frac{f}{\sum f}\right)$
234	234	234	$\left(1-\frac{3a}{\sum d}\right)\left(1-\frac{3e}{\sum e}\right)\left(1-\frac{3f}{\sum f}\right)$

		236	$\left(1-\frac{3a}{\sum d}\right)\left(1-\frac{3e}{\sum e}\right)\left(\frac{f}{\sum f}\right)$
		246	$\left(1-\frac{3a}{\sum d}\right)\left(1-\frac{3e}{\sum e}\right)\left(\frac{f}{\sum f}\right)$
		346	$\left(1-\frac{3a}{\sum d}\right)\left(1-\frac{3e}{\sum e}\right)\left(\frac{f}{\sum f}\right)$
	235	235	$\left(1-\frac{3a}{\sum d}\right)\left(\frac{e}{\sum e}\right)\left(1-\frac{3f}{\sum f}\right)$
		236	$\left(1-\frac{3a}{\sum d}\right)\left(\frac{e}{\sum e}\right)\left(\frac{f}{\sum f}\right)$
		256	$\left(1-\frac{3a}{\sum d}\right)\left(\frac{e}{\sum e}\right)\left(\frac{f}{\sum f}\right)$
		356	$\left(1-\frac{3a}{\sum d}\right)\left(\frac{e}{\sum e}\right)\left(\frac{f}{\sum f}\right)$
	245	245	$\left(1-\frac{3a}{\sum d}\right)\left(\frac{e}{\sum e}\right)\left(1-\frac{3f}{\sum f}\right)$
		246	$\left(1-\frac{3a}{\sum d}\right)\left(\frac{e}{\sum e}\right)\left(\frac{f}{\sum f}\right)$
		256	$\left(1-\frac{3a}{\sum d}\right)\left(\frac{e}{\sum e}\right)\left(\frac{f}{\sum f}\right)$
		456	$\left(1-\frac{3a}{\sum d}\right)\left(\frac{e}{\sum e}\right)\left(\frac{f}{\sum f}\right)$
	345	345	$\left(1-\frac{3a}{\sum d}\right)\left(\frac{e}{\sum e}\right)\left(1-\frac{3f}{\sum f}\right)$
		346	$\left(1-\frac{3a}{\sum d}\right)\left(\frac{e}{\sum e}\right)\left(\frac{f}{\sum f}\right)$
		356	$\left(1-\frac{3a}{\sum d}\right)\left(\frac{e}{\sum e}\right)\left(\frac{f}{\sum f}\right)$
		456	$\left(1-\frac{3a}{\sum d}\right)\left(\frac{e}{\sum e}\right)\left(\frac{f}{\sum f}\right)$

Table 6: 20 distinct triplets with their respective frequencies:

Sl. No.	sample	Frequency
1	123	1
2	124	1
3	134	1
4	234	1
5	125	2
6	135	2
7	145	2
8	235	2
9	245	2
10	345	2
11	126	4
12	136	4
13	146	4

14	236	4
15	246	4
16	346	4
17	156	6
18	256	6
19	356	6
20	456	6

Samples and Probabilities:

[FREQUENCY 1 EACH]

$$P(1\ 2\ 3) = \left(1 - \frac{3d}{\sum d}\right)\left(1 - \frac{3e}{\sum e}\right)\left(1 - \frac{3f}{\sum f}\right)$$

$$P(1\ 2\ 4) = \left(1 - \frac{3c}{\sum d}\right)\left(1 - \frac{3e}{\sum e}\right)\left(1 - \frac{3f}{\sum f}\right)$$

$$P(1\ 3\ 4) = \left(1 - \frac{3b}{\sum d}\right)\left(1 - \frac{3e}{\sum e}\right)\left(1 - \frac{3f}{\sum f}\right)$$

$$P(2\ 3\ 4) = \left(1 - \frac{3a}{\sum d}\right)\left(1 - \frac{3e}{\sum e}\right)\left(1 - \frac{3f}{\sum f}\right)$$

[FREQUENCY 2 EACH]

$$P(1\ 2\ 5) = \left(1 - \frac{3d}{\sum d}\right)\left(\frac{e}{\sum e}\right)\left(1 - \frac{3f}{\sum f}\right) + \left(1 - \frac{3c}{\sum d}\right)\left(\frac{e}{\sum e}\right)\left(1 - \frac{3f}{\sum f}\right)$$

$$= \left[\left(1 - \frac{3c}{\sum d}\right) + \left(1 - \frac{3d}{\sum d}\right)\right]\left(\frac{e}{\sum e}\right)\left(1 - \frac{3f}{\sum f}\right)$$

