

ALTERNATE ASYMMETRIC LAPLACE DISTRIBUTION: PROPERTIES AND APPLICATIONS

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SUMMARY

In this article, we introduce a one parameter asymmetric version of the standard Laplace distribution, named “the alternate asymmetric Laplace distribution (AALD)”. The proposed model extends the classical laplace law through the inclusion of a single shape parameter ρ . While most of the existing asymmetric models rely on multiple shape parameters and lack closed-form expressions for higher-order characteristics, the AALD, with its single parameter structure, offers analytical tractability and flexibility. Specifically, closed-form expressions are derived for cumulative distribution function, moment generating function, quantiles, entropy and order statistics, underscoring the theoretical richness of the model. The location-scale extension of AALD, named “the extended alternate asymmetric Laplace distribution (EAALD)” is introduced and studied in detail. The estimation and testing procedures for the parameters of EAALD is discussed and the performance of the obtained estimators is examined through a simulation study. Through applications to two real-world data sets from distinct domains, EAALD consistently provides a superior fit compared to the established asymmetric models, highlighting both its methodological novelty and practical relevance.

Keywords and phrases: Asymmetric Laplace distribution, Laplace distribution, Maximum likelihood estimation, Model selection, Simulation, Skew Laplace distribution, Generalized likelihood ratio test

1 Introduction

The Laplace distribution was introduced by French mathematician Pierre-Simon Laplace in 1774. It is a two-tailed version of the exponential distribution and hence, it is known in literature as the double exponential distribution. It is quite popular due to the properties of sharp peak at the mode, heavier than Gaussian tails, existence of all moments and infinite divisibility properties. In several practical situations, it is considered as an important alternative to the Gaussian based stochastic models. The classical Laplace distribution being one of the oldest distributions in the literature, is

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very popular in modelling detector relative efficiencies, extreme wind speeds, measurement errors, position errors in navigation, stock returns etc. For details see Hsu (1979), Hoaglin et al. (1983), Dadi and Marks (1987), Damsleth and El-Shaarawi (1989), Puig and Stephens (2000), Chen (2002), Andrews and Hampel (2015) etc.

Many researchers have made attempts to develop several extensions and variations of Laplace distribution in different ways. It can be extended in different forms like skewed, folded, generalized or multivariate versions that are very substantial in statistical theory and analysis. Some important works include Mathai (1993), Kozubowski and Podgórski (2000), Yu and Zhang (2005), Srivastava et al. (2006), Arslan (2010), Kozubowski and Nadarajah (2010), Cordeiro and Lemonte (2011), Kotz et al. (2012), Kozubowski et al. (2013), Liu and Kozubowski (2015) etc.

Through this article we propose a one parameter asymmetric version of the well-known Laplace distribution and named it as “the alternate asymmetric Laplace distribution (AALD).”

The novelty of the proposed model lies in its parsimony and analytical tractability. While several generalizations of the Laplace law, such as asymmetric, skew, and folded forms, have been introduced in the literature, most involve multiple shape parameters or lack closed-form expressions for higher-order characteristics. In contrast, the present alternate asymmetric Laplace distribution (AALD) introduces a single shape parameter ρ that simultaneously governs skewness, tail thickness, and departure from symmetry. The model seamlessly bridges the symmetric Laplace distribution ($\rho = 1$) and asymmetric heavy-tailed forms ($\rho > 0$), thereby offering a unified and interpretable structure. Despite its simplicity, the AALD retains closed-form expressions for the cdf, moments, moment generating function, quantile function, entropy, and order statistics-features rarely available in other asymmetric families. This balance between flexibility and analytical convenience makes the proposed model both theoretically appealing and practically useful in applications involving skewed or heavy-tailed data. The proposed model is particularly relevant for applications where skewness and heavy tails are prominent, such as environmental-engineering, reliability analysis, financial risk modelling, survival analysis and environmental data studies. In this paper, we have considered two real-world data sets from distinct domains, specifically from environmental-engineering and cancer research. The first data is from environmental-engineering and represents vinyl chloride data from clean upgradient ground-water monitoring wells with 34 observations. The second data denotes the survival times of a group of patients who received chemotherapy treatment alone and has 46 observations. For both the datasets, we have fitted certain well-known asymmetric models available in literature like asymmetric versions of Laplace distribution, alpha skew Laplace distribution, skew normal distribution, skew-t distribution, skew logistic distribution, modified skew logistic distribution and observed that the proposed model provides a superior fit compared to already existing models, thereby making it a practical alternative to the existing asymmetric models.

The remainder of this paper is organized as follows: In Section 2.1, we formally define the AALD and derive its cumulative distribution function, moments, generating functions, quantiles, entropy, reliability measures and order statistics. In Section 2.2, we propose the location-scale extension of AALD, named “the extended alternate asymmetric Laplace distribution (EAALD)” and studied its important statistical properties. The estimation of parameters of the EAALD is discussed in Section 2.3.1 followed by a testing procedure in Section 2.3.2. A simulation study, reported

in Section 3 evaluates the performance of the maximum likelihood estimators of the parameters. Section 4 illustrates the estimation and testing techniques using real-world data sets to demonstrate the practical applicability of the model. Finally, Section 5 concludes the article with a summary of findings and potential directions for future research.

2 Alternate asymmetric Laplace distribution

2.1 Definition and Properties

In this section, a new asymmetric version of Laplace distribution is introduced and present some of its important statistical properties.

Definition 2.1. A continuous random variable Y is said to follow alternate asymmetric Laplace distribution with parameter ρ (denoted by AALD(ρ)), if its probability density function (pdf) is of the following form for $0 < \rho < \infty$:

$$f(y; \rho) = \begin{cases} \frac{e^{\frac{y}{\rho}}}{(1+\rho)}, & y < 0 \\ \frac{e^{-y}}{(1+\rho)}, & y \geq 0. \end{cases} \quad (2.1)$$

Clearly, when $\rho = 1$, the pdf (2.1) reduces to standard Laplace distribution. The probability plots of the AALD(ρ) for particular values of parameter ρ are presented in Figure 1.

To validate the fact that the proposed AALD has a valid density function, it is enough to show that $\int_{-\infty}^{\infty} f(y) dy = 1$, i.e.,

$$\int_{-\infty}^0 f(y) dy + \int_0^{\infty} f(y) dy = \int_{-\infty}^0 \frac{e^{\frac{y}{\rho}}}{(1+\rho)} dy + \int_0^{\infty} \frac{e^{-y}}{(1+\rho)} dy = \frac{1}{(1+\rho)} \left(\frac{1}{\rho} + 1 \right) = 1.$$

Proposition 2.1. The cumulative distribution function (cdf) of the AALD(ρ) is given by

$$F(y; \rho) = \begin{cases} \frac{\rho e^{\frac{y}{\rho}}}{(1+\rho)}; & y < 0 \\ 1 - \frac{e^{-y}}{(1+\rho)}; & y \geq 0. \end{cases} \quad (2.2)$$

Proposition 2.2. The r^{th} raw moment μ'_r of the AALD(ρ) is given by

$$\mu'_r = \frac{r!}{1+\rho} \begin{cases} 1 - \rho^{r+1}; & r \text{ is odd} \\ 1 + \rho^{r+1}; & r \text{ is even.} \end{cases}$$

Corollary 2.1. The mean and variance of the AALD(ρ) are given by

$$\mu'_1 = 1 - \rho, \text{ and } \mu'_2 = 1 + \rho^2.$$

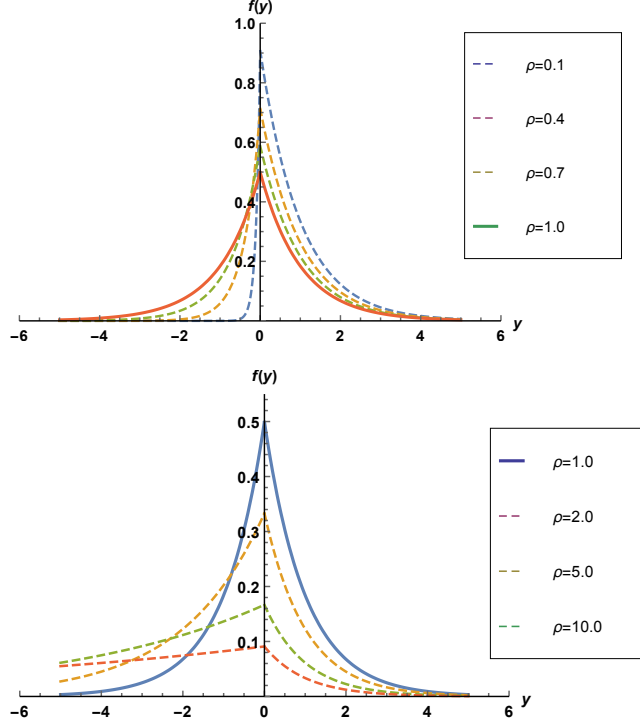


Figure 1: Probability density plot of the AALD for various choices of ρ .

