Characterizations of relative n-annihilators of nearlattices

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Abstract: In this paper we have introduced the notion of relative n-annihilators around a fixed element n of a nearlattice S which is used to generalize several results on relatively nearlattices. We have also given some characterizations of distributive and modular nearlattices in terms of relative n-annihilators.

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Introduction.

Relative annihilators in lattices and semi-lattices have been studied by many authors including [1], [2], [3] and [4]. Also [5] has used the annihilators in studying relative normal lattices. In this paper, we introduce the notion of relative annihilators around a fixed element n of a nearlattice S which is used to generalize several results on relatively nearlattices.

For a, b ∈ S, < a, b > denotes the relative annihilator, that is

< a, b > = \{x ∈ S: x ∧ a ≤ b\}. In presence of distributivity, it is easy to show that each relative annihilator is an ideal. Also note that < a, b > = < a, a ∧ b >. For detailed literature on this see [1] and [4]. Again for a, b ∈ L, where L is a lattice, recall that

< a, b >_d = \{x ∈ L: x ∨ a ≥ b\} is a relative dual annihilator. In presence of distributivity of L, < a, b >_d is a dual ideal (filter).

In case of a nearlattice it is not possible to define a dual relative annihilator ideal for any a and b. But if n is an upper element of S, then x ∨ n exists for all x ∈ S by the upper bound property of S. Then for any a ∈ (n], we can talk about dual relative annihilator ideal of the form < a, b >_d for any b ∈ S. That is, for any a ≤ n in S,

< a, b >_d = \{x ∈ S: x ∨ a ≥ b\}.

For a, b ∈ S and an upper element n ∈ S, we define,

< a, b >^n = \{x ∈ S: m(a, n, x) ∈ < b >^n\}

= \{x ∈ S: b ∧ n ≤ m(a, n, x) ≤ b ∨ n\}. 
We call $<a, b>^n$ the annihilator of a relative to b around the element n or simply a relative n-annihilator. It is easy to see that for all $a, b \in S$, $<a, b>^n$ is always a convex subset containing n. In presence of distributivity, it can easily be seen that $<a, b>^n$ is an n-ideal. If $0 \in S$, then putting $n = 0$, we have, $<a, b>^n = <a, b>$.

For two n-ideals A and B of a nearlattice S, $<A, B>$ denotes

$\{x \in S: m(a, n, x) \in B \text{ for all } a \in A\}$, when n is a medial element. In presence of distributivity, clearly $<A, B>$ is an n-ideal. Moreover, we can easily show that $<a, b>^n = <<a>_n, <b>_n>$. 

In this paper, we have given several characterizations of $<a, b>^n$. We have also given some characterizations of distributive and modular nearlattices in terms of relative n-annihilators.
1. Relative Annihilators around a central element of a Nearlattice.

We start with the following characterization of \(< a, b >^n\).

**Theorem 1.1** Let \(S\) be a nearlattice with a central element \(n\). Then for all \(a, b \in S\), the following conditions are equivalent.

(i) \(< a, b >^n\) is an \(n\)-ideal.

(ii) \(< a \wedge n, b \wedge n >_d\) is a filter and \(< a \vee n, b \vee n >\) is an ideal.

**Proof.** (i)⇒(ii). Suppose (i) holds. Let \(x, y \in < a \vee n, b \vee n >\) and \(x \vee y\) exists. Then \(x \wedge (a \vee n) \leq (b \vee n)\). Thus \((x \wedge (a \vee n)) \wedge n \leq (b \vee n)\), then by the neutrality of \(n\), \((x \vee n) \wedge (a \vee n) \leq (b \vee n)\).

Also \(m(x \vee n, n, a) = (x \vee n) \wedge (a \vee n) \leq b \vee n\). This implies \(x \vee y \in < a, b >^n\). Similarly, \(y \vee n \in < a, b >^n\). Since \(< a, b >^n\) is an \(n\)-ideal,

so \(x \vee y \vee n \in < a, b >^n\). This implies \(m(x \vee y \vee n, n, a) \leq b \vee n\). That is,

\((x \vee y) \wedge (a \vee n) \leq b \vee n\) and so \((x \vee y) \wedge (a \vee n) \leq b \vee n\). Therefore,

\(x \vee y \in < a \wedge n, b \vee n >\).

