International Journal of Statistical Sciences *ISSN 1683-5603*
Vol.24(1), March, 2024, pp 15-30
DOI: https://doi.org/10.3329/ijss.v24i1.71870

New Series of D-efficient Covariate Designs under BIBD set-up

Anurup Majumder[1*], Hiranmoy Das[2], Ankita Dutta[1] and Dikeshwar Nishad[3]

[1]Bidhan Chandra Krishi Viswavidyalaya, Krishi Viswavidyalaya-741252, India
[2]ICAR-Indian Institute of Vegetable Research, Varanasi-221305, India
[3]Pt. S. K. Shastri College of Agriculture and Research Station, Rajnandgaon-491441, India
[*]Correspondence should be addressed to Anurup Majumder
(Email: anurupbckv@gmail.com)

[Received December 13, 2023; Accepted February 08, 2024]

Abstract

In the present study, an effort has been made to construct D-efficient covariate designs in BIB design (v, b, r, k and λ) set-up when either one of k and r is odd or both k and r are odd numbers and Hadamard matrix of order k i.e., H_k does not exist. For all the developed D-efficient designs, the covariates are mutually orthogonal to each other. The methods of construction of D-efficient covariate designs are developed with the help of a new matrix viz., Special Array (Das *et. al.*, 2020). In this article, the series of developed D-efficient covariate designs are not available in the existing literature.

Keywords and Phrases: Hadamard Matrix, Optimal Covariate Designs (OCDs), D-efficiency and Kronecker product.

AMS Classification: 62K05.

1. Introduction

Optimal designs for covariate models are of relatively recent research interest but the concept was firstly introduced by Lopes Troya (1982a, 1982b). After a long gap, Das *et. al.* (2003) reinvestigated the topic and constructed Optimum Covariate Designs (OCDs) using Mutually Orthogonal Latin Squares (MOLS) and Hadamard matrices in the design set-up of RCBD and some series of BIBD. Rao *et al.* (2003) also revisited the problem in CRD and RCBD set-ups. They identified that the solutions of construction of OCDs by using Mixed Orthogonal Arrays (MOAs) and thereby giving further insights and some new solutions. Dutta (2004, 2009) and Dutta *et al.* (2007, 2009a, 2009b, 2010a) developed OCDs to different design set-ups. Dey and Mukerjee (2006) and Dutta *et al.* (2010b, 2014) developed some D-optimal covariate designs for estimation of regression coefficients in incomplete block design set-up, when global optimal designs do not exist. Das *et. al.* (2015) published an excellent book on 'Optimal Covariate Designs' covering developments in the topic of optimum covariates for different design set-ups. Furthermore, Das *et al.* (2020) has reported some new series of universal/global optimal covariate designs in CRD and

RCBD set-ups without the existence of Hadamard matrix of order v or b. Some new series of D-optimal covariate designs in CRD and RCBD set-ups were also reported by Das *et. al.* (2021). The main focus of the above works was aimed at development of optimal/most efficient estimation of covariates parameters of the ANCOVA model accommodating maximum number of covariates in optimal manners in different design settings. Optimality refers to attaining the least possible value of individual variances simultaneously for all the estimators of the covariate parameters. In the present study, efforts are given to make solutions to the limitations of the research works so far done in the topic.

2. Special Array (SA); Definition, Properties and Application (Das *et al*., 2020)

2.1 Definition: A square matrix with elements 1, -1 and 0 of order h with r (≥ 1) number of rows (and columns) with all elements 0, whose all the distinct row or column vectors except r rows (or columns) are mutually orthogonal is referred to as **Special Array** (SA) of order h. In SA, each row or column sum is zero except the first row or column. The simplest examples, one for order 3 and two for order 5 are given below:

$$\underset{r=1}{\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix}}, \quad \underset{r=3}{\begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 \end{pmatrix}} \quad \text{and} \quad \underset{r=1}{\begin{pmatrix} 1 & 1 & 0 & 1 & 1 \\ 1 & -1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & -1 & -1 \\ 1 & -1 & 0 & 1 & -1 \end{pmatrix}}$$

2.2 Properties:

Let the Special Array (SA) of order h be denoted as H_h^*, then

1) $\det(H_h^*)=0$; when $r \geq 1$and when r = 0, it becomes a Hadamard Matrix.
2) $H_h^* H_h^{*T} = H_h^{*T} H_h^*$
3) Let H_1^* and H_2^* be two SA of order h_1 and h_2, respectively. Then the Kronecker product of H_1^* and H_2^* is also a SA of order h_1h_2. For example,

$$
\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 & 0 & 1 & 1 \\ 1 & -1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & -1 & -1 \\ 1 & -1 & 0 & 1 & -1 \end{pmatrix} = \begin{pmatrix}
1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\
1 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 & -1 \\
1 & -1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & -1 & -1 \\
1 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 1 & 1 \\
1 & -1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & -1 & 1
\end{pmatrix}
$$

2.3 Application: Special Array is constructed from Hadamard matrix with r rows (and columns) with all elements zero in the middle is used to find out the optimum (Global or D) as well as D-efficient number of covariates in CRD, RBD and BIBD set-up.

