Characterizations of m-Normal Nearlattices in terms of Principal n-Ideals

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Abstract
A convex subnearlattice of a nearlattice S containing a fixed element n ∈ S is called an n-ideal. The n-ideal generated by a single element is called a principal n-ideal. The set of finitely generated principal n-ideals is denoted by P_n(S), which is a nearlattice. A distributive nearlattice S with 0 is called m-normal if its every prime ideal contains at most m number of minimal prime ideals. In this paper, we include several characterizations of those P_n(S) which form m-normal nearlattices. We also show that P_n(S) is m-normal if and only if for any m + 1 distinct minimal prime n-ideals P_o, P_1, ..., P_m of S, P_o ∨ ... ∨ P_m = S.

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Introduction
Lee in [9], also see Lakser [7], has determined the lattice of all equational subclasses of the class of all pseudo-complemented distributive lattices. They are given by
B_1 ⊂ B_o ⊂ ... ⊂ B_m ⊂ ... ⊂ B_ω, where all the inclusions are proper and B_ω is the class of all pseudo-complemented distributive lattices, B_1 consists of all one element algebra, B_o is the variety of Boolean algebras while B_m, for -1 ≤ m < ω, consists of all algebras satisfying the equation

\[(x_1 \land x_2 \land ... \land x_m)^* \lor \bigvee_{i=1}^{n} (x_1 \land x_2 \land ... \land x_{i-1} \land x_i^* \land x_{i+1} \land ... \land x_m)^* = 1\]

where \(x^*\) denotes the pseudo-complement of \(x\). Thus B_1 consists of all Stone algebras.
Davey [4] has independently given several characterizations of (sectionally) B_m and relatively B_m-lattices. On the other hand Cornish in [3] has studied...
distributive lattices (without pseudo-complementation) analogues to \( B_m \)-lattices and relatively \( B_m \)-lattices.

A distributive nearlattice \( S \) with 0 is called \( m \)-normal if each prime ideal of \( L \) contains at most \( m \)-minimal prime ideals. For a fixed element \( n \in S \), a convex subnearlattice containing \( n \) is called an \( n \)-ideal. An \( n \)-ideal generated by a finite number of elements \( a_1, a_2, \ldots, a_n \) is called a finitely generated \( n \)-ideal, denoted by \( < a_1, a_2, \ldots, a_n >_n \). The set of all finitely generated \( n \)-ideals is a nearlattice denoted by \( F_n(S) \). An \( n \)-ideal generated by a single element is called a principal \( n \)-ideal is denoted by \( P_n(S) \).

In this paper we include several characterizations of those \( P_n(S) \) which form \( m \)-normal nearlattices. We show that \( P_n(S) \) is \( m \)-normal if and only if for any \( m+1 \) distinct minimal prime \( n \)-ideals \( P_0, P_1, \ldots, P_m \) of \( S \), \( P_0 \lor \ldots \lor P_m = S \).

We start the paper with the following result on \( n \)-ideals due to Latif and Noor [8].

Lemma 1.1 For a central element \( n \in S \), \( P_n(S) \cong (n] \times [n) \).

Following result is also essential for the development of this paper, which is due to Ali [1,Theorem 1.1.12].

Lemma 1.2 Let \( S \) be a distributive near-lattice with an upper element \( n \) and let \( I, J \) be two \( n \)-ideals of \( S \). Then for any \( x \in I \lor J, x \lor n = i \lor j \) and \( x \land n = i' \land j' \) for some \( i, i', j, j' \in I \lor J \) with \( i, j \geq n \) and \( i', j' \leq n \).

Now we include the following result which is due to Noor and Ali [10] and this is a generalization of [2, Lemma 3.6].

A prime \( n \)-ideal \( P \) is said to be a minimal prime \( n \)-ideal belonging to \( n \)-ideal \( I \) if

(i) \( I \subseteq P \) and

(ii) There exists no prime \( n \)-ideal \( Q \) such that \( Q \neq P \) and \( I \subseteq Q \subseteq P \).
A prime n-ideal $P$ of a nearlattice $S$ is called a *minimal prime n-ideal* if there exists no prime n-ideal $Q$ such that $Q \neq P$ and $Q \subseteq P$. Thus a minimal prime n-ideal is a minimal prime n-ideal belonging to $\{n\}$.