Similarly,

$$P(1\ 3\ 5) = \left[\left(1 - \frac{3b}{\sum d}\right) + \left(1 - \frac{3d}{\sum d}\right)\right]\left(\frac{e}{\sum e}\right)\left(1 - \frac{3f}{\sum f}\right)$$

$$P(1\ 4\ 5) = \left[\left(1 - \frac{3b}{\sum d}\right) + \left(1 - \frac{3c}{\sum d}\right)\right]\left(\frac{e}{\sum e}\right)\left(1 - \frac{3f}{\sum f}\right)$$

$$P(2\ 3\ 5) = \left[\left(1 - \frac{3a}{\sum d}\right) + \left(1 - \frac{3d}{\sum d}\right)\right]\left(\frac{e}{\sum e}\right)\left(1 - \frac{3f}{\sum f}\right)$$

$$P(2\ 4\ 5) = \left[\left(1 - \frac{3a}{\sum d}\right) + \left(1 - \frac{3c}{\sum d}\right)\right]\left(\frac{e}{\sum e}\right)\left(1 - \frac{3f}{\sum f}\right)$$

$$P(3\ 4\ 5) == \left[\left(1 - \frac{3a}{\sum d}\right) + \left(1 - \frac{3b}{\sum d}\right)\right]\left(\frac{e}{\sum e}\right)\left(1 - \frac{3f}{\sum f}\right)$$

FREQUENCY 4 EACH

$$P(126) = \left(1 - \frac{3d}{\sum d}\right)\left(1 - \frac{3e}{\sum e}\right)\left(\frac{f}{\sum f}\right) + \left(1 - \frac{3d}{\sum d}\right)\left(\frac{e}{\sum e}\right)\left(\frac{f}{\sum f}\right) + \left(1 - \frac{3c}{\sum d}\right)\left(1 - \frac{3e}{\sum e}\right)\left(\frac{f}{\sum f}\right)$$

$$+ \left(1 - \frac{3c}{\sum d}\right)\left(\frac{e}{\sum e}\right)\left(\frac{f}{\sum f}\right)$$

$$= \left[\left(1 - \frac{3c}{\sum d}\right) + \left(1 - \frac{3d}{\sum d}\right)\right]\left(1 - \frac{2e}{\sum e}\right)\left(\frac{f}{\sum f}\right)$$

In the same way,

$$P(136) = \left[\left(1 - \frac{3b}{\sum d}\right) + \left(1 - \frac{3d}{\sum d}\right)\right]\left(1 - \frac{2e}{\sum e}\right)\left(\frac{f}{\sum f}\right)$$

$$P(146) = \left[\left(1 - \frac{3b}{\sum d}\right) + \left(1 - \frac{3c}{\sum d}\right)\right]\left(1 - \frac{2e}{\sum e}\right)\left(\frac{f}{\sum f}\right)$$

$$P(236) = \left[\left(1-\frac{3a}{\sum d}\right)+\left(1-\frac{3d}{\sum d}\right)\right]\left(1-\frac{2e}{\sum e}\right)\left(\frac{f}{\sum f}\right)$$
$$P(246) = \left[\left(1-\frac{3a}{\sum d}\right)+\left(1-\frac{3c}{\sum d}\right)\right]\left(1-\frac{2e}{\sum e}\right)\left(\frac{f}{\sum f}\right)$$
$$P(346) = \left[\left(1-\frac{3a}{\sum d}\right)+\left(1-\frac{3b}{\sum d}\right)\right]\left(1-\frac{2e}{\sum e}\right)\left(\frac{f}{\sum f}\right)$$