3. D-optimal as well as D-efficient covariates in BIB design set-up

Consider a BIBD (v, b, r, k, λ) set-up which can be written as

$$(\mathbf{Y},\ \mu\mathbf{1}+\mathbf{X_1}\boldsymbol{\tau}+\mathbf{X_2}\boldsymbol{\beta}+\mathbf{Z}\gamma,\ \sigma^2\mathbf{I}) \tag{3.1}$$

where, $\boldsymbol{\tau}$, $\boldsymbol{\beta}$ represent the vectors of treatment and block effects respectively and $\mathbf{X_1}^{\mathbf{nxv}}$, $\mathbf{X_2}^{\mathbf{nxb}}$ are the corresponding design matrices for treatment and block. Now, the problem is the estimation of the covariate parameters in $\boldsymbol{\gamma}$ optimally. By appropriate selection of the values of the covariates z_{ij} (the [i, j]th element of $\mathbf{Z}$), one can optimize the estimation of the parameters in $\boldsymbol{\gamma}$ while maintaining the properties of the design with regard to the treatment and block. Here, optimality refers to attaining the least possible value σ^2/n of individual variances for all the estimators of the parameters in $\boldsymbol{\gamma}$ simultaneously. Such a design is termed as globally optimal.

Based on the model (3.1), it is evident that for the estimation of the covariate effects orthogonal to the treatment and block effect contrasts, we must have

$$\mathbf{Z}'\mathbf{X}_1=\mathbf{0},\ \mathbf{Z}'\mathbf{X}_2=\mathbf{0} \tag{3.2}$$

and for most efficient estimation of the regression parameters, we must have

$$\mathbf{Z}'\mathbf{Z}=n\mathbf{I}_c \tag{3.3}$$

Dutta *et. al.* (2010) considered D-optimal design when n=2 (mod 4) in BIBD subject to the condition 3.2. Here, we have taken the situations or conditions (i) n = 0 (mod 4) with $\mathbf{Z}'\mathbf{X}_1= \mathbf{0}$ and

$\mathbf{Z'X}_2 \neq \mathbf{0}$ and (ii) n = 0 (mod 4) with $\mathbf{Z'X}_1 \neq \mathbf{0}$ and $\mathbf{Z'X}_2 \neq \mathbf{0}$. In both the cases, simultaneous estimation of ANOVA parameters and γ-parameters are not possible to estimate orthogonally and/or most efficiently. Here, the D-optimality criterion may be considered to give an efficient allocation of treatments and covariates in BIBD set-up. Based on the model (3.1), a block design for given b (such that $\mathbf{H}_b$ exist) and v, the reduced normal equation for estimation of γ is given by following Das *et. al.* (2015):

$$(\mathbf{Z'QZ})\gamma = \mathbf{Z'Qy}$$

$$\Rightarrow \quad \hat{\gamma} = (\mathbf{Z'QZ})^{-1}\mathbf{Z'Qy}$$

Where, $\mathbf{Q} = (\mathbf{I} - \mathbf{X}(\mathbf{X'X})^{-}\mathbf{X'})$, $\mathbf{X} = (\mathbf{X}_1, \mathbf{X}_2)$

Hence, the information matrix for γ is given by

$\mathbf{I}(\gamma) = \mathbf{Z'QZ}$

or, $\mathbf{I}(\gamma) = \mathbf{Z'}(\mathbf{I} - \mathbf{X}(\mathbf{X'X})^{-}\mathbf{X'})\mathbf{Z}$

or, $\mathbf{I}(\gamma) = \mathbf{Z'Z} - \mathbf{Z'X}(\mathbf{X'X})^{-}\mathbf{X'Z}$

or, $\det(\mathbf{I}(\gamma)) = \det(\mathbf{Z'Z} - \mathbf{Z'X}(\mathbf{X'X})^{-}\mathbf{X'Z})$

Since $\mathbf{Q}$ is non-negative definite, it follows that

$\mathbf{Z'QZ} \leq \mathbf{Z'Z}$ (in Lowener order sense; Pukelsheim 1993) and equality comes when $\mathbf{Z'X}_1 = \mathbf{0}$ and $\mathbf{Z'X}_2 = \mathbf{0}$.

But in the present situations (i) and (ii), it follows that

$\mathbf{Z'QZ} < \mathbf{Z'Z}$.

Now, the problem is that of selecting **Z-**matrix with $\left|z_{ij}^{(t)}\right| \leq 1$ satisfying the conditions either (i) or (ii), such that the covariate design will be either D-efficient or D-optimal, i.e., $\det(\mathbf{I}(\gamma))$ or $\det(\mathbf{Z'Z} - \mathbf{Z'X}(\mathbf{X'X})^{-}\mathbf{X'Z})$ should be maximum when $\mathbf{Z} \in \mathcal{Z}$, $\mathcal{Z} = \{\mathbf{Z}\text{: } z_{ij}^{(t)} \in [-1,\ 1]\ \forall\ i,\ j\ \}$. So, the contribution from $\mathbf{Z'Z}$ should be maximum and contribution from the part of $\mathbf{Z'X}(\mathbf{X'X})^{-}\mathbf{X'Z}$ should be minimum.