Following lemma will be needed for further development of this paper. This is [3, Lemma 3.6] and is easy to prove. So we omit the proof.

The following result is [4, Lemma 2.2] which also follows from the corresponding result for commutative semi-groups due to Kist [6].

**Lemma 1.3** Let $M$ be a prime ideal containing an ideal $J$ in a distributive medial nearlattice. Then $M$ is a minimal prime ideal belonging to $J$ if and only if for all $x \in M$, there exists $x' \notin M$ such that $x \land x' \in J$.

Now we generalize this result for n-ideals.

**Lemma 1.4** Let $n$ be a medial element and $M$ be a prime n-ideal containing an n-ideal $J$. Then $M$ is a minimal prime n-ideal belonging to $J$ if and only if for all $x \in M$ there exists $x' \notin M$ such that $m(x, n, x') \in J$.

**Proof.** Let $M$ be a minimal prime n-ideal belonging to $J$ and $x \in M$. Then by [11], $< < a >, J > \subsetneq M$. So there exists $x'$ with $m(x, n, x') \in J$ such that $x' \notin M$.

Conversely, suppose $x \in M$, then there exists $x' \notin M$ such that $m(x, n, x') \in J$. This implies $x' \notin M$, but $x' \in < < x >, J >$, that is $< < x >, J > \subsetneq M$. Hence by [10], $M$ is a prime n-ideal belonging to $J$.

Davey in [4, Corollary 2.3] used the following result in proving several equivalent conditions on $B_m$-lattices. On the other hand, Cornish in [3] has used this result in studying n-normal lattices.

**Proposition 1.5** Let $M_o, \ldots, M_n$ be $n+1$ distinct minimal prime ideals of a distributive nearlattice $S$. Then there exists $a_o, a_1, \ldots, a_n \in S$ such that $a_i \land a_j \in J$ $(i \neq j)$ and $a_j \notin M_j, j = 0, 1, \ldots, n$.

Now we generalize the above result in terms of n-ideals.

**Proposition 1.6** Let $S$ be a distributive nearlattice and $n \in S$ is medial. Suppose $M_o, \ldots, M_m$ be $m+1$ distinct minimal prime n-ideals containing n-ideal $J$. Then
there exists \(a_0, a_1, \ldots, a_n \in S\) such that \(m(a_i, n, a_j) \in J\) (\(i \neq j\)) and \(a_j \not\in M_j\) (\(j = 0, 1, \ldots, m\)).

**Proof.** Let \(n = 1\). Let \(x_0 \in M_1 - M_0\) and \(x_1 \in M_0 - M_1\). Then by Lemma 1.3, there exists \(x_1' \not\in M_0\) such that \(m(x_0, n, x_1') \in J\). Hence \(a_1 = x_1, a_0 = m(x_0, n, x_1')\) are the required elements.

Observe that \(m(a_0, n, a_1) = m(m(x_0, n, x_1'), n, x_1)\)

\[
= (x_0 \land x_1 \land x_1') \lor (x_0 \land n) \lor (x_1 \land n) \lor (x_1' \land n)
\]

\[
= (x_0 \land m(x_1, n, x_1')) \lor (x_0 \land n) \lor (m(x_1, n, x_1') \land n)
\]

\[
= m(x_0, n, m(x_1, n, x_1'))
\]

Now, \(m(x_1, n, x_1') \land n \leq m(x_0, n, m(x_1, n, x_1'))\)

\[
\leq m(x_1, n, x_1') \lor n
\]

and \(m(x_1, n, x_1') \in J\), so by convexity \(m(a_0, n, a_1) \in J\).

Assume that, the result is true for \(n = m-1\), and let \(M_0, \ldots, M_m\) be \(m+1\) distinct minimal prime \(n\)-ideals. Let \(b_0, b_1, \ldots, b_m \in M_m - \bigcup_{j=0}^{m} M_j\) satisfy

\(m(b_i, n, b_j) \in J\) (\(i \neq j\)) and \(b_j \not\in M_j\). Now choose \(b_m \in M_m - \bigcup_{j=0}^{m-1} M_j\) and by Lemma 1.4, let \(b'_m\) satisfy \(b'_m \not\in M_m\) and \(m(b_m, n, b'_m) \in J\). Clearly,

\(a_0 = m(b_j, n, b_m)\) (\(j = 0, \ldots, m-1\)) and \(a_m = b'_m\), establish the result. \(\square\)

Let \(J\) be an \(n\)-ideal of a distributive lattice \(L\). A set of elements \(x_0, \ldots, x_n \in L\) is said to be **pairwise in** \(J\) if \(m(x_i, n, x_j) = n\) for all \(i \neq j\).