Proposition 2.3. *The measure of skewness and kurtosis of the AALD(ρ) are given by*

$$\gamma_1 = \frac{\mu_3}{\mu_2^{3/2}} = \frac{2(1 - \rho^3)}{(1 + \rho^2)^{3/2}}, \text{ and } \gamma_2 = \frac{\mu_4}{\mu_2^2} - 3 = \frac{6(1 + \rho^4)}{(1 + \rho^2)^2}.$$

Using the Proposition 2.2, we have computed the values of the 2nd, 3rd and 4th central moments (μ_2, μ_3 and μ_4) to obtain the expressions for the measure of skewness γ_1 and measure of kurtosis γ_2 of AALD. We have computed the values of mean, variance, γ_1 and γ_2 of AALD for some particular choices of ρ and presented in the Table A1 in Appendix. Figure 4 in Appendix presents the skewness and kurtosis plots of AALD(ρ). From Table A1 and Figure 4, it is observed that $\gamma_1 > 0$ for $0 < \rho < 1$, $\gamma_1 < 0$ for $1 < \rho < \infty$ and $\gamma_1 = 0$ for $\rho = 1$, indicating that AALD is positively skewed (or right-skewed) for $0 < \rho < 1$, negatively skewed (or left-skewed) for $1 < \rho < \infty$ and symmetric for $\rho = 1$. Further, the coefficient of kurtosis of AALD, γ_2 is always greater than 3, indicating that AALD is leptokurtic in nature for all values of the parameter. Also, when $\rho=1$, the AALD reduces to standard Laplace distribution with $\gamma_1 = 0$ and $\gamma_2 = 3$.

Proposition 2.4. *For any $t \in (-1/\rho, 1)$, the moment generating function (mgf) of the AALD(ρ) is*

the following:

$$M_Y(t) = \frac{1}{1 - [\rho t^2 + (1 - \rho)t]}.$$

Corollary 2.2. For any $t \in R$ and $i = \sqrt{-1}$, the characteristic function of the AALD(ρ) is given by

$$\Phi_Y(t) = \frac{1}{1 + \rho t^2 - (1 - \rho)it}.$$

Proposition 2.5. The n^{th} cumulant K_n of the AALD(ρ) is given by

$$K_n = \begin{cases} (n-1)!(1 - \rho^n); & n \text{ is odd} \\ (n-1)!(1 + \rho^n); & n \text{ is even.} \end{cases}$$

In particular, first four cumulants of the AALD are as follows

$$\begin{aligned} K_1 &= 1 - \rho, \\ K_2 &= 1 + \rho^2, \\ K_3 &= 2(1 - \rho^3), \\ K_4 &= 6(1 + \rho^4). \end{aligned}$$

Now we derive the expression for quantiles of alternate asymmetric Laplace distribution using the cdf obtained in (2.2) which is available in closed form.

Proposition 2.6. The quantile function of the AALD(ρ) is given by

$$Q(t) = Y_t = \begin{cases} \rho \log \left[\frac{(1+\rho)t}{\rho} \right]; & 0 < t \leq \frac{\rho}{1+\rho} \\ -\log [(1 + \rho)(1 - t)]; & \frac{\rho}{1+\rho} \leq t < 1. \end{cases}$$

In particular, the first quartile of the AALD(ρ) is

$$Q_1 = Y_{0.25} = \begin{cases} \log \left[\frac{4}{3(1+\rho)} \right]; & 0 < \rho \leq \frac{1}{3} \\ \rho \log \left[\frac{(1+\rho)}{4\rho} \right]; & \rho \geq \frac{1}{3}. \end{cases}$$

The second quartile (median) of the AALD(ρ) is

$$Q_2 = Y_{0.50} = \begin{cases} \log \left[\frac{2}{1+\rho} \right]; & 0 < \rho \leq 1 \\ \rho \log \left[\frac{(1+\rho)}{2\rho} \right]; & \rho \geq 1, \end{cases}$$

and the third quartile of AALD(ρ) is

$$Q_3 = Y_{0.75} = \begin{cases} \log \left[\frac{4}{(1+\rho)} \right]; & 0 < \rho \leq 3 \\ \rho \log \left[\frac{3(1+\rho)}{4\rho} \right]; & \rho \geq 3. \end{cases}$$

Next we present the expression of a well-known entropy measure used in information theory as a measure of variation of uncertainty. We derive the expression for Rényi entropy of the alternate asymmetric Laplace distribution through the following proposition.

Proposition 2.7. *The Rényi entropy of the AALD(ρ) is given by*

$$I_R(\gamma) = \frac{1}{\gamma - 1} \log \left[\frac{1}{\gamma(1 + \rho)^{\gamma-1}} \right],$$

where $\gamma > 0$ and $\gamma \neq 1$.

Certain reliability measures such as survival function, hazard rate function, mean residual life function are presented through the following propositions.

Proposition 2.8. *The survival function of AALD(ρ) is given by*

$$S(y) = \begin{cases} \frac{1+\rho-\rho e^{\frac{y}{\rho}}}{(1+\rho)}; & y < 0 \\ \frac{e^{-y}}{1+\rho}; & y > 0. \end{cases}$$

The proof immediately follows from the definition of survival function.

Proposition 2.9. *The failure rate function or hazard rate function of the AALD(ρ) is given by*

$$h(y) = \begin{cases} \frac{e^{\frac{y}{\rho}}}{1+\rho-\rho e^{\frac{y}{\rho}}}; & y < 0 \\ 1; & y > 0. \end{cases}$$

The proof directly follows from the definition of hazard rate function and is, hence, omitted.

Proposition 2.10. *The mean residual life function (MRLF) of AALD(ρ) is given by*

$$m(y) = \begin{cases} \frac{\rho^2 e^{\frac{y}{\rho}} - \rho^2 - \rho y - y}{1+\rho-\rho e^{\frac{y}{\rho}}}; & y < 0 \\ 1; & y > 0. \end{cases}$$

In continuous distribution theory, the tail probabilities are of considerable significance both in theoretical and applied contexts. They describe the likelihood that a continuous random variable takes the values in the extreme regions of its domain- far from the central tendency of the distribution. These probabilities are widely used in reliability, survival analysis, insurance, risk management, climatology, hydrology etc. The expression for tail probabilities of the AALD is as follows.

Proposition 2.11. *The tail probability of the AALD(ρ) for left tail and right tail are*

$$P(Y < y) = F(y) = \frac{\rho e^{\frac{y}{\rho}}}{(1 + \rho)}; y \leq 0, \text{ and } P(Y > y) = 1 - F(y) = \frac{e^{-y}}{(1 + \rho)};$$

respectively, with the corresponding tail decay rate $(1/\rho)$ and 1.

The tail probability determines how quickly the tail of a distribution vanishes as $y \rightarrow \infty$ or $y \rightarrow -\infty$. The tail probabilities often becomes very small as the value of y increases. So, on a linear scale, small probabilities near zero becomes visually compressed making it hard to see how the tail behaves. Inorder to have better visibility, we use semilog plot (or semilogarithmic plot) to visualize the tail probabilities of AALD and presented in Figures 5-7 in Appendix. Here, x -axis is linear and y -axis is logarithmic which effectively plots the tail probabilities versus y -values. The slop of the plot denotes the rate of tail decay. For AALD, both left and right tails exhibit exponential decay, but with different tail decay rates. For left tail, the tail decay rate is $\frac{1}{\rho}$ and it depends on the value of the parameter ρ . Also, it exhibits heavier tails for $0 < \rho < 1$. For right tail, the tail decay rate is 1 for all values of the parameter ρ . For $\rho = 1$, both tails decay at same exponential rate 1. Therefore, AALD has heavier tails than normal distribution, whose tails decays superexponentially.