3.1 Conditions for D-efficiency

We have already observed that when n=0 (mod 4) with either $\mathbf{Z'X}_1 = \mathbf{0}$ and $\mathbf{Z'X}_2 \neq \mathbf{0}$ or $\mathbf{Z'X}_1 \neq \mathbf{0}$ and $\mathbf{Z'X}_2 \neq \mathbf{0}$, it is impossible to estimate γ-components most efficiently in the sense of attaining the lower bound σ^2/n to the variance of the estimated covariate parameters. Thus, in both the cases, the first problem is that of choosing a matrix $\mathbf{Z}^{nxc} = (z_{ij}^{(t)})$ with $z_{ij}^{(t)} \in [-1,\ 1]\ \forall\ i,\ j$ such that the contribution of $\mathbf{Z'Z}$ is maximum and secondly, the contribution from the part of $\mathbf{Z'X}(\mathbf{X'X})^{-}\mathbf{X'Z}$ should be minimum subject to the condition either (i) or (ii). A necessary condition for maximization of $\det(\mathbf{Z'Z})$ where $\mathbf{Z} \in \mathcal{Z}$, is that $z_{ij}^{(t)} = \pm 1\ \forall\ i,\ j,\ t$ (Lemma 4.4.1 of Das *et. al.*, 2015). Based on the necessary condition, we can restrict to the class $\mathcal{Z}^* = \{\mathbf{Z}\text{: } z_{ij}^{(t)} = \pm 1\ \forall\ i,\ j,\ t\ \}$ for finding D-efficient design.

Conjecture 3.1.1: A design with covariate matrix $\mathbf{Z}^* \in \mathcal{Z}^*$ is D-efficient in the sense of maximizing $\det(\mathbf{Z'Z})$ and $\det(\mathbf{I}(\gamma))$ with the contribution of $\mathbf{Z'Z}$ is maximum and the contribution from the part of $\mathbf{Z'X}(\mathbf{X'X})^{-}\mathbf{X'Z}$ should be minimum subject to the conditions either (i) or (ii), if it satisfies $\mathbf{Z^{*\prime}Z^{*}} = n\mathbf{I}_c$ and $a_{lm} = \pm 1$ and $a_{lg} = \pm 1$, where a_{lm} and a_{lg} are the elements of $\mathbf{Z'X}_1$ and $\mathbf{Z'X}_2$ respectively, $l = 1,2,...,c$; $m = 1,2,...,v$ and $g = 1,2,...,b$.

Proof: Based on the necessary condition, we can restrict to the class Z* for maximization of det($\mathbf{Z'Z}$). For any $\mathbf{Z} \in Z^*$, we can write

$$\det(\mathbf{Z'Z}) = \det\begin{pmatrix} n & s_{12} & \cdots & s_{1c} \\ s_{12} & n & \cdots & s_{2c} \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ s_{1c} & s_{2c} & \cdots & n \end{pmatrix}$$

where, $s_{tt'} = \sum_i \sum_j z_{ij}^{(t)} z_{ij}^{(t')}$, $t \neq t' = 1, 2, \ldots, c$. The det($\mathbf{Z'Z}$) will be maximum whenever it is possible to construct $\mathbf{Z'Z} = n\mathbf{I}_c$ i.e., the covariates are mutually orthogonal to each other. So, all off-diagonal elements of $\mathbf{Z'Z}$ can be zero. As the elements of $\mathbf{Z'X_1}$ and $\mathbf{Z'X}_2$ will be either +1 or -1 depending on the conditions (i) and (ii), the contribution from the part of $\mathbf{Z'X(X'X)^{-}X'Z}$ will be minimum. Hence, the statement in the above conjecture is proved.

Now, we can represent any column of **Z*** in the form of a matrix **U** of order vxb corresponding to the incidence matrix of the block design.

Based on the conditions (i) and (ii) and $\mathbf{Z^{*\prime}Z^{*}} = n\mathbf{I}_c$, thus, in terms of **U** matrix, the conditions reduce to:

$\mathbf{C_1^*}$: Each **U**-matrix has all row-sums equal to either zero or ±1 depending on the conditions (i) and (ii) respectively;

$\mathbf{C_2^*}$: Each **U**-matrix has all column-sums equal to either +1 or -1;

$\mathbf{C_3^*}$: The grand total of all the entries in the Hadamard product of any two distinct **U**-matrices reduces to zero.

3.2 The method of construction for D-efficient U-matrix when $\mathbf{H}_{k-1}$ exists

For construction of D-efficient **U** matrices of order vxb from a BIBD (v, b, r, k, λ) where r (even or odd number) and k (always odd number) and $\mathbf{H}_k$ do not exist, we follow the steps given below.

Step 1. Construct the BIB design D (v, b, r, k, λ).

Step 2. Construct the incidence matrix (**N**) of order vxb from the design D.

Step 3. Let us consider a Hadamard matrix of order b, $\mathbf{H}_b$.

$\mathbf{H}_b = (\mathbf{1}, \mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_{b-1})$

Step 4. Let us construct a Special Array $\mathbf{H}_k^*$ of order k from $\mathbf{H}_{k-1}$ with one row and column with all zero elements in middle, i.e., $(\mathbf{1}^*, \mathbf{h}_1^*, \mathbf{h}_2^*, \ldots, \mathbf{h}_{(k-1)/2-1}^*, \mathbf{0}, \mathbf{h}_{(k-1)/2}^*, \ldots, \mathbf{h}_{k-2}^*)$.