FREQUENCY 6 EACH

$$P(156) = \left(1-\frac{3d}{\sum d}\right)\left(\frac{e}{\sum e}\right)\left(\frac{f}{\sum f}\right)+\left(1-\frac{3d}{\sum d}\right)\left(\frac{e}{\sum e}\right)\left(\frac{f}{\sum f}\right)+\left(1-\frac{3c}{\sum d}\right)\left(\frac{e}{\sum e}\right)\left(\frac{f}{\sum f}\right)+\left(1-\frac{3c}{\sum d}\right)\left(\frac{e}{\sum e}\right)\left(\frac{f}{\sum f}\right)$$
$$+\left(1-\frac{3b}{\sum d}\right)\left(\frac{e}{\sum e}\right)\left(\frac{f}{\sum f}\right)+\left(1-\frac{3b}{\sum d}\right)\left(\frac{e}{\sum e}\right)\left(\frac{f}{\sum f}\right)$$
$$= 2\left[\left(1-\frac{3b}{\sum d}\right)+\left(1-\frac{3c}{\sum d}\right)+\left(1-\frac{3d}{\sum d}\right)\right]\left(\frac{e}{\sum e}\right)\left(\frac{f}{\sum f}\right)$$

Similarly we can deduce the following

$$P(256) = 2\left[\left(1-\frac{3a}{\sum d}\right)+\left(1-\frac{3c}{\sum d}\right)+\left(1-\frac{3d}{\sum d}\right)\right]\left(\frac{e}{\sum e}\right)\left(\frac{f}{\sum f}\right)$$
$$P(356) = 2\left[\left(1-\frac{3a}{\sum d}\right)+\left(1-\frac{3b}{\sum d}\right)+\left(1-\frac{3d}{\sum d}\right)\right]\left(\frac{e}{\sum e}\right)\left(\frac{f}{\sum f}\right)$$
$$P(456) = 2\left[\left(1-\frac{3a}{\sum d}\right)+\left(1-\frac{3b}{\sum d}\right)+\left(1-\frac{3c}{\sum d}\right)\right]\left(\frac{e}{\sum e}\right)\left(\frac{f}{\sum f}\right)$$

5. ILLUSTRATIVE EXAMPLE 1: N=6

X: [10 8 12 11 15 12]
$\sum d = 41, \sum e = 56, \sum f = 68$

For k=3: $\pi_1 = \pi_2 = \pi_3 = \pi_{12} = \pi_{13} = \pi_{23} = 1.$

For k=4:

Table 7

Sl. No.	sample (s)	Freq	P(s)
1	123	1	0.195121951
2	124	1	0.12195122
3	134	1	0.414634146
4	234	1	0.268292683
			1

Table 8: Computation of π_i

i	π_i	p_i	$3p_i$
1	0.73171	0.24390	0.731707
2	0.58537	0.19512	0.585366
3	0.87805	0.29268	0.878049
4	0.80488	0.26829	0.804878
	3	1	3

From the table it is seen that $\pi_i = np_i$ for all i where $p_i = \frac{X_i}{\sum X_i}$.

Table 9: $\mathbf{\Pi - matix}$

i \ j	1	2	3	4
1	**0.73171**	0.31707	0.60976	0.53659
2	0.31707	**0.58537**	0.46341	0.39024
3	0.60976	0.46341	**0.87805**	0.68293
4	0.53659	0.39024	0.68293	**0.80488**
$\sum_{j(\neq i)} \boldsymbol{\pi_{ij}}$	1.46341	1.17073	1.75610	1.60976
$(\boldsymbol{n-1})\boldsymbol{\pi_i}$	1.46341	1.17073	1.75610	1.60976

$$From\ the\ computation, \quad \sum_{j(\neq i)} \boldsymbol{\pi_{ij}} = (n-1)\pi_i \quad for\ all\ i.$$

For k=5:

Table 10

Sl. No.	sample (s)	Freq	P(s)
1	123	1	0.038327526
2	124	1	0.023954704
3	134	1	0.081445993
4	234	1	0.052700348
5	125	2	0.084930314
6	135	2	0.163327526
7	145	2	0.143728223
8	235	2	0.12412892
9	245	2	0.104529617
10	345	2	0.182926829
			1

Table 11: Computation of $\boldsymbol{\pi_i}$

$\boldsymbol{i}$	$\boldsymbol{\pi_i}$	$\boldsymbol{p_i}$	$\boldsymbol{3p_i}$
1	0.53571	0.17857	0.53571
2	0.42857	0.14286	0.42857
3	0.64286	0.21429	0.64286
4	0.58929	0.19643	0.58929
5	0.80357	0.26786	0.80357
	3	1	3

From the table it is seen that $\pi_i = np_i$ for all i.