Moreover, for \(x \in < a \wedge n, b \vee n >\) and \(t \leq x\) \((t \in S)\).

Obviously, \(t \wedge (a \vee n) \leq b \vee n\), and so \(t \in < a \vee n, b \vee n >\).

Hence \(< a \wedge n, b \vee n >\) is an ideal.

A dual proof of above shows that \(< a \wedge n, b \wedge n >_d\) is a filter.

(ii)⇒(i). Suppose (ii) holds and \(x, y \in < a, b >^n\).

Then \(b \wedge n \leq (x \wedge a) \vee (x \wedge n) \vee (a \wedge n) \leq b \vee n\), and \(b \wedge n \leq (y \wedge a) \vee (y \wedge n) \vee (a \wedge n) \leq b \vee n\). So, \(b \vee n \leq [(x \wedge a) \vee (x \wedge n) \vee (a \wedge n)] \wedge n = (x \wedge n) \vee (a \wedge n)\). This implies \(x \wedge n \in < a \wedge n, b \wedge n >_d\). Similarly,

\(y \wedge n \in < a \wedge n, b \wedge n >_d\). Since \(< a \wedge n, b \wedge n >_d\) is a filter, so we have, \(x \wedge y \wedge n \in < a \wedge n, b \wedge n >_d\). Thus, \((x \wedge y \wedge n) \vee (a \wedge n) \geq (b \wedge n)\).

But \(m(x \wedge y \wedge n, n, a) = (x \wedge y \wedge n) \vee (a \wedge n) \geq (b \wedge n)\), and

so \(x \wedge y \wedge n \in < a, b >^n\). Again, by neutrality of \(n\), \((x \wedge n) \wedge (a \wedge n) = (x \wedge a) \wedge n \leq (b \vee n)\). Similarly, \((y \vee n) \wedge (a \wedge n) \leq (b \vee n)\).
Thus \(((x \land y) \lor n) \land (a \lor n) \leq (b \lor n)\).

But \(((x \land y) \lor n) \land (a \lor n) = m((x \land y) \lor n, n, a)\), as \(n\) is neutral.

Therefore, \((x \land y) \lor n \in < a, b >^n\) and so by the convexity of \(< a, b >^n\),
\(x \land y \in < a, b >^n\).

A dual proof of above shows that \(x \lor y \in < a, b >^n\). Clearly, \(< a, b >^n\) contains \(n\).

Therefore, \(< a, b >^n\) is an \(n\) –ideal. \(\Box\)

**Proposition 1.2** Let \(S\) be a nearlattice with a central element \(n\). Then for all \(a, b \in S\), the following conditions hold.

(i) \(< a \lor n, b \lor n >\) is an ideal if and only if \([n]\) is a distributive subnearlattice of \(S\).

(ii) \(< a \land n, b \land n >\) is a filter if and only if \([n]^d\) is a distributive subnearlattice of \(S\).

**Proof.** Suppose for all \(a, b \in S\), \(< a \lor n, b \lor n >\) is an ideal. Thus for all \(p, q \in [n]\), \(< p, q > \cap [n]\) is an ideal in the subnearlattice \([n]\). Then by \([1.1]\), \([n]\) is distributive.

Conversely, suppose \([n]\) is distributive. Let \(x, y \in < a \lor n, b \lor n >\) and \(x \lor y\) exists. Then \(x \land (a \lor n) \leq b \lor n\). Since \(n\) is neutral, so \((x \lor n) \land (a \lor n) = [x \land (a \lor n)] \lor n \leq b \lor n\) implies that \(x \lor n \in < a \lor n, b \lor n >\).