$$H_k^* = \begin{pmatrix} 1\ldots\ \ 1\ 0\ \ 1\ldots1 \\ \ldots\ldots\ \ 0\ \ldots\ldots \\ \ldots\ldots\ \ 0\ \ldots\ldots \\ \ldots\ldots\ 0\ \ldots\ldots \\ 1\ldots\ {-1}\ 0\ {-1}\ldots1 \\ 0\ldots\ \ 0\ 0\ \ 0\ldots0 \\ 1\ldots\ \ 1\ 0\ {-1}\ldots{-1} \\ \ldots\ldots\ \ 0\ \ldots\ldots \\ \ldots\ldots\ \ 0\ \ldots\ldots \\ \ldots\ldots\ \ 0\ \ldots\ldots \\ 1\ldots\ {-1}\ 0\ \ 1\ldots{-1} \end{pmatrix}$$

Step 5. Using $\mathbf{H_k}$*****and $\mathbf{H_b}$, by Kronecker product of these two matrices, we get (k-2) sets of (b-1) **U*** matrices of order kxb (without consider the first column and one column with all zeros). In each of the **U*** matrix there are one row with all elements zero in the middle and all the **U*** matrices are mutually orthogonal to each other.

$U^* = h_i^* \otimes h'_j$, $\otimes$ denotes the Kronecker product

where i=1,2,…,(k-2) and j=1,2,…,(b-1).

Step 6. In each of (k-2) sets, let us replace the row with all zero elements of **U*** matrix by (j+1)th row of $\mathbf{H}_b$. In that way, we get (k-2) sets of (b-1) mutually orthogonal **U**** matrices of order kxb. All the **U**** matrix has all row-sums equal to zero and all column-sums equal to either +1 or -1.

Step 7. The non-zero elements of the incidence matrix (**N**) are replaced by the elements (+1 or -1) of each of the **U**** matrix from ith set column wise. Thus, we get (b-1) **U***** matrix of order vxb. It has been observed that the column sum of the **U***** matrix is always equal to either +1 or -1. But, the row totals of the **U***** matrices are must be either zero or ±1 (it may not be true for all **U***** matrices) depending on r (even or odd). If the row totals are either zero or ±1 is true for all U*** matrices depending on r (even or odd), we may get directly atmost (b-1) **U** matrices of order vxb satisfying the conditions $\mathbf{C}_1$*-$\mathbf{C}_3$* simultaneously for D-efficiency from **U***** matrices. On the other hand, when it is not true for all U*** matrices i.e., some or all row totals has been violated in some or all **U***** matrices, to construct D-efficient **U**-matrix of order vxb, the elements of non-zero row sums or the elements of row sums neither +1 nor -1 depending on r (even or odd) of **U***** matrices has been rearranged column wise by trial and error method such that the resulting **U** matrices satisfying the conditions $\mathbf{C}_1$*-$\mathbf{C}_3$* simultaneously for D-efficiency. So, we may get atmost (b-1) **U** matrix of order vxb depending on **U***** matrices.

For easy understanding of the above steps, the following example will be useful.

Example 3.2.1: Let us consider a BIBD with parameters; v = 4, b = 8, r = 6, k = 3 and λ = 4. The construction of D-optimum **U**-matrix of order 4x8 is the following:

Step 1. Lay out of the BIBD (v = 4, b = 8, r = 6, k = 3 and λ = 4).

Blk 1	Blk 2	Blk 3	Blk 4	Blk 5	Blk 6	Blk 7	Blk 8
1	2	1	1	1	2	1	1
2	3	3	2	2	3	3	2
3	4	4	4	3	4	4	4

Step 2. Construction of the incidence matrix (**N**) of order 4x8 from the above design is

$$N = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}$$

Step 3. Let us consider a Hadamard matrix of order 8, $\mathbf{H_8}$.

$$H_8 = (1, h_1, h_2, h_3, h_4, h_5, h_6, h_7) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{pmatrix}$$

Step 4. Let us construct a Special Array $\mathbf{H_3*}$ from $\mathbf{H_2}$ with one row and column with all zero elements in middle.

$$H_3^* = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

Step 5. Using $\mathbf{H_3*}$ and $\mathbf{H_8}$, by Kronecker product of these two matrices, we get 7**U*** matrices of order 3x8 (without consider the first column and one column with all zeros). In each of the **U*** matrix there are one row with all elements zero in the middle and all the **U*** matrices are mutually orthogonal to each other. The $\mathbf{U}_1$* matrix is presented here.

$$U_1^* = h_1^* \otimes h'_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \end{pmatrix}$$

Similarly, we can easily construct the others.

Step 6. Let the zero elements of **U*** matrix be replaced by (j+1)th row of $\mathbf{H}_8$, where j = 1, 2,...,7. In that way, 7 mutually orthogonal **U**** matrices of order 3x8 be constructed. All the **U**** matrix has all row-sums equal to zero and all column-sums equal to either +1 or -1. Here, the $\mathbf{U}_1$** matrix is given below:

$$U_1^{**} = \begin{pmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \end{pmatrix}$$

Similarly, the others be easily constructed.