Table 12: $\mathbf{\Pi - matix}$

i \ j	1	2	3	4	5
1	**0.53571**	0.14721	0.28310	0.24913	0.39199
2	0.14721	**0.42857**	0.21516	0.18118	0.31359
3	0.28310	0.21516	**0.64286**	0.31707	0.47038
4	0.24913	0.18118	0.31707	**0.58929**	0.43118

5	0.39199	0.31359	0.47038	0.43118	**0.80357**
$\sum_{j(\neq i)} \boldsymbol{\pi_{ij}}$	1.07143	0.85714	1.28571	1.17857	1.60714
$(\boldsymbol{n}-\mathbf{1})\boldsymbol{\pi_i}$	1.07143	0.85714	1.28571	1.17857	1.60714

$$Again, \quad \sum_{j(\neq i)} \boldsymbol{\pi_{ij}} = (n-1)\pi_i \quad for\ all\ i.$$

For k=6:

Table 13

Sl. No.	sample (s)	Freq	P(s)
1	123	1	0.018036483
2	124	1	0.011272802
3	134	1	0.038327526
4	234	1	0.024800164
5	125	2	0.039967206
6	135	2	0.076860012
7	145	2	0.067636811
8	235	2	0.058413609
9	245	2	0.049190408
10	345	2	0.086083214
11	126	4	0.025978684
12	136	4	0.049959008
13	146	4	0.043963927
14	236	4	0.037968846
15	246	4	0.031973765
16	346	4	0.055954089
17	156	6	0.069174011
18	256	6	0.055339209
19	356	6	0.083008813
20	456	6	0.076091412
			1

Table 14: Computation of $\boldsymbol{\pi_i}$

$\boldsymbol{i}$	$\boldsymbol{\pi_i}$	$\boldsymbol{p_i}$	$\boldsymbol{3p_i}$
1	**0.44118**	0.14706	0.44118
2	**0.35294**	0.11765	0.35294
3	**0.52941**	0.17647	0.52941
4	**0.48529**	0.16176	0.48529
5	**0.66176**	0.22059	0.66176
6	**0.52941**	0.17647	0.52941
	3	1	3

From the table it is again seen that $\pi_i = np_i$ for all i.

Table 15: Π − matix

i\j	1	2	3	4	5	6
1	**0.44118**	0.09526	0.18318	0.16120	0.25364	0.18908
2	0.09526	**0.35294**	0.13922	0.11724	0.20291	0.15126
3	0.18318	0.13922	**0.52941**	0.20516	0.30437	0.22689
4	0.16120	0.11724	0.20516	**0.48529**	0.27900	0.20798
5	0.25364	0.20291	0.30437	0.27900	**0.66176**	0.28361
6	0.18908	0.15126	0.22689	0.20798	0.28361	**0.52941**
$\sum_{j(\neq i)} \boldsymbol{\pi_{ij}}$	0.88235	0.70588	1.05882	0.97059	1.32353	1.05882
$(\boldsymbol{n-1})\boldsymbol{\pi_i}$	0.88235	0.70588	1.05882	0.97059	1.32353	1.05882

$It\ is\ again\ verified\ that\ \sum_{j(\neq i)} \boldsymbol{\pi_{ij}} = (n-1)\pi_i\ \ for\ all\ i.$

ILLUSTRATIVE EXAMPLE 2: n=5, N=20

X:	20	22	23	25	26	27	29	30	31	33	34	36	37	39	40	42	43	45	47	50

For k=5 :$\pi_i = 1, i = 1,2, \dots ,5;\ \ \pi_{ij} =\ 1\ for\ i \neq j,\ and\ i,j = 1,2, \dots ,5.$

For k=6 :

Table 16

Sl. No.	S	P(s)
1	12345	**0.055944**
2	13456	**0.230769**
3	12456	**0.195804**
4	12356	**0.125874**
5	12346	**0.090909**
6	23456	**0.300699**
Total		**1**

Table 17: $\boldsymbol{\pi_i}$ values

$\boldsymbol{i}$	1	2	3	4	5	6	Sum
X_i	20	22	23	25	26	27	143
$5p_i$	0.699301	0.769231	0.804196	0.874126	0.909091	0.944056	
π_i	0.699301	0.769231	0.804196	0.874126	0.909091	0.944056	5