The next result is [3, Lemma 2.3] which was suggested by Hindman in [5, Theorem 1.8].

**Lemma 1.7** Let \(J\) be an ideal in a distributive nearlattice \(S\). For a given positive integer \(n \geq 2\), the following conditions are equivalent.

(i) For any \(x_1, \ldots, x_n \in S\) which are 'pairwise in \(J\)' that is

\(x_i \land x_j \in J\) for any \(i \neq j\), there exists \(k\) such that \(x_k \in J\).
(ii) For any ideals \( J_1, \ldots, J_n \) in \( S \) such that \( J_i \cap J_j \subseteq J \) for any \( i \neq j \), there exists \( k \) such that \( J_k \subseteq J \).

(iii) \( J \) is the intersection of at most \( n-1 \) distinct prime ideals. □

Our next result is a generalization of above result. This result will be needed in proving the next theorem which is the main result of this section. In fact, the following lemma is very useful in studying those \( P_n(S) \) which are \( m \)-normal.

**Lemma 1.8** Let \( J \) be an \( n \)-ideal in a distributive nearlattice \( S \) and \( n \in S \) is medial.

For a given positive integer \( m \geq 2 \), the following conditions are equivalent.

(i) For any \( x_1, \ldots, x_n \in S \) with \( m(x_i, n, x_j) \in J \) (that is, they are pairwise in \( J \)) for any \( i \neq j \), there exists \( k \) such that \( x_k \in J \).

(ii) For any \( n \)-ideals \( J_1, \ldots, J_m \) in \( S \) such that \( J_i \cap J_j \subseteq J \) for any \( i \neq j \), there exists \( k \) such that \( J_k \subseteq J \).

(iii) \( J \) is the intersection of at most \( m-1 \) distinct prime \( n \)-ideals.

**Proof.** (i) and (ii) are easily seen to be equivalent.

(iii)⇒(i). Suppose \( P_1, P_2, \ldots, P_k \) are \( k \) (\( 1 \leq k \leq m-1 \)) distinct prime \( n \)-ideals such that \( J = P_1 \cap P_2 \cap \cdots \cap P_k \). Let \( x_1, x_2, \ldots, x_m \in S \) be such that \( m(x_i, n, x_j) \in J \) for all \( i \neq j \). Suppose no element \( x_i \) is a member of \( J \). Then for each \( r \) (\( 1 \leq r \leq k \)) there is at most one \( i \) (\( 1 \leq i \leq m \)) such that \( x_i \in P_r \). Since \( k < m \), there is some \( i \) such that \( x_i \in P_1 \cap P_2 \cap \cdots \cap P_k \).

(i)⇒(iii). Suppose (i) holds for \( m = 2 \), then it implies that \( J \) is a prime \( n \)-ideal. Then (iii) is trivially true. Thus we may assume that there is a largest integer \( t \) with \( 2 \leq t < m \) such that the condition (i) does not hold for \( J \) (consequently condition (i) holds for \( t+1, t+2, \ldots, m \)). Then for some \( 2 \leq t < m \) we may suppose that there exist elements \( a_1, a_2, \ldots, a_t \in L \) such that
\[
m(a_i, n, a_j) \in J \quad \text{for} \quad i \neq j, i = 1, 2, \ldots, t, \quad j = 1, 2, \ldots, t, \quad \text{yet} \quad a_1, a_2, \ldots, a_t \notin J.
\]
As \( S \) is a distributive lattice, \( <a_i >_n, J> \) is an \( n \)-ideal for any \( i \in \{1, 2, \ldots, t\} \). Each \( <a_i >_n, J> \) is in fact a prime \( n \)-ideal.
Firstly \( <a_i>n_n J > \neq S\), since \( a_i \notin J \). Secondly, suppose that \( b \) and \( c \) are in \( S \) and \( m(b, n, c) \in <a_i>n_n J > \). Consider the set of \( t+1 \) elements \( \{a_1, a_2, \ldots, a_{i-1}, m(b, n, a_i), m(c, n, a_i), a_{i+1}, \ldots, a_t\} \). This set is pairwise in \( J \) and so, either \( m(b, n, a_i) \in J \) or \( m(c, n, a_i) \in J \). Since condition (i) holds for \( t+1 \).