Step 7. The non-zero elements of the incidence matrix (**N**) are replaced by the elements (+1 or -1) of each of the **U**** matrix column wise. Thus, we get 7**U***** matrix of order 4x8. It has been observed that the column sum of the **U***** matrix is always equal to either +1 or -1. But, in this case, the row totals of the **U***** matrices are must be zero as r = 6. In this case, we get directly 4**U** matrices of order 4x8 from **U***** matrices satisfying the conditions $\mathbf{C}_1$*-$\mathbf{C}_3$* simultaneously for D-optimality or D- efficiency covariate designs. The desired 4**U** matrices are given below:

$$U_1^{***} = \begin{pmatrix} 1 & 0 & 1 & 1 & -1 & 0 & -1 & -1 \\ 1 & 1 & 0 & 1 & -1 & -1 & 0 & -1 \\ -1 & 1 & 1 & 0 & 1 & -1 & -1 & 0 \\ 0 & -1 & -1 & -1 & 0 & 1 & 1 & 1 \end{pmatrix} = U_1$$

$$U_2^{***} = \begin{pmatrix} 1 & 0 & 1 & -1 & -1 & 0 & -1 & 1 \\ 1 & -1 & 0 & -1 & -1 & 1 & 0 & 1 \\ -1 & -1 & 1 & 0 & 1 & 1 & -1 & 0 \\ 0 & 1 & -1 & 1 & 0 & -1 & 1 & -1 \end{pmatrix} = U_2$$

$$U_3^{***} = \begin{pmatrix} 1 & 0 & -1 & -1 & -1 & 0 & 1 & 1 \\ 1 & 1 & 0 & -1 & -1 & -1 & 0 & 1 \\ -1 & 1 & -1 & 0 & 1 & -1 & 1 & 0 \\ 0 & -1 & 1 & 1 & 0 & 1 & -1 & -1 \end{pmatrix} = U_3$$

$$U_4^{***} = \begin{pmatrix} 1 & 0 & -1 & 1 & -1 & 0 & 1 & -1 \\ 1 & -1 & 0 & 1 & -1 & 1 & 0 & -1 \\ -1 & -1 & -1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & -1 & -1 & 1 \end{pmatrix} = U_4$$

In this example, it has been verified that $\mathbf{Z'X_1}=\mathbf{0}$ and $\mathbf{Z'X_2} \neq \mathbf{0}$ as r even and k odd number, $\mathbf{Z'Z} = n\mathbf{I}_c$, $\mathbf{Z'QZ} < \mathbf{Z'Z}$; where $\mathbf{Q} = (\mathbf{I} - \mathbf{X}(\mathbf{X'X})^{-}\mathbf{X'})$, $\mathbf{X} = (\mathbf{X}_1, \mathbf{X}_2)$, which gives the set of four covariates which are orthogonal to each other. The determinant value of I(γ) of the above design has been compared with the Determinant value of global optimal value of the above design assuming that $\mathbf{Z'QZ} = \mathbf{Z'Z}$ for evaluating the efficiency value of the design**.** In this case, $\mathbf{X}_1$, $\mathbf{X}_2$ and $\mathbf{Z}$ are the following:

$$X_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, X_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, Z = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

Now, we can easily find out $\mathbf{Z'X}_1$, $\mathbf{Z'X}_2$ and $\mathbf{Z'Z}$. These are given below:

$$Z'X_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; Z'X_2 = \begin{pmatrix} 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{pmatrix}$$

$$Z'Z = \begin{pmatrix} 24 & 0 & 0 & 0 \\ 0 & 24 & 0 & 0 \\ 0 & 0 & 24 & 0 \\ 0 & 0 & 0 & 24 \end{pmatrix}$$

So, the information matrix for γ is $\mathbf{I}(\gamma) = \mathbf{Z'QZ} = \mathbf{Z'Z} - \mathbf{Z'X}(\mathbf{X'X})^{-}\mathbf{X'Z} =$

$$\begin{pmatrix} 21.3333336 & 0 & -5.55112E\text{-}17 & 0 \\ 0 & 21.3333336 & 0 & 0 \\ 1.66533E\text{-}16 & 1.11022E\text{-}16 & 21.3333336 & -1.11022E\text{-}16 \\ 0 & 0 & 0 & 21.3333336 \end{pmatrix}$$

$$\cong \begin{pmatrix} 21.33 & 0 & 0 & 0 \\ 0 & 21.33 & 0 & 0 \\ 0 & 0 & 21.33 & 0 \\ 0 & 0 & 0 & 21.33 \end{pmatrix}$$

The determinant value of the information matrix for γ i.e., $\det(\mathbf{I}(\gamma)) = (21.33)^4 = 206996.70$, which is compared with $\det(\mathbf{I}(\gamma)) = (24)^4$, assuming that $\mathbf{I}(\gamma) = \mathbf{Z'QZ} = \mathbf{Z'Z} = 331776$ for the set of four covariates in a BIBD ($v = 4$, $b = 8$, $r = 6$, $k = 3$ and $\lambda = 4$). Thus, the efficiency value of the above covariate design with respect to global optimal design will be 62.47%.

3.3 The method of construction for D-efficient U-matrix when $\mathbf{H}_{k-t}$ exists ($t \geq 2$):

For construction of D-efficient **U** matrices of order vxb from a BIBD (v, b, r, k, λ) where r (even or odd number) and k (always odd number) and $\mathbf{H}_k$ do not exist, we follow the steps given below.

Step 1. Construct the BIB design D (v, b, r, k, λ).

Step 2. Construct the incidence matrix (**N**) of order vxb from the design D.

Step 3. Let us consider a Hadamard matrix of order b, $\mathbf{H}_b$.