Table 18: Π − matix:

j \i	1	2	3	4	5	6
1		0.468531	0.503497	0.573427	0.608392	0.643357
2	0.468531		0.573427	0.643357	0.678322	0.713287
3	0.503497	0.573427		0.678322	0.713287	0.748252
4	0.573427	0.643357	0.678322		0.783217	0.818182
5	0.608392	0.678322	0.713287	0.783217		0.853147
6	0.643357	0.713287	0.748252	0.818182	0.853147	
$\sum_{j(\neq i)} \boldsymbol{\pi_{ij}}$	2.797203	3.076923	3.216783	3.496503	3.636364	3.776224
$(n-1)\pi_i$	2.797203	3.076923	3.216783	3.496503	3.636364	3.776224

For k=7:

Table 19

Sl. No.	Parental sample	s	P(s)
1	12345	12345	0.008782
2	12345	13457	0.009432
3	12345	12457	0.009432
4	12345	12357	0.009432
5	12345	12347	0.009432
6	12345	23457	0.009432
7	13456	13456	0.036225
8	13456	14567	0.038909
9	13456	13567	0.038909
10	13456	13467	0.038909
11	13456	13457	0.038909
12	13456	34567	0.038909
13	12456	12456	0.030737
14	12456	14567	0.033013
15	12456	12567	0.033013
16	12456	12467	0.033013
17	12456	12457	0.033013
18	12456	24567	0.033013
19	12356	12356	0.019759
20	12356	13567	0.021223
21	12356	12567	0.021223
22	12356	12367	0.021223
23	12356	12357	0.021223
24	12356	23567	0.021223
25	12346	12346	0.014271
26	12346	13467	0.015328
27	12346	12467	0.015328
28	12346	12367	0.015328
29	12346	12347	0.015328
30	12346	23467	0.015328
31	23456	23456	0.047203
32	23456	24567	0.050699
33	23456	23567	0.050699
34	23456	23467	0.050699
35	23456	23457	0.050699
36	23456	34567	0.050699
Total			**1**

Table 20: Values of $\pi_i's$

i	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**sum**
X_i	20	22	23	25	26	27	29	172
$5p_i$	0.581395	0.639535	0.668605	0.726744	0.755814	0.784884	0.843023	
π_i	0.581395	0.639535	0.668605	0.726744	0.755814	0.784884	0.843023	5

Table 21: $\Pi -$ matix:

j \i	1	2	3	4	5	6	7
1		0.310538	0.333713	0.380062	0.403236	0.426411	0.471621
2	0.310538		0.380062	0.426411	0.449585	0.472760	0.518784
3	0.333713	0.380062		0.449585	0.472760	0.495934	0.542365
4	0.380062	0.426411	0.449585		0.519109	0.542283	0.589527
5	0.403236	0.449585	0.472760	0.519109		0.565458	0.613108
6	0.426411	0.472760	0.495934	0.542283	0.565458		0.636689
7	0.471621	0.518784	0.542365	0.589527	0.613108	0.636689	
$\sum_{j(\neq i)} \boldsymbol{\pi_{ij}}$	2.325581	2.558140	2.674419	2.906977	3.023256	3.139535	3.372093
$(n-1)\pi_i$	2.325581	2.558140	2.674419	2.906977	3.023256	3.139535	3.372093

For k=8:

Table 22

Sl. No.	Parental sample	s	P(s)
1	12345	12345	0.002261
2	12345	13458	0.001304
3	12345	12458	0.001304
4	12345	12358	0.001304
5	12345	12348	0.001304
6	12345	23458	0.001304
7	13457	13457	0.002428
8	13457	14578	0.001401
9	13457	13578	0.001401
10	13457	13478	0.001401
11	13457	13458	0.001401
12	13457	34578	0.001401
13	12457	12457	0.002428
14	12457	14578	0.001401
15	12457	12578	0.001401
16	12457	12478	0.001401
17	12457	12458	0.001401
18	12457	24578	0.001401
19	12357	12357	0.002428
20	12357	13578	0.001401
21	12357	12578	0.001401
22	12357	12378	0.001401
23	12357	12358	0.001401
24	12357	23578	0.001401
25	12347	12347	0.002428
26	12347	13478	0.001401
27	12347	12478	0.001401
28	12347	12378	0.001401
29	12347	12348	0.001401
30	12347	23478	0.001401