Next, we derive the distribution of order statistics and moments of k^{th} order statistics of alternate asymmetric Laplace distribution.

Suppose $Y_1, Y_2, \dots, Y_n$ be a random sample of size n from AALD(ρ) with pdf and cdf given by (2.1) and (2.2) respectively. Let $Y_{1:n} < Y_{2:n} < \dots < Y_{n:n}$ denote corresponding order statistics. Then for $k > 1$, the pdf and q^{th} order moment of k^{th} order statistic $Y_{k:n}$ are presented through the following propositions.

Proposition 2.12. *The pdf of k^{th} order statistic of the AALD(ρ) is given by*

$$f_{Y_{k:n}}(y) = \frac{1}{B(k, n-k+1)} \begin{cases} \frac{\rho^{k-1}}{(1+\rho)^k} e^{\frac{ky}{\rho}} \left[1 - \frac{\rho e^{\frac{y}{\rho}}}{1+\rho}\right]^{n-k}; & y < 0 \\ \frac{1}{(1+\rho)^{n-k+1}} e^{-(n-k+1)y} \left[1 - \frac{e^{-y}}{1+\rho}\right]^{k-1}; & y \geq 0, \end{cases}$$

where $B(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$.

Corollary 2.3. The pdf and cdf of the largest order statistic $Y_{n:n} = \max(Y_1, Y_2, \dots, Y_n)$ are

$$f_{Y_{n:n}}(y) = \frac{n}{(1+\rho)^n} \begin{cases} \rho^{n-1} e^{\frac{ny}{\rho}}; & y < 0 \\ e^{-y}(1+\rho - e^{-y})^{n-1}; & y \geq 0, \end{cases}$$

and

$$F_{Y_{n:n}}(y) = \frac{1}{(1+\rho)^n} \begin{cases} \rho^n e^{\frac{ny}{\rho}}; & y < 0 \\ (1+\rho - e^{-y})^n; & y \geq 0. \end{cases}$$

Corollary 2.4. The pdf and cdf of the smallest order statistic $Y_{1:n} = \min(Y_1, Y_2, \dots, Y_n)$ are given by

$$f_{Y_{1:n}}(y) = \frac{n}{(1+\rho)^n} \begin{cases} e^{\frac{y}{\rho}} \left[1 + \rho - \rho e^{\frac{y}{\rho}} \right]^{n-1}; & y < 0 \\ e^{-ny}; & y \geq 0, \end{cases}$$

and

$$F_{Y_{1:n}}(y) = \begin{cases} 1 - \frac{\left[1 + \rho - \rho e^{\frac{y}{\rho}} \right]^n}{(1+\rho)^n}; & y < 0 \\ 1 - \frac{e^{-ny}}{(1+\rho)^n}; & y \geq 0. \end{cases}$$

Proposition 2.13. The q^{th} raw moment of k^{th} order statistic $Y_{k:n}$ of the AALD(ρ) is given by

$$\begin{aligned} E(Y_{k:n}^q) &= \frac{\Gamma(q+1)}{B(k, n-k+1)} \frac{\rho^{k-1}}{(1+\rho)^k} \sum_{j=0}^{n-k} \binom{n-k}{j} [\rho/(1+\rho)]^j (-1)^{j+q} [\rho/(j+k)]^{(q+1)} \\ &+ \frac{\Gamma(q+1)}{B(k, n-k+1)(1+\rho)^{n-k+1}} \sum_{i=0}^{k-1} \binom{k-1}{i} \frac{[1/(1+\rho)]^i (-1)^i}{(n-k+i+1)^{q+1}}. \end{aligned}$$

2.2 Location-scale extension

In this section, we present the definition and some important statistical properties of location-scale extension of the alternate asymmetric Laplace distribution.

Definition 2.2. Let the random variable Y follows AALD(ρ) with pdf (2.1), then $Z = \mu + \sigma Y$ is said to have an extended asymmetric Laplace distribution (denoted by EAALD(ρ, μ, σ)) with parameters ρ , μ and σ , with pdf

$$f(z; \rho, \mu, \sigma) = \begin{cases} \frac{e^{\frac{(z-\mu)}{\rho\sigma}}}{\sigma(1+\rho)}; & z < \mu \\ \frac{e^{-\frac{(z-\mu)}{\sigma}}}{\sigma(1+\rho)}; & z \geq \mu, \end{cases} \quad (2.3)$$

for $\rho > 0$, $-\infty < \mu < \infty$ and $\sigma > 0$.

It is to be noted that,

- i) When $\mu = 0$ and $\sigma = 1$, the EAALD reduces to AALD.
- ii) When $\rho = 1$, the EAALD reduces to classical Laplace distribution with location parameter μ and scale parameter σ .
- iii) When $\rho = 1$, $\mu = 0$ and $\sigma = 1$, the EAALD reduces to standard Laplace distribution.

Proposition 2.14. The cdf of EAALD(ρ, μ, σ) with pdf (2.3) is given by

$$F(z; \rho, \mu, \sigma) = \begin{cases} \frac{\rho}{(1+\rho)} e^{\frac{(z-\mu)}{\rho\sigma}}; & z < \mu \\ 1 - \frac{e^{-\frac{(z-\mu)}{\sigma}}}{(1+\rho)}; & z \geq \mu. \end{cases} \quad (2.4)$$

Proposition 2.15. The r^{th} raw moment μ'_r of the EAALD(ρ, μ, σ) is given by

$$\mu'_r = \frac{1}{\sigma(1+\rho)} \left[e^{-\frac{\mu}{\rho\sigma}} (-1)^r (\rho\sigma)^{r+1} \Gamma(r+1, -\frac{\mu}{\rho\sigma}) + e^{\frac{\mu}{\sigma}} \sigma^{r+1} \Gamma(r+1, \frac{\mu}{\sigma}) \right].$$

Corollary 2.5. The mean and variance of the EAALD(ρ, μ, σ) are given by

$$\mu'_1 = \mu + \sigma(1 - \rho), \text{ and } \mu_2 = \sigma^2(1 + \rho^2).$$

Proposition 2.16. The measure of skewness and kurtosis of the EAALD(ρ, μ, σ) are given by

$$\gamma_1 = \frac{\mu_3}{\mu_2^{3/2}} = \frac{2\sigma^2(1 - \rho^3)}{(1 + \rho^2)^{3/2}}, \text{ and } \gamma_2 = \frac{\mu_4}{\mu_2^2} - 3 = \frac{6(1 + \rho^4)}{(1 + \rho^2)^2}.$$

Proposition 2.17. For any $t \in (-1/\rho, 1)$, the moment generating function (mgf) of the EAALD(ρ, μ, σ) is the following:

$$M_Z(t) = \frac{e^{t\mu}}{1 - [\rho t^2 \sigma^2 + (1 - \rho)t\sigma]}.$$

Corollary 2.6. For any $t \in R$ and $i = \sqrt{-1}$, the characteristic function of the EAALD(ρ, μ, σ) is given by

$$\Phi_Z(t) = \frac{e^{it\mu}}{1 + \rho t^2 \sigma^2 - (1 - \rho)it\sigma}.$$

Proposition 2.18. The n^{th} cumulant K_n of the EAALD(ρ, μ, σ) is given by

$$K_n = \begin{cases} \mu + (n-1)!\sigma(1 - \rho^n); & n = 1 \\ (n-1)!\sigma^n(1 - \rho^n); & n \text{ is odd and } n \neq 1 \\ (n-1)!\sigma^n(1 + \rho^n); & n \text{ is even.} \end{cases}$$