$\mathbf{H}_b = (\mathbf{1}, \mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_{b-1})$

Step 4. Let us construct a Special Array $\mathbf{H}_k^*$ of order k from $\mathbf{H}_{k-t}$ with t (>1, odd number and $\mathbf{H}_{t-1}$ exist) rows and columns with all zero elements in middle, i.e., $(\mathbf{1}^*, \mathbf{h}_1^*, \mathbf{h}_2^*, \ldots, \mathbf{h}_{(k-t)/2-1}^*, 0, \ldots, 0, \mathbf{h}_{(k-t)/2}^*, \ldots, \mathbf{h}_{k-t-1}^*)$.

$$\begin{array}{ccc} & \overset{t}{\longleftrightarrow} & \\ H_k^* = & \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 & 1 & 1 \\ 1 & -1 & 0 & \cdots & 0 & -1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 1 & 0 & \cdots & 0 & -1 & -1 \\ 1 & -1 & 0 & \cdots & 0 & 1 & -1 \end{pmatrix} & \updownarrow t \end{array}$$

Step 5. Using $\mathbf{H_k}$* and $\mathbf{H_b}$, by Kronecker product of these two matrices, we get (k-t-1) set of (b-1) $\mathbf{U}_{ij}$* matrices of order kxb (without consider the first column and the t columns with all zeros), where i=1,2,…,(k-t-1) and j=1,2,…,(b-1). In each of the $\mathbf{U}_{ij}$* matrix there are t rows with all elements zero in the middle.

$U_{ij}^* = h_i^* \otimes h_j'$, $\otimes$ denotes the Kronecker product

Step 6. As $\mathbf{H}_b$ and $\mathbf{H}_{t-1}$ both are exists, following the Step 6 of the Method 3.2, we can construct (t-2) sets of (b-1) mutually orthogonal **U**** matrices of order txb. All the **U**** matrix has all row-sums equal to zero and all column-sums equal to either +1 or -1.
Step 7. Consider any set i (i=1,2,…,k-t-1), in each **U*** matrix of ith set, insert the **U**** matrix of order txb from i′th set (i′=1,2,...,t-2) in the t rows with all elements zero in the middle of **U*** matrix, such that all the t rows with all elements zero has been replaced by +1 or -1 till **U**** and **U*** matrices has been utilized totally. Let the resulting matrix be **U*****. For example, insert $\mathbf{U}_{i'1}$** matrix in $\mathbf{U}_{i1}$*, then insert $\mathbf{U}_{i'2}$** matrix in $\mathbf{U}_{i2}$* and so on repeat the procedure with other **U**** matrices in the remaining **U*** matrices till **U**** and **U*** matrices has been used totally for the ith set. So, we get (b-1) **U***** matrices of order kxb, which are orthogonal to each other and all the **U***** matrix has all row-sums equal to zero and all column-sums equal to either +1 or -1 for ith set.

Step 8. The non-zero elements of the incidence matrix (**N**) are replaced by the elements (+1 or -1) of each of the **U***** matrix from ith set column wise. Thus, we get (b-1) **U****** matrix of order vxb. It has been observed that the column sum of the **U****** matrix is always equal to either +1 or -1. But, the row totals of the **U****** matrices are must be either zero or ±1 (it may not true for all **U****** matrices) depending on r (even or odd). If the row totals are either zero or ±1 is true for all **U****** matrices depending on r (even or odd), we may get directly atmost (b-1) **U** matrices of order vxb satisfying the conditions $\mathbf{C}_1$*-$\mathbf{C}_3$* simultaneously for D-efficiency from **U****** matrices. On the other hand, when it is not true for all U**** matrices i.e., some or all row totals has been violated in some or all U**** matrices, to construct D-efficient **U**-matrix of order vxb, the elements of non-zero row sums or the elements of row sums neither +1 nor -1 depending on r (even or odd) of **U****** matrices has been rearranged column wise by trial and error method such that the resulting **U** matrices satisfying the conditions $\mathbf{C}_1$*-$\mathbf{C}_3$* simultaneously for D-efficiency. So, we may get atmost (b-1) **U** matrix of order vxb depending on **U****** matrices.
For easy understanding of the above steps, the following example will be useful.

Example 3.3.1: Let us consider a BIBD with parameters; v = 8, b = 8, r = 7, k = 7 and λ = 6. The construction of D-efficient**U**-matrix of order 8x8 is the following:

Step 1. Lay out of the BIBD (v = b = 8, r = k = 7 and λ = 6).

Blk 1	Blk 2	Blk 3	Blk 4	Blk 5	Blk 6	Blk 7	Blk 8
1	2	1	1	1	1	1	1
2	3	3	2	2	2	2	2
3	4	4	4	3	3	3	3
4	5	5	5	5	4	4	4
5	6	6	6	6	6	5	5
6	7	7	7	7	7	7	6
7	8	8	8	8	8	8	8

Step 2. Construction of the incidence matrix (**N**) of order 8x8 from the above design is

$$N = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Step 3. Let us consider a Hadamard matrix of order 8, $\mathbf{H}_8$.

$$H_8 = (1, h_1, h_2, h_3, h_4, h_5, h_6, h_7) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{pmatrix}$$

Step 4. Let us construct a Special Array $\mathbf{H}_7^*$ of order 7 from $\mathbf{H}_4$ with t (=3, odd number and $\mathbf{H}_{t-1}$ exist) rows and columns with all zero elements in middle, i.e., $(\mathbf{1^*, h_1^*, 0, 0, 0, h_2^*, h_3^*})$.

$$\mathrm{H}_7^* = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

Step 5. Using $\mathbf{H_7^*}$ and $\mathbf{H_8}$, by Kronecker product of these two matrices, we get 3 set of 7$\mathbf{U}_{ij}$* matrices of order 7x8 (without consider the first column and the 3 columns with all zeros), where i=1,2,3 and j=1,2,…,7. In each of the $\mathbf{U}_{ij}$* matrix there are 3 rows with all elements zero in the middle. Here, $\mathbf{U}_{11}$* matrix is shown below:

$$\mathrm{U}_{11}^* = \mathrm{h}_1^* \otimes \mathrm{h}'_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \end{pmatrix}$$

Similarly, we can easily construct the others.