That is, \( b \in <a_i>n_n J > \) or \( c \in <a_i>n_n J > \) and so \( <a_i>n_n J > \) is prime.

Clearly, \( J \subseteq \bigcap_{1 \leq i \leq t} <a_i>n_n J > \). If \( w \in \bigcap_{1 \leq i \leq t} <a_i>n_n J > \). Then \( w, a_1, a_2, \ldots, a_t \) are pairwise in \( J \) and so \( w \in J \). Hence \( J = \bigcap_{1 \leq i \leq t} <a_i>n_n J > \) is the intersection of \( t <m \) prime \( n \)-ideals.

An ideal \( J \neq S \) satisfying the equivalent conditions of Lemma 1.7. is called an \( m \)-prime ideal. Similarly, an \( n \)-ideal \( J \neq S \) satisfying the equivalent conditions of Lemma 1.8. is called an \( m \)-prime \( n \)-ideal.

For \( a, b \in S, <a, b > = \{x \in S : x \wedge a \leq b\} \) is known as annihilator of \( a \) relative to \( b \) or simply a relative annihilator. In presence of distributivity, it is easy to show that each relative annihilator is an ideal. Again for \( a, b \in L \), where \( L \) is a lattice, we define \( <a, b >_d = \{x \in L : x \vee a \geq b\} \) is a relative dual annihilator. In presence of distributivity of \( L \), \( <a, b >_d \) is a dual ideal (filter).

For \( a, b \in S \) and an upper element \( n \in S \), we define, \( <a, b >^n = \{x \in S : m(a, n, x) \in <b >_n\} = \{x \in S : b \wedge n \leq m(a, n, x) \leq b \vee n\} \).

We call \( <a, b >^n \) the annihilator of \( a \) relative to \( b \) around the element \( n \) or simply a relative \( n \)-annihilator. It is easy to see that for all \( a, b \in S, <a, b >^n \) is always a convex subset containing \( n \). In presence of distributivity, it can easily be seen that \( <a, b >^n \) is an \( n \)-ideal. If \( 0 \in S \), then putting \( n = 0 \), we have, \( <a, b >^n = <a, b > \).
For two n-ideals A and B of a nearlattice S, \( < A, B > \) denotes \( \{ x \in S: m(a, n, x) \in B \text{ for all } a \in A \} \), when n is a medial element. In presence of distributivity, clearly \( < A, B > \) is an n-ideal.

Now we generalize a result of Davey in [4, Proposition 3.1.].

**Theorem 1.9** Let J be an n-ideal of a distributive nearlattice S and n be a central element of S. Then the following conditions are equivalent.

(i) For any \( m+1 \) distinct prime n-ideals \( P_0, P_1, \ldots, P_m \) belonging to J, \( P_0 \lor P_1 \lor \ldots \lor P_m = S \).

(ii) Every prime n-ideal containing J contains at most \( m \) distinct minimal prime n-ideals belonging to J.

(iii) If \( a_0, a_1, \ldots, a_m \in S \) with \( m(a_i, n, a_j) \in J \) (i ≠ j) then \( \lor_j < < a_j >_n, J > = S \).

**Proof.** (i)⇒(ii) is obvious.

(ii)⇒(iii). Assume \( a_0, a_1, \ldots, a_m \in S \) with \( m(a_i, n, a_j) \in J \) and \( \lor_j < < a_j >_n, J > \neq S \). It follows that \( a_j \not\in J \), for all j. Then by [8], there exists a prime n-ideal P such that \( \lor_j < < a_j >_n, J > \subseteq P \). But by [11], we know that P is either a prime ideal or a prime filter.

Suppose P is a prime ideal. For each j, let \( F_j = \{ x \land y: x \geq a_j, x, y \geq n, y \not\in P \} \).