Sl. No.	Parental sample	s	P(s)
31	23457	23457	0.002428
32	23457	24578	0.001401
33	23457	23578	0.001401
34	23457	23478	0.001401
35	23457	23458	0.001401
36	23457	34578	0.001401
37	13456	13456	0.009325
38	13456	14568	0.005380
39	13456	13568	0.005380
40	13456	13468	0.005380
41	13456	13458	0.005380
42	13456	34568	0.005380
43	14567	14567	0.010016
44	14567	15678	0.005779
45	14567	14678	0.005779
46	14567	14578	0.005779
47	14567	14568	0.005779
48	14567	45678	0.005779
49	13567	13567	0.010016
50	13567	15678	0.005779
51	13567	13678	0.005779
52	13567	13578	0.005779
53	13567	13568	0.005779
54	13567	35678	0.005779
55	13467	13467	0.010016
56	13467	14678	0.005779
57	13467	13678	0.005779
58	13467	13478	0.005779
59	13467	13468	0.005779
60	13467	34678	0.005779
61	13457	13457	0.010016
62	13457	14578	0.005779
63	13457	13578	0.005779
64	13457	13478	0.005779
65	13457	13458	0.005779
66	13457	34578	0.005779
67	34567	34567	0.010016
68	34567	35678	0.005779
69	34567	34678	0.005779
70	34567	34578	0.005779
71	34567	34568	0.005779
72	34567	45678	0.005779
73	12456	12456	0.007912
74	12456	14568	0.004565
75	12456	12568	0.004565
76	12456	12468	0.004565

Sl. No.	Parental sample	s	P(s)
77	12456	12458	0.004565
78	12456	24568	0.004565
79	14567	14567	0.008499
80	14567	15678	0.004903
81	14567	14678	0.004903
82	14567	14578	0.004903
83	14567	14568	0.004903
84	14567	45678	0.004903
85	12567	12567	0.008499
86	12567	15678	0.004903
87	12567	12678	0.004903
88	12567	12578	0.004903
89	12567	12568	0.004903
90	12567	25678	0.004903
91	12467	12467	0.008499
92	12467	14678	0.004903
93	12467	12678	0.004903
94	12467	12478	0.004903
95	12467	12468	0.004903
96	12467	24678	0.004903
97	12457	12457	0.008499
98	12457	14578	0.004903
99	12457	12578	0.004903
100	12457	12478	0.004903
101	12457	12458	0.004903
102	12457	24578	0.004903
103	24567	24567	0.008499
104	24567	25678	0.004903
105	24567	24678	0.004903
106	24567	24578	0.004903
107	24567	24568	0.004903
108	24567	45678	0.004903
109	12356	12356	0.005087
110	12356	13568	0.002935
111	12356	12568	0.002935
112	12356	12368	0.002935
113	12356	12358	0.002935
114	12356	23568	0.002935
115	13567	13567	0.005463
116	13567	15678	0.003152
117	13567	13678	0.003152
118	13567	13578	0.003152
119	13567	13568	0.003152
120	13567	35678	0.003152
121	12567	12567	0.005463
122	12567	15678	0.003152
123	12567	12678	0.003152

Sl. No.	Parental sample	s	P(s)
124	12567	12578	0.003152
125	12567	12568	0.003152
126	12567	25678	0.003152
127	12367	12367	0.005463
128	12367	13678	0.003152
129	12367	12678	0.003152
130	12367	12378	0.003152
131	12367	12368	0.003152
132	12367	23678	0.003152
133	12357	12357	0.005463
134	12357	13578	0.003152
135	12357	12578	0.003152
136	12357	12378	0.003152
137	12357	12358	0.003152
138	12357	23578	0.003152
139	23567	23567	0.005463
140	23567	25678	0.003152
141	23567	23678	0.003152
142	23567	23578	0.003152
143	23567	23568	0.003152
144	23567	35678	0.003152
145	12346	12346	0.003674
146	12346	13468	0.002119
147	12346	12468	0.002119
148	12346	12368	0.002119
149	12346	12348	0.002119
150	12346	23468	0.002119
151	13467	13467	0.003946
152	13467	14678	0.002276
153	13467	13678	0.002276
154	13467	13478	0.002276
155	13467	13468	0.002276
156	13467	34678	0.002276
157	12467	12467	0.003946
158	12467	14678	0.002276
159	12467	12678	0.002276
160	12467	12478	0.002276
161	12467	12468	0.002276
162	12467	24678	0.002276
163	12367	12367	0.003946
164	12367	13678	0.002276
165	12367	12678	0.002276
166	12367	12378	0.002276
167	12367	12368	0.002276
168	12367	23678	0.002276
169	12347	12347	0.003946