Step 6. As $\mathbf{H}_8$ and $\mathbf{H}_2$ both are exists, following the Step 6 of the Method 3.2, we can construct one set of 8 mutually orthogonal **U**** matrices of order 3x8. All the **U**** matrix has all row-sums equal to zero and all column-sums equal to either +1 or -1. The $\mathbf{U}_1$** matrix is presented here

$$\mathbf{U}_1^{**} = \begin{pmatrix} \mathbf{1} & \mathbf{-1} & \mathbf{1} & \mathbf{-1} & \mathbf{1} & \mathbf{-1} & \mathbf{1} & \mathbf{-1} \\ \mathbf{1} & \mathbf{-1} & \mathbf{1} & \mathbf{-1} & \mathbf{1} & \mathbf{-1} & \mathbf{1} & \mathbf{-1} \\ \mathbf{-1} & \mathbf{1} & \mathbf{-1} & \mathbf{1} & \mathbf{-1} & \mathbf{1} & \mathbf{-1} & \mathbf{1} \end{pmatrix}$$

Similarly, we can easily construct the others.

Step 7. Here, we consider 1st set, in each **U*** matrix of 1st set, insert the **U**** matrix of order 3x8 in the 3 rows with all elements zero in the middle of **U*** matrix, such that all the 3 rows with all elements zero has been replaced by +1 or -1 till **U**** and **U*** matrices has been utilized properly. Let the resulting matrix be **U*****. So, in this example, insert $\mathbf{U}_1$** matrix in $\mathbf{U}_{i1}$*, then insert $\mathbf{U}_2$** matrix in $\mathbf{U}_{i2}$* and so on repeat the procedure with other **U**** matrices in the remaining **U*** matrices till 7**U**** and 7**U*** matrices has been used totally. Now, we get 7**U***** matrices of order 7x8, which are orthogonal to each other and all the **U***** matrix has all row-sums equal to zero and all column-sums equal to either +1 or -1. Here, $\mathbf{U}_1$** matrix has been inserted in the first matrix of first set i.e., $\mathbf{U}_{11}$* and we get the following $\mathbf{U}_1$*** matrix of order 7x8.

$$
U_1^{***} = \begin{pmatrix}
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
-1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
-1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
-1 & 1 & -1 & 1 & -1 & 1 & -1 & 1
\end{pmatrix} \rightarrow \mathbf{U}_1^{**}
$$

Similarly, we can easily construct the others.

Step 8. The non-zero elements of the incidence matrix (**N**) are replaced by the elements (+1 or -1) of each of the **U***** matrix column wise. Thus, we get 7 **U****** matrix of order 8x8. It has been observed that the column sum of the **U****** matrix is always equal to either +1 or -1 and the row totals of the **U******matrices are must be ±1 as r=7 (it is not true for all the **U****** matrices). Now, we get directly 2**U** matrices of order 8x8 satisfying the conditions $\mathbf{C}_1$*-$\mathbf{C}_3$* simultaneously for D-efficiency from 7**U****** matrices. For the remaining **U****** matrices where the row totals (some or all) are not ±1, to construct more D-efficient **U**-matrix of order 8x8, the elements of those rows of **U****** matrices has been rearranged column wise by trial and error method such that the resulting **U** matrices satisfying the conditions $\mathbf{C}_1$*-$\mathbf{C}_3$* simultaneously for D-efficiency. So, here we get in total 3 **U** matrices of order 8x8. These are the followings.

$$
U_1^{***} = \begin{pmatrix}
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
-1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
-1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
-1 & 1 & -1 & 1 & -1 & 1 & -1 & 1
\end{pmatrix}, \quad N = \begin{pmatrix}
1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{pmatrix}
$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & -1 & 0 & 1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 0 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & 0 & -1 & 1 & -1 \\ -1 & -1 & 1 & -1 & 1 & 0 & -1 & 1 \\ 1 & 1 & -1 & 1 & -1 & 1 & 0 & -1 \\ -1 & -1 & 1 & -1 & 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \end{pmatrix} = \mathrm{U}_1^{*****} = \mathrm{U}_1$$

Similarly, we get,

$$\mathrm{U}_2^{*****} = \begin{pmatrix} 1 & 0 & -1 & -1 & 1 & 1 & -1 & -1 \\ -1 & 1 & 0 & 1 & -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & 0 & 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 0 & 1 & -1 & -1 \\ -1 & 1 & -1 & -1 & 1 & 0 & 1 & 1 \\ 1 & -1 & 1 & 1 & -1 & -1 & 0 & -1 \\ -1 & 1 & -1 & -1 & 1 & 1 & -1 & 0 \\ 0 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \end{pmatrix} = \mathrm{U}_2$$

From $\mathbf{U}_3$****, we get $\mathbf{U}_3$ by trial and error method with respect to the conditions $\mathbf{C}_1$*-$\mathbf{C}_3$* for D-efficiency.

$$\mathrm{U}_3 = \begin{pmatrix} 1 & 0 & -1 & 1 & 1 & -1 & -1 & 1 \\ -1 & 1 & 0 & -1 & -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 0 & 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & 0 & -1 & -1 & 1 \\ -1 & -1 & -1 & 1 & 1 & 0 & 1 & -1 \\ 1 & 1 & -1 & -1 & -1 & 1 & 0 & 1 \\ -1 & -1 & 1 & 1 & 1 & -1 & -1 & 0 \\ 0 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \end{pmatrix}$$

From the other two sets of **U*** matrices, we will may also get at most 7**U** matrices satisfying the conditions $\mathbf{C}_1$*-$\mathbf{C}_3$* simultaneously for D-efficiency by the similar way.