In particular, cumulants of the EAALD are as follows

$$\begin{aligned} K_1 &= \mu + \sigma(1 - \rho), \\ K_2 &= \sigma^2(1 + \rho^2), \\ K_3 &= 2\sigma^3(1 - \rho^3), \\ K_4 &= 6\sigma^4(1 + \rho^4), \\ K_8 &= 5040\sigma^8(1 + \rho^8). \end{aligned}$$

Proposition 2.19. *The quantile function of the EAALD(ρ, μ, σ) is given by*

$$Q^*(t) = Z_t = \begin{cases} \mu + \rho\sigma \log \left[\frac{(1+\rho)t}{\rho} \right] & ; 0 < t \leq \frac{\rho}{1+\rho} \\ \mu - \sigma \log [(1+\rho)(1-t)] & ; \frac{\rho}{1+\rho} \leq t < 1. \end{cases}$$

In particular, the first quartile of the EAALD(ρ, μ, σ) is

$$Q_1^* = Z_{0.25} = \begin{cases} \mu - \sigma \log \left[\frac{3(1+\rho)}{4} \right] & ; 0 < \rho \leq \frac{1}{3} \\ \mu + \rho\sigma \log \left[\frac{(1+\rho)}{4\rho} \right] & ; \rho \geq \frac{1}{3}. \end{cases}$$

The second quartile (median) of the EAALD(ρ, μ, σ) is

$$Q_2^* = Z_{0.50} = \begin{cases} \mu - \sigma \log \left[\frac{(1+\rho)}{2} \right] & ; 0 < \rho \leq 1 \\ \mu + \rho\sigma \log \left[\frac{(1+\rho)}{2\rho} \right] & ; \rho \geq 1, \end{cases}$$

and the third quartile of EAALD(ρ, μ, σ) is

$$Q_3^* = Z_{0.75} = \begin{cases} \mu - \sigma \log \left[\frac{(1+\rho)}{4} \right] & ; 0 < \rho \leq 3 \\ \mu + \rho\sigma \log \left[\frac{3(1+\rho)}{4\rho} \right] & ; \rho \geq 3. \end{cases}$$

Proposition 2.20. *The Rényi entropy of the EAALD(ρ, μ, σ) is given by*

$$I_R(\gamma) = \frac{1}{\gamma - 1} \log \left[\frac{1}{\gamma(1+\rho)^{\gamma-1}} \right],$$

where $\gamma > 0$ and $\gamma \neq 1$.

Proposition 2.21. *The survival function of EAALD(ρ, μ, σ) is given by*

$$S(z) = \begin{cases} \frac{1+\rho-\rho e^{-\frac{(z-\mu)}{\rho\sigma}}}{(1+\rho)} & ; z < \mu \\ \frac{1}{1+\rho} e^{-\frac{(z-\mu)}{\sigma}} & ; z > \mu. \end{cases}$$

The proof immediately follows from the definition of survival function.

Proposition 2.22. *The failure rate function or hazard rate function of the EAALD(ρ, μ, σ) is given by*

$$h(z) = \begin{cases} \frac{e^{-\frac{(z-\mu)}{\rho\sigma}}}{\sigma \left(1+\rho-\rho e^{-\frac{(z-\mu)}{\rho\sigma}} \right)} & ; z < \mu \\ \frac{1}{\sigma} & ; z > \mu. \end{cases}$$

Proposition 2.23. The mean residual life function (MRLF) of EAALD(ρ, μ, σ) is given by

$$m(z) = \begin{cases} \frac{(1+\rho)\mu - (1+\rho)z - \rho^2\sigma(1 - e^{\frac{(z-\mu)}{\rho\sigma}})}{1 + \rho - \rho e^{\frac{(z-\mu)}{\rho\sigma}}}; & z < \mu \\ \sigma & ; z > \mu. \end{cases}$$

Proposition 2.24. The tail probability of the EAALD(ρ, μ, σ) for left tail and right tail are

$$P(Z < z) = F(z) = \frac{\rho e^{\frac{(z-\mu)}{\rho\sigma}}}{(1+\rho)}; z \leq \mu \text{ and } P(Z > z) = 1 - F(z) = \frac{e^{\frac{(z-\mu)}{\sigma}}}{(1+\rho)}; z \geq \mu,$$

with exponential tail decay rates $1/(\rho\sigma)$ and $1/\sigma$ respectively.

Let $Z_1, Z_2, \dots, Z_n$ be a random sample of size n from EAALD(ρ, μ, σ) with pdf and cdf given by (2.3) and (2.4) respectively. Let $Z_{1:n} < Z_{2:n} < \dots < Z_{n:n}$ denote corresponding order statistics.

Proposition 2.25. The pdf of k^{th} order statistic of the EAALD(ρ, μ, σ) is given by

$$f_{Z_{k:n}}(z) = \frac{1}{B(k, n-k+1)} \begin{cases} \frac{\rho^{k-1}}{\sigma(1+\rho)^k} e^{\frac{k(z-\mu)}{\rho\sigma}} \left[1 - \frac{\rho}{1+\rho} e^{\frac{(z-\mu)}{\rho\sigma}}\right]^{n-k} & ; z < \mu \\ \frac{1}{\sigma(1+\rho)^{n-k+1}} e^{-\frac{(n-k+1)(z-\mu)}{\sigma}} \left[1 - \frac{1}{1+\rho} e^{-\frac{(z-\mu)}{\sigma}}\right]^{k-1} & ; z \geq \mu, \end{cases}$$

$$\text{where } B(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}.$$

Proposition 2.26. The q^{th} raw moment of k^{th} order statistic $Z_{k:n}$ of the EAALD(ρ, μ, σ) is given by

$$\begin{aligned} E(Z_{k:n}^q) &= \frac{1}{B(k, n-k+1)} \frac{\sigma^q \rho^{k+q}}{(1+\rho)^k} \sum_{j=0}^{n-k} \binom{n-k}{j} \left[\rho/(1+\rho)\right]^j (-1)^{j+q} \frac{e^{-\frac{(j+k)\mu}{\rho\sigma}}}{(j+k)^{q+1}} \Gamma(q+1, -\frac{(j+k)\mu}{\rho\sigma}) \\ &+ \frac{1}{B(k, n-k+1)} \frac{\sigma^q}{(1+\rho)^{n-k+1}} \\ &\times \sum_{i=0}^{k-1} \binom{k-1}{i} \left(\frac{-1}{1+\rho}\right)^i \frac{e^{\frac{(n-k+i+1)\mu}{\sigma}}}{(n-k+i+1)^{q+1}} \Gamma(q+1, \frac{(n-k+i+1)\mu}{\sigma}). \end{aligned}$$

2.3 Estimation and Testing

In this section, we present the estimation and testing procedures for the parameters of the extended alternate asymmetric Laplace distribution.

2.3.1 Maximum Likelihood Estimation

Here, we consider the estimation of the parameters ρ, μ and σ of the EAALD using the maximum likelihood estimation (MLE) procedure.

Let $Z_1, Z_2, Z_3, \dots, Z_n$ be a random sample of size n from EAALD with pdf (2.3). Let $Z_{(1)} < Z_{(2)} < \dots < Z_{(n)}$ denote the corresponding ordered sample. Then, the log-likelihood function $l(\rho, \mu, \sigma)$ of the sample is given by

$$l(\rho, \mu, \sigma) = -n \log \sigma - n \log(1 + \rho) + \frac{1}{\rho \sigma} \sum_{I_1} (z_i - \mu) - \frac{1}{\sigma} \sum_{I_2} (z_i - \mu), \quad (2.5)$$

where $\sum_{I_i}$ denotes the summation over the set I_i such that $I_1 = \{i : Z_i < \mu, \text{ for } i = 1, 2, \dots, m\}$ and $I_2 = \{i : Z_i \geq \mu, \text{ for } i = m+1, \dots, n\}$. Here, the log-likelihood function is a continuous function in each parameter, but is not differentiable at $Z_i = \mu; i = 1, 2, \dots, n$. Thus, we assume that $Z_i \neq \mu; i = 1, 2, \dots, n$ to obtain the likelihood equations.