Let \( x_1 \land y_1, x_2 \land y_2 \in F_j \).

Then \( (x_1 \land y_1) \land (x_2 \land y_2) = (x_1 \land x_2) \land (y_1 \land y_2) \).

Now, \( x_1 \land x_2 \geq a_j \) and \( y_1 \land y_2 = m(y_1, n, y_2) \). So \( t \geq x \land y \) implies \( t = (t \lor x) \land (t \lor y) \). Since \( y \not\in P \), so \( t \lor y \not\in P \). Hence \( t \in F_j \), and so \( F_j \) is a dual ideal.

We now show that \( F_j \cap J = \phi \), for all \( j = 0, 1, 2, \ldots, m \). If not let \( b \in F_j \cap J \), then \( b = x \land y, x \geq a_j, x, y \geq n, y \not\in P \). Hence \( m(a_j, n, y) = (a_j \land n) \lor n \lor (a_j \land y) = (a_j \land y) \lor n = (a_j \lor n) \land (y \lor n) \). But \( (a_j \lor n) \land (y \lor n) \in F_j \) and
Again, \( m(a_j, n, y) \in J \) with \( y \notin P \) implies \( \langle < a_j >_n, J > \subseteq P \), which is a contradiction. Hence \( F_j \cap J = \phi \) for all \( j \). For each \( j \), let \( P_j \) be a minimal prime \( n \)-ideal belonging to \( J \) and \( F_j \cap P_j = \phi \). Let \( y \in P_j \). If \( y \notin P \), then \( y \lor n \notin P \).

Then \( m(a_j, n, y \lor n) = (a_j \lor n) \land (y \lor n) \in F_j \).

But \( m(a_j, n, y \lor n) \in \langle y \lor n \rangle_n \subseteq \langle y \rangle_n \subseteq P_j \), which is a contradiction. Hence \( P_j \cap F_j = \phi \) for all \( j \). For each \( j \), let \( P_j \) be a minimal prime \( n \)-ideal belonging to \( J \) and \( F_j \cap P_j = \phi \). Let \( y \in P_j \). If \( y \notin P \), then \( y \lor n \notin P \).

Now, \( a_j \lor n = (a_j \lor n) \land (a_j \lor n \lor y) \in F_j \) for any \( y \notin P \). This implies \( F_j \cap F_j \neq \emptyset \), which is a contradiction. So, \( a_j \notin P_j \). But \( m(a_i, n, a_j) \in J \subseteq P_j \) \( (i \neq j) \) which implies \( a_i \notin P_j \) \( (i \neq j) \) as \( P_j \) is prime. It follows that \( P_j \) form a set of \( m+1 \) distinct minimal prime \( n \)-ideals belonging to \( J \) and contained in \( P \). This contradicts (ii).

Therefore, \( \forall j \langle < a_j >_n, J > = S \).

Similarly, if \( P \) is filter, then a dual proof of above also shows that \( \forall j \langle < a_j >_n, J > = S \), and hence (iii) holds.

(iii) \( \Rightarrow \) (i). Let \( P_0, P_1, \ldots, P_m \) be \( m+1 \) distinct minimal prime \( n \)-ideals belonging to \( J \). Then by Proposition 1.6, there exists \( a_o, a_1, \ldots, a_m \in S \) such that \( m(a_i, n, a_j) \in J \) \( (i \neq j) \) and \( a_j \notin P_j \). This implies \( \langle < a_j >_n, J > \subseteq P_j \) for all \( j \). Then by (iii), \( \langle < a_o >_n, J > \lor \langle < a_1 >_n, J > \lor \ldots \lor \langle < a_m >_n, J > \subseteq P_o \lor P_1 \lor \ldots \lor P_m \), which implies \( P_o \lor P_1 \lor \ldots \lor P_m = S \).

For a prime \( n \)-ideal \( P \) of \( S \), \( n(P) = \{ x \in S : m(x, n, y) = n \} \) for some \( y \in S-P \).

Clearly, \( n(P) \) is an \( n \)-ideal and \( n(P) \subseteq P \). Our next result is a nice extension of above result in terms of \( n \)-ideals.

**Theorem 1.10.** Let \( S \) be a distributive nearlattice with a central element \( n \). Then the following conditions are equivalent.