Similarly, \(y \lor n \in < a \lor n, b \lor n >\). Then \((x \lor y) \land (a \lor n) \leq (x \lor y \lor n) \land (a \lor n) = [(x \lor n) \land (a \lor n)] \lor [(y \lor n) \land (a \lor n)]\) as \([n]\) is distributive.
\(\leq (b \lor n)\).

Therefore, \(x \lor y \in < a \lor n, b \lor n >\). Since \(< a \lor n, b \lor n >\) has always the hereditary property, so \(< a \lor n, b \lor n >\) is an ideal.

(ii) can be proved dually. \(\Box\)

By Theorem 1.1 and above result and using \([8,\text{ theorem 1.5.2}]\), we have the following result.
Theorem 1.3 Let \( S \) be a nearlattice with a central element \( n \). Then for all \( a, b \in S \), 
\(< a, b >^n \) is an \( n \)-ideal if and only if \( P_n(S) \) is distributive nearlattice. \( \square \)

Recall that a nearlattice \( S \) is distributive if for all \( x, y, z \in S \), 
\( x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \) provided \( y \vee z \) exists.[3] has given an alternative definition of distributivity of \( S \). A nearlattice \( S \) is distributive if and only if for all \( t, x, y, z \in S \), 
\( t \wedge ((x \wedge y) \vee (x \wedge z)) = (t \wedge x \wedge y) \vee (t \wedge x \wedge z) \).

Similarly, by [4], a nearlattice \( S \) is modular if and only if for all \( t, x, y, z \in S \) with \( z \leq x \), 
\( x \wedge ((t \wedge y) \vee (t \wedge z)) = (x \wedge t \wedge y) \vee (t \wedge z) \).

Since for a sesquimedial element \( n \), \( S \) is distributive if and only if \( P_n(S) \) is distributive,
we have the following Corollary, which is a generalization of [1, Theorem 1] and a result of [6]. This also generalizes a result of [7, theorem 3.1.3].

Corollary 1.4 Suppose \( S \) is a nearlattice. Then for a central element \( n \in S \), \(< a, b >^n \) is an \( n \)-ideal for all \( a, b \in S \) if and only if \( S \) is distributive. \( \square \)


Theorem 1.5 Let \( n \) be a central element of a nearlattice \( S \). Then the following conditions are equivalent.

(i) \( S \) is distributive.

(ii) \(< a \vee n, b \vee n > \) is an ideal and \(< a \wedge n, b \wedge n >_a \) is a filter whenever \(< b >_n \subseteq < a >_n \).

Proof. (i)\( \Rightarrow \) (ii). Suppose (i) holds. That is, \( S \) is distributive. Then by Corollary 1.4, 
\(< a, b >^n \) is an \( n \)-ideal for all \( a, b \in S \). Thus by Theorem 1.1, (ii) holds.

(ii)\( \Rightarrow \) (i). Suppose (ii) holds and let \( x, y, z \in [n] \) and \( y \vee z \) exists.
Clearly, \((x \land y) \lor (x \land z) \leq x\). So, \(\langle x, (x \land y) \lor (x \land z) \rangle\) is an ideal as \\
\(\langle (x \land y) \lor (x \land z) \rangle \subseteq \langle x \rangle\). Since \(x \land y \leq (x \land y) \lor (x \land z)\), \\
so \(y \in \langle x, (x \land y) \lor (x \land z) \rangle\). Similarly, \(z \in \langle x, (x \land y) \lor (x \land z) \rangle\).

Hence \(y \lor z \in \langle x, (x \land y) \lor (x \land z) \rangle\) and so \([n]\) is distributive. Using the other part of (ii) we can similarly show that \([n]\) is also distributive. Thus by [8, theorem 1.5.2], \(P_n(S)\) is distributive and so \(S\) is distributive. \(\Box\)

**Theorem 1.6** Let \(n\) be a central element of a nearlattice \(S\). Then the following conditions are equivalent.

(i) \(P_n(S)\) is modular.