Assume that $\hat{\rho}, \hat{\mu}$ and $\hat{\sigma}$ be the maximum likelihood estimator (MLE) of the parameters ρ, μ and σ of the EAALD. On differentiating the log-likelihood function given in equation (2.5) with respect to the parameters and equating to zero, we obtain the following likelihood equations.

$$\frac{\partial l(\rho, \mu, \sigma)}{\partial \rho} = 0 \Rightarrow -\frac{n}{(1 + \rho)} - \sum_{I_1} \frac{(z_i - \mu)}{\rho^2 \sigma} = 0, \quad (2.6)$$

$$\frac{\partial l(\rho, \mu, \sigma)}{\partial \sigma} = 0 \Rightarrow -\frac{n}{\sigma} - \sum_{I_1} \frac{(z_i - \mu)}{\rho \sigma^2} + \sum_{I_2} \frac{(z_i - \mu)}{\sigma^2} = 0. \quad (2.7)$$

On solving the equation (2.6), we obtain the maximum likelihood estimator of parameter ρ of EAALD as

$$\begin{aligned} n\rho^2\sigma + \sum_{I_1} (z_i - \mu)\rho + \sum_{I_1} (z_i - \mu) &= 0 \\ \Rightarrow \hat{\rho} &= \frac{1}{2n\hat{\sigma}} \left[-\sum_{I_1} (z_i - \hat{\mu}) \pm \left(\left\{ \sum_{I_1} (z_i - \hat{\mu}) \right\}^2 - 4n\sigma \sum_{I_1} (z_i - \hat{\mu}) \right)^{\frac{1}{2}} \right]. \end{aligned}$$

Since the range of variation of the parameter ρ is $0 < \rho < \infty$, $\left(\left\{ \sum_{I_1} (z_i - \hat{\mu}) \right\}^2 - 4n\sigma \sum_{I_1} (z_i - \hat{\mu}) \right)^{\frac{1}{2}}$ should be greater than $-\sum_{I_1} (z_i - \hat{\mu})$. Therefore, the MLE of the parameter ρ of EAALD is

$$\hat{\rho} = \frac{1}{2n\hat{\sigma}} \left[-\sum_{I_1} (z_i - \hat{\mu}) + \left(\left\{ \sum_{I_1} (z_i - \hat{\mu}) \right\}^2 - 4n\sigma \sum_{I_1} (z_i - \hat{\mu}) \right)^{\frac{1}{2}} \right].$$

On solving the equation (2.7), we obtain the maximum likelihood estimator of parameter σ of EAALD as

$$n\rho\sigma + \sum_{I_1} (z_i - \mu) - \rho \sum_{I_2} (z_i - \mu) = 0 \Rightarrow \hat{\sigma} = \frac{1}{n} \sum_{I_2} (z_i - \hat{\mu}) - \frac{1}{n\hat{\rho}} \sum_{I_1} (z_i - \hat{\mu}).$$

It is to be noted that, the log-likelihood function in (2.5) is not differentiable at $z_i = \mu; i = 1, 2, \dots, n$ so, we obtain the MLE of μ as follows. Let us consider the log-likelihood function

in (2.5) in which only the last two terms $(1/\rho\sigma)\sum_{I_1}(z_i - \mu)$ and $(1/\sigma)\sum_{I_2}(z_i - \mu)$ depends on parameter μ . Let us arrange the data such that, $z_{(1)} < z_{(2)} < \dots < z_{(m)} < \dots < z_{(n)}$ be the order statistics. Consider two consecutive order statistics such that $z_{(m)} < \mu < z_{(m+1)}$. On each interval $(z_{(m)}, z_{(m+1)})$, the log-likelihood function is a linear function in μ . Therefore, the derivative of log-likelihood function with respect to parameter μ is

$$\frac{\partial l(\rho, \mu, \sigma)}{\partial \mu} = \frac{1}{\rho\sigma}n_- - \frac{1}{\sigma}n_+,$$

where n_- and n_+ be the number of observations below and above μ such that $n_- + n_+ = n$. Here, m observations are below μ and $(n - m)$ observations are above parameter μ .

$$\frac{\partial l(\rho, \mu, \sigma)}{\partial \mu} = \frac{1}{\rho\sigma}m - \frac{1}{\sigma}(n - m) = \frac{1}{\sigma} \left[\frac{m}{\rho} - (n - m) \right]. \quad (2.8)$$

This derivative is a constant within the interval and depends only on the number of data points on each side of μ . To determine the MLE of parameter μ , we equate derivative in (2.8) to zero and yields

$$m = \frac{\rho}{1 + \rho}n.$$

Further, the fraction of observations below $\hat{\mu}$ is $\rho/(1 + \rho)$. Therefore, the MLE of parameter μ of EAALD is obtained as

$$\hat{\mu} = \text{empirical quantile of order } \frac{\rho}{1 + \rho}.$$

The uniqueness of the obtained maximum likelihood estimators is established by showing that Hessian matrix to be a negative definite in all interior points of the parameter space. The Hessian matrix denoted by $H(\hat{\Theta})$ (where $\Theta = (\rho, \mu, \sigma)$), is the second order partial derivative of the log-likelihood function with respect to parameters ρ and σ , computed at $\rho = \hat{\rho}$, $\mu = \hat{\mu}$ and $\sigma = \hat{\sigma}$. For EAALD, the Hessian matrix is given by

$$H(\hat{\Theta}) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

where

$$\begin{aligned} a_{11} &= \frac{\partial^2 l(\rho, \mu, \sigma)}{\partial \rho^2} = \frac{n}{(1 + \rho)^2} + 2 \sum_{I_1} \frac{(z_i - \mu)}{\sigma \rho^3}, \\ a_{12} &= a_{21} = \frac{\partial^2 l(\rho, \mu, \sigma)}{\partial \rho \partial \sigma} = \sum_{I_1} \frac{(z_i - \mu)}{\sigma^2 \rho^2}, \\ a_{22} &= \frac{\partial^2 l(\rho, \mu, \sigma)}{\partial \sigma^2} = \frac{n}{\sigma^2} + 2 \sum_{I_1} \frac{(z_i - \mu)}{\rho \sigma^3} - 2 \sum_{I_2} \frac{(z_i - \mu)}{\sigma^3}, \end{aligned}$$

and it is computed at $\rho = \hat{\rho}$, $\mu = \hat{\mu}$ and $\sigma = \hat{\sigma}$.

Next, we test the significance of the additional parameter ρ of the proposed model EAALD using generalized likelihood ratio test procedure.

2.3.2 Generalized likelihood ratio test

In this section, we use the generalized ratio test (GLRT) for testing the significance of the parameter ρ of the EAALD. Here, we test the null hypothesis $H_0: \rho = 1, \mu = 0, \sigma = 1$ against the alternative $H_1: \rho \neq 1, \mu \neq 0, \sigma \neq 1$. The test statistic is given by,

$$LRT = -2\ln \left[\frac{Sup_{\theta \in \Theta} L(\hat{\Theta}^*|z)}{Sup_{\theta \in \Theta} L(\hat{\Theta}|z)} \right], \quad (2.9)$$

where $\hat{\Theta}$ is the maximum likelihood estimates of $\Theta=(\rho,\mu,\sigma)$ with no restriction and $\hat{\Theta}^*$ is the maximum likelihood estimates of $\Theta=(\rho,\mu,\sigma)$ when $\rho = 1, \mu = 0$ and $\sigma = 1$. The test statistic given in (2.9) asymptotically follows chi-square distribution with one degree of freedom.

3 Simulation

In this section, we analyse the performance of the maximum likelihood estimators of EAALD by conducting a simulation study. Using the software MATHEMATICA, a finite sample of size (n) is generated using inversion method and the parameters are estimated using maximum likelihood estimation procedure. The initial values of the parameter used to generate data represent different skewness scenarios, with (i) $\rho = 0.100, \mu = -5.000, \sigma = 2.000$ and (ii) $\rho = 0.030, \mu = -2.000, \sigma = 11.000$ representing positive skewness and (iii) $\rho = 4.000, \mu = -12.000, \sigma = 7.000$ and (iv) $\rho = 1.100, \mu = -15.000, \sigma = 8.500$ representing negative skewness. For each parameter setting, independent samples of sizes ranging from 20 to 500 are generated from EAALD and computed their mean square errors and absolute bias. We considered 1000 replications per scenario to ensure the robustness of the results. Tables 1–2 presents the mean value of maximum likelihood estimators of ρ, μ and σ of the EAALD along with their respective mean squared errors (MSE) and absolute bias. From Tables 1–2, it can be observed that as the sample size increases, absolute bias and MSE of the estimator decreases and the mean value of the estimator approaches to its original value.