(i) For any \( m+1 \) distinct minimal prime \( n \)-ideals \( P_0, P_1, \ldots, P_m \).
\[ P_0 \lor P_1 \lor \ldots \lor P_m = S. \]

(ii) Every prime \( n \)-ideal contains at most \( m \) minimal prime \( n \)-ideals.

(iii) For any \( a_0, a_1, \ldots, a_m \in S \) with \( m(a_i, n, a_j) = n \) for \( i \neq j \),
\[ i = 0, 1, 2, \ldots, m, \ j = 0, 1, 2, \ldots, m, \ \langle a_0 \rangle_n^* \lor \langle a_1 \rangle_n^* \lor \ldots \lor \langle a_m \rangle_n^* = S. \]

(iv) For each prime \( n \)-ideal \( P \), \( n(P) \) is an \( m+1 \)-prime \( n \)-ideal.

**Proof.** (i) \(\Rightarrow\) (ii), (ii) \(\Rightarrow\) (iii) and (iii) \(\Rightarrow\) (i) easily hold by Theorem 1.9, replacing \( J \) by \( \{n\} \). To complete the proof we need to show that (iv) \(\Rightarrow\) (iii) and (ii) \(\Rightarrow\) (iv).

(iv) \(\Rightarrow\) (iii). Suppose (iv) holds and \( x_0, x_1, \ldots, x_m \) are \( m+1 \) elements of \( S \) such that \( m(x_i, n, x_j) = n \) for \( i \neq j \). Suppose that \( \langle x_0 \rangle_n^* \lor \ldots \lor \langle x_m \rangle_n^* \neq S \). Then by Stone’s separation theorem in [9], there is a prime \( n \)-ideal \( P \) such that \( \langle x_0 \rangle_n^* \lor \ldots \lor \langle x_m \rangle_n^* \subseteq P \). Hence \( x_0, x_1, \ldots, x_m \in S - n(P) \). This contradicts (iv) by Lemma 1.8, since \( m(x_i, n, x_j) = n \) for all \( i \neq j \). Thus (iii) holds.

(ii) \(\Rightarrow\) (iv). This follows immediately from Lemma 1.8.

**Proposition 1.11** Let \( S \) be a distributive medial nearlattice and \( n \in S \) is a central element. If the equivalent conditions of Theorem 1.10 hold, then for any \( m+1 \) elements \( x_0, x_1, \ldots, x_m \) and \( m(x_i, n, x_j) = n \) for \( i \neq j \),
\[ \bigvee_{0 \leq i \leq m} \langle x_0 \rangle_n \lor \langle x_1 \rangle_n \lor \ldots \lor \langle x_m \rangle_n \]

\[ = \langle x_0 \rangle_n \lor \langle x_1 \rangle_n \lor \ldots \lor \langle x_m \rangle_n \lor \langle x_i \rangle_n \lor \langle x_{i+1} \rangle_n \lor \ldots \lor \langle x_m \rangle_n \].

**Proof.** Let \( \langle b_i \rangle_n = \langle x_0 \rangle_n \lor \langle x_1 \rangle_n \lor \ldots \lor \langle x_i \rangle_n \lor \langle x_{i+1} \rangle_n \lor \ldots \lor \langle x_m \rangle_n \) for each \( 0 \leq i \leq m \). Suppose \( x \in (\langle x_0 \rangle_n \lor \langle x_1 \rangle_n \lor \ldots \lor \langle x_m \rangle_n) \).

Then \( \langle x \rangle_n \lor \langle b_0 \rangle_n \lor \ldots \lor \langle x_m \rangle_n = \{n\} \). For all \( i \neq j \),
\( \langle x \rangle_n \lor \langle b_j \rangle_n \) \(\lor\) \( \langle x \rangle_n \lor \langle b_j \rangle_n = \{n\} \).

So \( \langle x \rangle_n \lor \langle b_0 \rangle_n \lor \ldots \lor \langle x_m \rangle_n = S \).

Thus \( x \in (\langle x \rangle_n \lor \langle b_0 \rangle_n \lor \ldots \lor \langle x_m \rangle_n) \lor \ldots \lor (\langle x \rangle_n \lor \langle b_m \rangle_n) \lor \{n\} \).