(ii) For \(a, b \in S\) with \(\langle b \rangle_n \subseteq \langle a \rangle_n\), \(x \in \langle b \rangle_n\) and \(y \in \langle a, b \rangle_n\)

imply \(x \land y, x \lor y \in \langle a, b \rangle_n\) if \(x \lor y\) exists in \(S\).

**Proof.** (i)⇒(ii). Suppose \(P_n(S)\) is modular. Then by [8, theorem 1.5.2], \([n]\) and \((n)\) are modular. Here \(\langle b \rangle_n \subseteq \langle a \rangle_n\), so \(a \land n \leq b \land n \leq n \leq b \lor n \leq a \lor n\). Since \(x \in \langle b \rangle_n\), so \(b \land n \leq x \leq b \lor n\).

Hence \(a \land n \leq b \land n \leq x \land n \leq x \lor n \leq b \lor n \leq a \lor n\).

Now, \(y \in \langle a, b \rangle_n\) implies \(m(y, n, a) \in \langle b \rangle_n\).

Thus, \((y \land a) \lor (y \land n) \lor (a \land n) \leq b \lor n\), and so by the neutrality of \(n\),

\(((y \land a) \lor (y \land n) \lor (a \land n)) \lor n = (y \land n) \land (a \lor n) \leq b \lor n\).

Thus, using the modularity of \([n]\) and the existence of \(x \lor y\),

\[m(x \lor y \land n, n, a) = (x \lor y \lor n) \land (a \lor n)\]

\[= [(a \lor n) \land (y \lor n)] \lor (x \lor n) \text{ as } x \lor n \leq b \lor n \leq a \lor n.\]

This implies \(m(x \lor y \lor n, n, a) \leq b \lor n\) and so \(x \lor y \lor n \in \langle a, b \rangle_n\). Since \(n\) is neutral, so \(a \land n \leq b \land n \leq x \land n\) implies that

\(b \land n \leq (x \land n) \lor (y \land n) \lor (a \land n)\)

\[= ((x \lor y) \land n) \lor (a \land n)\]

\[= m((x \lor y) \land n, n, a)\]

\[\leq b \lor n.\]
Therefore, \((x \lor y) \land n \in < a, b >^n\). Hence by convexity of \(< a, b >^n\),
\(x \lor y \in < a, b >^n\).
Again, using the modularity of \([n]\), a dual proof of above shows that
\(x \land y \in < a, b >^n\). Hence (ii) holds.

\((ii) \Rightarrow (i)\). Suppose (ii) holds. Let \(x, y, z \in [n]\) with \(x \leq z\) and whenever \(x \lor y\) exists. Then
\(x \lor (y \land z) \leq z\). This implies \(< x \lor (y \land z) >_n \subseteq < z >_n\).
Now, \(x \leq x \lor (y \land z)\) implies \(x \in < x \lor (y \land z) >_n\).
Again, \(y \land z \leq x \lor (y \land z)\) implies \(m(y, n, z) = y \land z \in < x \lor (y \land z) >_n\).
Hence \(y \in < z, x \lor (y \land z) >_n\). Thus by (ii), \(x \lor y \in < z, x \lor (y \land z) >_n\). That is, \((x \lor y) \land z \leq x \lor (y \land z)\) and so \((x \lor y) \land z = x \lor (y \land z)\). Therefore, \([n]\) is modular.
Similarly, using the condition (ii) we can easily show that \([n]\) is also modular. Hence by
[8, theorem 1.5.2], \(P_n(S)\) is modular. \(\Box\)

We conclude this paper with the following characterization of minimal prime \(n\)-ideals belonging to an \(n\)-ideal. Since the proof of this is almost similar to [8, theorem 2.1.4], we omit the proof.

**Theorem 1.7**  Let \(S\) be a distributive nearlattice and \(P\) be a prime \(n\)-ideal of \(S\) belonging to an \(n\)-ideal \(J\). Then the following conditions are equivalent.

(i) \(P\) is minimal prime \(n\)-ideal belonging to \(J\).

(ii) \(x \in P\) implies \(< < x >_n, J > \not\in P\). \(\Box\)
REFERENCES