4 Applications

In this section, we numerically illustrate the usefulness of the proposed model EAALD using two real-world data sets from distinct fields- environmental-engineering and cancer research. The first dataset is taken from Etage et al. (2023), represents vinyl chloride data from clean upgradient ground-water monitoring wells in (g/L). It has 34 observations. The second dataset represents the survival times (in years) of a group of patients who received chemotherapy treatment alone. The data is taken from Chesneau and Artault (2021) and has 46 observations. Further, the proposed model EAALD is compared with the existing models - asymmetric Laplace distribution (ALD^1) of Kozubowski and Podgórski (2000), asymmetric Laplace distribution (ALD^2) of Yu and Zhang (2005), alpha skew Laplace distribution (ASLD) of Shams Harandi and Alamatsaz (2013), skew normal distribution (SND) of Azzalini (2005), skew- t distribution (StD) of Jones and Faddy (2003), skew logistic distribution (S/D) of Nadarajah (2009), modified skew logistic distribution (MS/D) of

Table 1: The parameter estimates, MSEs and absolute bias for the simulated samples of EAALD.

Initial value of the parameters (positive skewness)	Sample size (n)	Parameters	Estimates	MSE	Absolute bias
(i)	20	ρ	0.001	9.700×10^{-3}	0.099
		μ	-5.368	0.156	0.368
		σ	2.059	0.189	0.059
	50	ρ	0.002	9.600×10^{-3}	0.098
		μ	-5.182	0.083	0.182
		σ	2.056	0.121	0.056
	100	ρ	0.003	9.500×10^{-3}	0.097
		μ	-5.108	0.035	0.108
		σ	2.041	0.074	0.041
	200	ρ	0.004	9.300×10^{-3}	0.096
		μ	-5.029	0.004	0.029
		σ	1.987	0.007	0.013
	500	ρ	0.013	8.200×10^{-3}	0.087
		μ	-5.006	0.002	0.006
		σ	1.997	0.002	0.003
(ii)	20	ρ	1.000×10^{-4}	1.700×10^{-2}	2.990×10^{-2}
		μ	-2.105	0.333	0.105
		σ	10.803	1.041	0.197
	50	ρ	2.000×10^{-4}	8.000×10^{-4}	2.980×10^{-2}
		μ	-2.098	0.092	0.098
		σ	10.974	0.497	0.026
	100	ρ	4.000×10^{-4}	7.000×10^{-4}	2.960×10^{-2}
		μ	-2.084	0.045	0.084
		σ	11.009	0.214	0.009
	200	ρ	5.300×10^{-3}	3.000×10^{-4}	2.470×10^{-2}
		μ	-1.989	0.042	0.011
		σ	10.995	0.035	0.005
	500	ρ	1.220×10^{-3}	1.000×10^{-4}	1.780×10^{-2}
		μ	-1.992	0.009	0.008
		σ	11.002	0.003	0.002

Table 2: The parameter estimates, MSEs and absolute bias for the simulated samples of EAALD.

Initial value of the parameters (negative skewness)	Sample size (n)	Parameters	Estimates	MSE	Absolute bias
(iii)	20	ρ	3.929	0.955	0.071
		μ	-12.231	3.802	0.231
		σ	6.772	1.1622	0.228
	50	ρ	4.069	0.383	0.069
		μ	-11.843	0.7290	0.157
		σ	6.815	0.919	0.185
	100	ρ	3.949	0.122	0.051
		μ	-12.083	0.309	0.083
		σ	7.060	0.470	0.0603
	200	ρ	3.956	0.087	0.044
		μ	-11.943	0.131	0.057
		σ	7.026	0.118	0.026
	500	ρ	4.037	0.067	0.037
		μ	-12.016	0.103	0.016
		σ	7.020	0.117	0.020
(iv)	20	ρ	1.011	0.081	0.089
		μ	-15.361	1.651	0.361
		σ	7.994	1.899	0.506
	50	ρ	1.033	0.040	0.067
		μ	-15.274	0.896	0.274
		σ	8.406	0.901	0.094
	100	ρ	1.165	0.033	0.065
		μ	-15.119	0.450	0.119
		σ	8.431	0.313	0.069
	200	ρ	1.038	0.031	0.062
		μ	-15.099	0.217	0.099
		σ	8.451	0.260	0.049
	500	ρ	1.081	0.024	0.019
		μ	-15.061	0.177	0.061
		σ	8.536	0.242	0.036

Kumar and Manju (2015) and extended asymmetric double lindley distribution (EADLD) of Kumar and Jose (2025) to highlight the suitability of the model.

For examining the preliminary aspects of the data sets, we have computed the descriptive statistics such as minimum value, first quartile (Q_1), second quartile (Q_2), third quartile (Q_3), maximum value, mean, mode, standard deviation (S.D) and coefficient of variation (C.V) given in Table A2 in Appendix.

Using the MATHEMATICA software, we have fitted EAALD to both data sets by utilizing maximum likelihood estimation procedure discussed in Section 2.3.1. We have also considered the fitting of the models, ALD^1 , ALD^2 , ASLD, SND, S_tD , S/D , MS/D and EADLD, for comparison and the numerical results are presented in Table 3 and Table 4 respectively. We have computed the P-value and Kolmogorov Smirnov Statistic (KSS) value associated in both the cases. For model comparison, we have used some well known information measures: Akaike's Information Criterion (AIC), Bayesian Information Criterion (BIC) and corrected Akaike's Information Criterion (AICc). From Tables 3-4, we can observe that EAALD has the maximum P-value, least KSS value, least AIC, BIC and AICc values, indicating a superior fit compared to the existing models. Therefore, it can be learnt that EAALD is more appropriate for modelling these data sets compared to the existing alternate models and hence, it can be viewed as an alternative to ALD^1 , ALD^2 , ASLD, SND, S_tD , S/D , MS/D and EADLD. We have plotted the fitted density plot as well as empirical and theoretical cdf plots corresponding to ALD^1 , ALD^2 , ASLD, SND, S_tD , S/D , MS/D , EADLD and EAALD for both the data sets and presented in Figures 2-3.

Rather than solely depending on the model with minimum AIC, we have computed the Akaike weight to provide a probabilistic context to support model choice. The concept of Akaike weight (or Probability of betterment), introduced by Burnham and Anderson (2002), provides a way to quantify the relative likelihood that a model is the best among the set of candidate models based on AIC values. It is the probability that model j is best among the set of candidate models, given the data and models considered, and is given by

$$w_j = \frac{\exp(-\delta_j/2)}{\sum_{r=1}^R \exp(-\delta_r/2)}, \quad (4.1)$$

where w_j is the weight for model j , R is the total number of models and $\delta_j = \text{AIC of } j^{th} \text{ model} - \text{AIC of model having minimum value}$. Here, the Akaike weights has been computed for ALD^1 , ALD^2 , ASLD, SND, S_tD , S/D , MS/D , EADLD, EAALD and we denote it as $w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8$ and w_9 respectively. The percentage chance that the model is best among candidate models is obtained as $w_j \times 100$. The numerical results are presented in Table A3 in Appendix. From Table A3, it is seen that EAALD has 98.178% and 91.202% probability that it is best among the set of models considered for analyzing these data sets. These results also highlight the suitability of EAALD in modelling these data sets.