Sl. No.	**Parental sample**	**s**	**P(s)**
170	12347	13478	0.002276
171	12347	12478	0.002276
172	12347	12378	0.002276
173	12347	12348	0.002276
174	12347	23478	0.002276
175	23467	23467	0.003946
176	23467	24678	0.002276
177	23467	23678	0.002276
178	23467	23478	0.002276
179	23467	23468	0.002276
180	23467	34678	0.002276
181	23456	23456	0.012151
182	23456	24568	0.007010
183	23456	23568	0.007010
184	23456	23468	0.007010
185	23456	23458	0.007010
186	23456	34568	0.007010
187	24567	24567	0.013051
188	24567	25678	0.007530
189	24567	24678	0.007530
190	24567	24578	0.007530
191	24567	24568	0.007530
192	24567	45678	0.007530
193	23567	23567	0.013051
194	23567	25678	0.007530
195	23567	23678	0.007530
196	23567	23578	0.007530
197	23567	23568	0.007530
198	23567	35678	0.007530
199	23467	23467	0.013051
200	23467	24678	0.007530
201	23467	23678	0.007530
202	23467	23478	0.007530
203	23467	23468	0.007530
204	23467	34678	0.007530
205	23457	23457	0.013051
206	23457	24578	0.007530
207	23457	23578	0.007530
208	23457	23478	0.007530
209	23457	23458	0.007530
210	23457	34578	0.007530
211	34567	34567	0.013051
212	34567	35678	0.007530
213	34567	34678	0.007530
214	34567	34578	0.007530
215	34567	34568	0.007530
216	34567	45678	0.007530
Total			**1**

Table 23: Values of $\pi_i's$

i	1	2	3	4	5	6	7	8	Sum
X_i	20	22	23	25	26	27	29	30	202
$5p_i$	0.495050	0.544554	0.569307	0.618812	0.643564	0.668317	0.717822	0.742574	5
π_i	0.495050	0.544554	0.569307	0.618812	0.643564	0.668317	0.717822	0.742574	5

Table 24: $\Pi -$ matix:

j \i	1	2	3	4	5	6	7	8
1		0.218299	0.234590	0.267172	0.283463	0.299754	0.331536	0.345383
2	0.218299		0.267172	0.299754	0.316045	0.332336	0.364689	0.379922
3	0.234590	0.267172		0.316045	0.332336	0.348627	0.381266	0.397191
4	0.267172	0.299754	0.316045		0.364918	0.381209	0.414420	0.431729
5	0.283463	0.316045	0.332336	0.364918		0.397500	0.430997	0.448998
6	0.299754	0.332336	0.348627	0.381209	0.397500		0.447573	0.466268
7	0.331536	0.364689	0.381266	0.414420	0.430997	0.447573		0.500806
8	0.345383	0.379922	0.397191	0.431729	0.448998	0.466268	0.500806	
$\sum_{j(\neq i)} \pi_{ij}$	1.980198	2.178218	2.277228	2.475248	2.574257	2.673267	2.871287	2.970297
$(n-1)\pi_i$	1.980198	2.178218	2.277228	2.475248	2.574257	2.673267	2.871287	2.970297

Concluding Section

This study turned out to be an interesting exercise for the two young researchers in our group. Also, the derivations opened up slowly and logically.

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References

[1] Horvitz, D. G. and Thompson, D. J. (1952): A Generalization of Sampling without Replacement from a Finite Universe. Journal of the American Statistical Association, 47,663-685.

[2] Chao, M. T. (1982): A General Purpose Unequal Probability Sampling Plan. Biometrika, 69(3), 653–656.

[3] Brewer, K. R.W. and Hanif, M. (1983): Sampling with Unequal Probabilities. Lecture Notes in Statistics, No. 15, Springer-Verlag, New York.

[4] Hedayat, A. S. and Sinha, K. Bikas (1991): Design and inference in finite population sampling. Wiley, New York.