Remark 3.1: Let a BIB design D* (v, b, r, k, λ) exist with atleast c* (≥ 2) D-efficient **U**-matrices. Then for the BIB design D** (v**=v, b**=mb, r**=mr, k**=k, λ**=mλ) obtained by repeating each block m (≥ 2) times, atleast c**= mc* D-efficient **U**-matrices can be constructed whenever $\mathbf{H}_m$ exists.

For example, a BIB design D* (v = b = 4, r = k = 3 and λ = 2), atleast c* = 2 D-efficient **U**-matrices be available. So, based on the Remark 3.1, we can construct atleast c**= 4 D-efficient **U**-matrices for the BIB design D** (v** = 4, b** = 8, r** = 6, k** = 3 and λ** = 4), where m=2 and $\mathbf{H}_2$ exists.

4. Conclusion

Two new series of D-efficient covariate designs in BIB design set-up are developed when either k or r is odd or both k and r are odd numbers and Hadamard matrix of order k i.e., $\mathbf{H}_k$ does not exist. All the developed designs are constructed with the help of a new matrix viz., Special Array (as defined in section 2). We also propose a conjecture (3.1.1) and **a**ccording to the conjecture, $\det(\mathbf{I}(\gamma)) = \det(\mathbf{Z}'\mathbf{Z} - \mathbf{Z}'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-}\mathbf{X}'\mathbf{Z})$ may or may not be the unique maximum value. If the value is unique maximum, then the design will be D- optimal, otherwise it will be D- efficient design. The proof of the claim is not yet established or found in available literatures.

Acknowledgement: The authors are thankful and also acknowledging the suggestions and comments of the reviewer for the preparation of revised manuscript of present article.

References

[1] Das, H., Dutta, A., Nishad, D., Majumder, A. (2021): New D-optimal Covariate Designs in CRD and RCBD set-ups. J. Ind. Soc.Agric. Stat., 75(2), 157-168.

[2] Das, H., Majumder, A., Kumar, M., Nishad, D. (2020): New Series of Optimal Covariate Designs in CRD and RCBD set-ups. J. Ind. Soc.Agric. Stat., 74(1), 41-50.

[3] Das, K., Mandal, N. K., Sinha, B. K. (2003): Optimal experimental designs with covariates. J. Stat. Plan. Inference, 115, 273-285.

[4] Das, P., Dutta, G., Mandal, N. K., Sinha, B. K. (2015): Optimal covariate design-Theory and applications. Springer, India.

[5] Dey, A., Mukerjee, R. (2006): D-optimal designs for covariate models. Statistics, 40, 297-305.

[6] Dutta, G. (2004): Optimum choice of covariates in BIBD set-up. Cal.Stat. Assoc. Bull., 55, 39-55.

[7] Dutta, G. (2009): Optimum designs for covariates models. (Ph.D. Thesis submitted to the University of Calcutta, West Bengal)

[8] Dutta, G., Das, P., Mandal, N. K. (2007): Optimum choice of covariates for a series of SBIBDs obtained through projective geometry. J. Modern Appld. Stat. Methods, 6, 649-656.

[9] Dutta, G., Das, P., Mandal, N. K. (2009a): Optimum covariate designs in split-plot and strip-plot design set-ups. J. Appld. Stat., 36, 893-906.

[10] Dutta, G., Das, P., Mandal, N. K. (2009b): Optimum covariate designs in partially balanced incomplete block (PBIB) design set-ups. J. Stat. Plan. Inference, 139, 2823-2835.

[11] Dutta, G., Das, P., Mandal, N. K. (2010a): Optimum covariate designs in binary proper equireplicate block design set-up. Discrete Mathematics, 310, 1037-1049.

[12] Dutta, G., Das, P., Mandal, N. K. (2010b): D-optimal designs for covariate parameters in block design set-up. Commun. Stat. Theory Methods, 39, 3434-3443.

[13] Dutta, G., Das, P., Mandal, N. K. (2014): D-Optimal designs for covariate models. Commun. Stat. Theory Methods, 43, 165-174.

[14] Pukelsheim F. (1993): Optimal design of experiments. Wiley, New York.

[15] Rao, P.S.S.N.V.P., Rao, S. B., Saha, G. M., Sinha, B. K. (2003): Optimal designs for covariates' models and mixed orthogonal arrays. Electron Notes. Discret. Math., 15, 157-160.

[16] Troya, Lopes, J. (1982a): Optimal designs for covariate models. J. Stat. Plan. Inference, 6, 373-419.

[17] Troya, Lopes, J. (1982b): Cyclic designs for a covariate model. J. Stat. Plan. Inference, 7, 49-75.