The uniqueness of the obtained MLEs of the parameters of EAALD is established by showing the Hessian matrix $H(\hat{\Theta})$ to be a negative definite. We have computed the Hessian matrix $H(\hat{\Theta})$ and their corresponding eigen values using the MLEs $\hat{\rho}$, $\hat{\mu}$ and $\hat{\sigma}$ for both the datasets. For data set 1, the

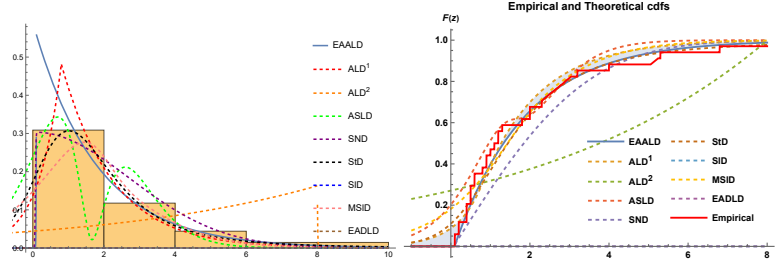


Figure 2: The fitted density plot and empirical and theoretical cdf plot of vinyl chloride data.

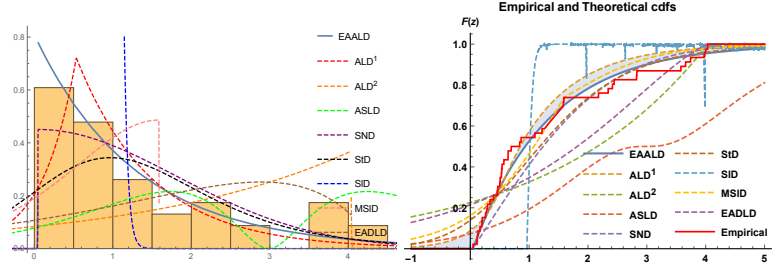


Figure 3: The fitted density plot and empirical and theoretical cdf plot of chemotherapy data.

Hessian matrix is obtained as

$$\begin{bmatrix} -251.172 & -0.232 \\ -0.232 & -10.626 \end{bmatrix}$$

with its corresponding eigen values $-251.172, -10.626$. Similarly, for data set 2, the Hessian matrix is obtained as

$$\begin{bmatrix} -22732.680 & -16.905 \\ -16.905 & -27.966 \end{bmatrix}$$

with its corresponding eigen values $-22732.700, -27.953$. Since, all eigen values are negative, the obtained Hessian matrix $H(\hat{\Theta})$ is a negative definite for both the datasets. Hence, the MLEs obtained for the parameters of EAALD for both the data sets are unique.

Next, we adopt the generalized likelihood ratio test procedure discussed in Section 2.3.2, to test the null hypothesis $H_0: \rho = 1, \mu = 0, \sigma = 1$ against the alternative $H_1: \rho \neq 1, \mu \neq 0, \sigma \neq 1$. The numerical results obtained are presented in Table A4 in Appendix. Based on the computed values of the likelihood ratio test statistic (LRT) and their associated P-values in Table A4, it is evident that the null hypothesis is rejected for both data sets under consideration. These results highlight the significance of the parameter ρ of the EAALD and affirm the suitability of the proposed model EAALD in analyzing the two data sets considered in this article.

5 Conclusion and Discussion

This study proposes a simple yet flexible “alternate asymmetric Laplace distribution (AALD)” that extends the classical Laplace law through a single shape parameter (ρ). The parameter governs both skewness and tail thickness, enabling a smooth transition from the symmetric Laplace model ($\rho = 1$) to asymmetric heavy-tailed forms ($\rho > 0$). Unlike many multi-parameter generalizations, the AALD maintains analytical tractability while providing sufficient flexibility for modelling asymmetric data. Closed-form expressions have been derived for the pdf, cdf, moments, moment generating function, quantiles, entropy, and order statistics, demonstrating the internal coherence and theoretical richness of the model. Its compact form allows efficient computation and straightforward parameter estimation using standard inferential techniques. The location-scale extension of AALD, named “the extended alternate asymmetric Laplace distribution (EAALD)” is introduced and the parameters are estimated using maximum likelihood estimation procedure. To evaluate the significance of the parameter, we employ GLRT. The practical applicability of EAALD is demonstrated through analyses of real-world data sets from environmental-engineering and cancer research, illustrating its superior fit compared to existing models. The proposed model is particularly relevant for applications where skewness and heavy tails are prominent, such as environmental-engineering, reliability analysis, financial risk modelling, survival analysis and environmental data studies. Owing to its balance between parsimony and flexibility, the model offers a practical and interpretable alternative to existing asymmetric distributions. In addition, a simulation study is conducted to evaluate the performance of the maximum likelihood estimators under various sample sizes and parameter settings.

In essence, the proposed AALD offers a rare combination of simplicity, interpretability, and analytical completeness. With a single shape parameter (ρ), it flexibly captures a wide spectrum of distributional forms - right-skewed and heavy-tailed when ($0 < \rho < 1$), symmetric when ($\rho = 1$), and left-skewed and heavy-tailed when ($\rho > 1$). This versatility, combined with closed-form expressions for important characteristics, makes the AALD a robust and elegant extension of the Laplace family, suitable for both theoretical developments and practical applications involving asymmetric or heavy-tailed data. Future research may explore its regression, multivariate, and mixture extensions to broaden its applicability.

Table 3: Estimated values of the parameters of ALD^1 , ALD^2 , $ASLD$, SND , SfD , $MSfD$, $EADLD$ and $EAALD$ with respective log-likelihood value, P-value, KSS value, AIC, BIC and AICc values for vinyl dataset.

Estimates of the parameters	ALD^1 (μ, σ)	ALD^2 (γ, μ, σ)	$ASLD$ (α, μ, σ)	SND (λ, μ, σ)	SfD (a, b, μ)	SfD (λ, μ, σ)	$MSfD$ ($\beta, \delta, \mu, \sigma$)	$EADLD$ (θ, k, μ, σ)	$EAALD$ (ρ, μ, σ)
$\hat{\rho}$	-	-	-	-	-	-	-	-	0.003
$\hat{\lambda}$	-	-	-	177.478	-	168463.934	-	-	-
$\hat{\alpha}$	-	-	4.151	-	-	-	-	-	-
$\hat{\beta}$	-	-	-	-	-	-	-0.578	-	-
$\hat{\delta}$	-	-	-	-	-	-	-2.886	-	-
$\hat{\gamma}$	-	0.998	-	-	-	-	-	-	-
$\hat{a}$	-	-	-	-	3.301	-	-	-	-
$\hat{b}$	-	-	-	-	1.290	-	-	-	-
$\hat{\theta}$	-	-	-	-	-	-	-	218.616	-
$\hat{k}$	-	-	-	-	-	-	-	0.116	-
$\hat{\mu}$	0.799	8.000	1.534	0.072	0.188	-1.000×10^{-4}	2.032	8.001	0.100
$\hat{\sigma}$	0.957	0.009	0.481	2.631	-	1.393	1.063	0.002	1.785
l	-61.897	-95.650	-69.288	-57.738	-62.028	-5.805×10^7	-67.638	-2.314×10^8	-53.696
P-value	0.585	2×10^{-5}	0.051	0.048	0.567	3.775×10^{-11}	0.139	2.034×10^{-31}	0.798
KSS	0.128	0.396	0.227	0.229	0.130	0.588	0.193	1.000	0.106
AIC	127.794	197.300	144.577	121.475	130.056	116095234.133	143.276	462869322.911	113.392
BIC	130.847	201.879	149.156	126.054	134.635	116095238.713	149.382	462869329.016	117.971
AICc	128.181	198.100	145.377	122.275	130.856	116095234.933	144.656	462869324.289	114.192

Table 4: Estimated values of the parameters of ALD^1 , ALD^2 , $ASLD$, SND , StD , MS/D , $EADLD$ and $EAALD$ with respective log-likelihood value, P-value, KSS value, AIC, BIC and AICc values for chemotherapy dataset.