Hence by Lemma 1.2, \( x \lor n = a_0 \lor \ldots \lor a_m \) where \( a_i \in (\langle x \rangle_n \lor \langle b_i \rangle_n \lor \ldots \lor \langle x_m \rangle_n) \) and \( a_i \geq n \) for \( i = 0, 1, \ldots, m \). Then \( x \lor n = (a_0 \lor (x \lor n)) \lor \ldots \lor (a_m \lor (x \lor n)) \).
Now $a_i < x > \cap < b_i > \cap < a_i >_n \cap < b_i >_n = \{n\}$. Then by a routine calculation we find that $(a_i \wedge x \wedge b_i) \vee n = n$
Thus $< a_i >_n \cap < b_i >_n = [n, (a_i \wedge x \wedge b_i) \vee n] = \{n\}$ implies that $a_i \wedge (x \vee n) \in < b_i >_n^*$ and so $x \vee n \in < b_o >_n^* \vee < b_1 >_n^* \vee \ldots \vee < b_m >_n^*$. By a dual proof of above and using Theorem 1.3.7, we can easily show that $x \wedge n < b_o >_n^* \vee < b_1 >_n^* \vee \ldots \vee < b_m >_n^*$.
Thus by convexity, $x \in < b_o >_n^* \vee < b_1 >_n^* \vee \ldots \vee < b_m >_n^*$. 
This proves that L.H.S. $\subseteq$ R.H.S. The reverse inclusion is clear. 

**Theorem 1.12** Let $S$ be a distributive nearlattice and $n \in S$ is central. Then the following conditions are equivalent.

(i) $P_n(S)$ is $m$-normal.

(ii) Every prime $n$-ideal contains at most $m$ minimal prime $n$-ideals.

(iii) For any $m+1$ distinct minimal prime $n$-ideals $P_o, \ldots, P_m$;

$$P_o \vee \ldots \vee P_m = S.$$ 

(iv) If $m(a_i, n, a_j) = n$, this implies $< a_o >_n^* \vee \ldots \vee < a_m >_n^* = S$.

(v) For each prime $n$-ideal $P$, $n(P)$ is an $m+1$ prime $n$-ideal.

**Proof.** (i)$\Rightarrow$(ii). Let $P_n(S)$ be $m$-normal, since $n$ is central, $P_n(S) \cong (n)^d \times [n)$, so both $(n)^d$ and $[n)$ are $m$-normal. Suppose $P$ is any prime $n$-ideal of $S$. Then by [10], either $P \supseteq (n)$ or $P \supseteq [n)$. Without loss of generality, suppose $P \supseteq [n)$.

Then by [10], $P$ is prime ideal of $S$. Hence by [2, Lemma 3.4], $P = P \cap [n)$ is a prime ideal of $[n)$. Since $[n)$ is $m$-normal, so by [3] $P = P \cap [n)$ contains at most $m$ minimal prime ideals $R_1, R_2, \ldots, R_m$ of $[n)$. Therefore, $P$ contains at most $m$ minimal prime ideals $T_1, T_2, \ldots, T_m$ of $S$ where $R_1 = T_1 \cap [n), R_2 = T_2 \cap [n), \ldots, R_m = T_m \cap [n)$. Since $n \in R_1, \ldots, R_m$, $n \in T_1, \ldots, T_m$, hence $T_1, \ldots, T_m$ are minimal prime $n$-ideals of $S$. Thus (ii) holds.

(ii)$\Rightarrow$(i). Suppose (ii) holds. Let $P_1$ be a prime ideal in $[n)$. Then by [2, Lemma 3.4], $P_1 = P \cap [n)$ for some prime ideal $P$ of $S$. Since
n ∈ P₁ ⊆ P, so P is prime n-ideal. Therefore, P contains at most m minimal prime n-ideals R₁,..., Rₘ of S. Thus by [2, Lemma 3.4], P₁ contains at most m minimal prime ideals T₁ = R₁ ∩ [n), T₂ = R₂ ∩ [n),……., Tₘ = Rₘ ∩ [n) of [n).

Hence by Theorem 1.10, [n] is m-normal. Similarly, we can prove that (n]d is also m-normal. Thus by Lemma 1.1, Pₙ(S) is m-normal.

(ii) ⇔ (iii) has already been proved in Theorem 1.10

References