Estimates of the parameters	ALD^1 (μ, σ)	ALD^2 (γ, μ, σ)	$ASLD$ (α, μ, σ)	SND (λ, μ, σ)	StD (a, b, μ)	StD (λ, μ, σ)	MS/D ($\beta, \delta, \mu, \sigma$)	$EADLD$ (θ, k, μ, σ)	$EAALD$ (ρ, μ, σ)
$\hat{\rho}$	-	-	-	-	-	-	-	-	0.002
$\hat{\lambda}$	-	-	-	1436.024	-	96272.228	-	-	-
$\hat{\alpha}$	-	-	-76226.727	-	-	-	-	-	-
$\hat{\beta}$	-	-	-	-	-	-	-	-	-
$\hat{\delta}$	-	-	-	-	-	-	-	-	-
$\hat{\gamma}$	-	0.998	-	-	-	-	-	-	-
$\hat{a}$	-	-	-	-	6.212	-	-	-	-
$\hat{b}$	-	-	-	-	3.053	-	-	-	-
$\hat{\theta}$	-	-	-	-	-	-	-	0.237	-
$\hat{k}$	-	-	-	-	-	-	-	293.498	-
$\hat{\mu}$	0.534	4.033	2.986	0.044	-0.017	0.047	1.589	4.033	0.047
$\hat{\sigma}$	0.639	0.005	0.627	1.771	-	0.961	0.759	0.001	1.280
l	-66.699	-91.915	-95.452	-59.705	-71.501	-2.810×10^7	-72.086	-85.930	-57.364
P-value	0.332	4.485×10^{-5}	3.705×10^{-8}	0.026	0.144	1.063×10^{-12}	0.120	8.016×10^{-4}	0.659
KSS	0.136	0.334	0.431	0.213	0.165	0.543	0.171	0.286	0.104
AIC	137.398	189.829	196.905	125.411	149.003	56204126.689	152.171	179.860	120.729
BIC	141.055	195.315	202.391	130.897	154.489	56204132.175	159.486	187.175	126.215
AICc	137.677	190.401	197.476	125.982	149.574	56204127.261	153.147	180.836	121.300

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Conflict of Interest The authors declare they have no conflicts of interest.

Data Availability The authors confirm that the data supporting the findings of this study are available within the article.

A1 Appendix

A1.1 Proof of Proposition 2.1

For $y < 0$, the cdf is

$$F(y; \rho) = \int_{-\infty}^y \frac{e^{\frac{y}{\rho}}}{(1 + \rho)} dy = \frac{\rho e^{\frac{y}{\rho}}}{(1 + \rho)}.$$

And for $y \geq 0$, the cdf of the AALD is

$$\begin{aligned} F(y; \rho) &= \int_{-\infty}^0 \frac{e^{\frac{y}{\rho}}}{(1 + \rho)} dy + \int_0^y \frac{e^{-y}}{(1 + \rho)} dy \\ &= \frac{1}{(1 + \rho)} [(1 + \rho) - e^{-y}] \\ &= 1 - \frac{e^{-y}}{(1 + \rho)}. \end{aligned}$$

A1.2 Proof of Proposition 2.2

The r^{th} moment of AALD is obtained as

$$\begin{aligned} \mu'_r &= E(Y^r) \\ &= \frac{1}{1 + \rho} \left[\int_{-\infty}^0 y^r e^{\frac{y}{\rho}} dy + \int_0^{\infty} y^r e^{-y} dy \right] \\ &= \frac{1}{1 + \rho} [(-1)^r \rho^{r+1} \Gamma(r + 1) + \Gamma(r + 1)] \\ &= \frac{r!}{1 + \rho} \begin{cases} 1 - \rho^{r+1}; r \text{ is odd} \\ 1 + \rho^{r+1}; r \text{ is even} \end{cases}. \end{aligned}$$

A1.3 Proof of Proposition 2.4

By definition, the moment generating function of AALD is

$$\begin{aligned}
 M_Y(t) &= E(e^{tY}) \\
 &= \frac{1}{1+\rho} \left[\int_{-\infty}^0 e^{(t+\frac{1}{\rho})y} dy + \int_0^{\infty} e^{(t-1)y} dy \right] \\
 &= \frac{1}{1+\rho} \left[\frac{1}{\left(t+\frac{1}{\rho}\right)} + \frac{1}{(1-t)} \right] \\
 &= \frac{1}{(1-t)} \cdot \frac{1}{(1+\rho t)} \\
 &= \frac{1}{1 - [\rho t^2 + (1-\rho)t]}; \quad t \in \left(-\frac{1}{\rho}, 1\right).
 \end{aligned}$$

A1.4 Proof of Proposition 2.5

The cumulant generating function of AALD is

$$K_Y(t) = \sum_{n=1}^{\infty} K_n \frac{t^n}{n!} \quad (\text{A1.1})$$

$$\begin{aligned}
 &= \log [M_Y(t)] \\
 &= \log \left[\frac{1}{1 - [\rho t^2 + (1-\rho)t]} \right] \\
 &= -\log [1 - [\rho t^2 + (1-\rho)t]]. \quad (\text{A1.2})
 \end{aligned}$$

On expanding the logarithmic term in (A1.2), we get

$$\begin{aligned}
 K_Y(t) &= [\rho t^2 + (1-\rho)t] + \frac{[\rho t^2 + (1-\rho)t]^2}{2} + \frac{[\rho t^2 + (1-\rho)t]^3}{3} + \frac{[\rho t^2 + (1-\rho)t]^4}{4} + \dots \\
 &= [\rho t^2 + (1-\rho)t] + \frac{[\rho^2 t^4 + (1-\rho)^2 t^2 + 2\rho(1-\rho)t^3]}{2} \\
 &\quad + \frac{[\rho t^2 + (1-\rho)t] [\rho^2 t^4 + (1-\rho)^2 t^2 + 2\rho(1-\rho)t^3]}{3} \\
 &\quad + \frac{[\rho^2 t^4 + (1-\rho)^2 t^2 + 2\rho(1-\rho)t^3] [\rho t^2 + (1-\rho)t]}{4} + \dots
 \end{aligned}$$

Now, we equate the coefficients of $\frac{t^n}{n!}$ in the right hand side of (A1.1) and (A1.2) to obtain the n^{th} cumulant of the AALD as

$$K_n = \begin{cases} (n-1)!(1-\rho^n); & n \text{ is odd} \\ (n-1)!(1+\rho^n); & n \text{ is even} \end{cases}. \quad (\text{A1.3})$$

A1.5 Proof of Proposition 2.6

For $0 < t \leq \frac{\rho}{1+\rho}$,

$$F(y_t) = t \Rightarrow \frac{\rho e^{\frac{y_t}{\rho}}}{1+\rho} = t \Rightarrow y_t = \rho \log \left[\frac{t(1+\rho)}{\rho} \right].$$

However, if $\frac{\rho}{1+\rho} \leq t < 1$,

$$\frac{1+\rho - e^{-y_t}}{1+\rho} = t \Rightarrow y_t = -\log [(1+\rho)(1-t)].$$

A1.6 Proof of Proposition 2.7

By definition, Rényi entropy of a distribution with pdf $(\cdot)$ is

$$I_R(\gamma) = \frac{1}{\gamma-1} \log \int_{-\infty}^{\infty} f^\gamma(x) dx,$$

where $\gamma > 0$ and $\gamma \neq 1$. Thus, the Rényi entropy of AALD with pdf (2.1) is

$$I_R(\gamma) = \frac{1}{\gamma-1} \log \int_{-\infty}^{\infty} f^\gamma(y) dy,$$

in which

$$\int_{-\infty}^{\infty} f^\gamma(y) dy = \frac{1}{(1+\rho)^\gamma} \left[\int_{-\infty}^0 e^{\frac{\gamma y}{\rho}} dy + \int_0^{\infty} e^{-\gamma y} dy \right] = \frac{1}{\gamma(1+\rho)^{\gamma-1}}.$$

Thus the entropy of AALD is obtained as

$$I_R(\gamma) = \frac{1}{\gamma-1} \log \left[\frac{1}{\gamma(1+\rho)^{\gamma-1}} \right].$$

A1.7 Proof of Proposition 2.10

By definition, the mean residual life function is given by

$$m(y) = E[Y - y | Y > y].$$

For $y < 0$, MRLF of the AALD is

$$\begin{aligned} m(y) &= \frac{1}{1-F(y)} \int_y^0 (1-F(t)) dt \\ &= \frac{1}{1-\frac{\rho e^{\frac{y}{\rho}}}{1+\rho}} \int_y^0 \left(1 - \frac{\rho e^{\frac{t}{\rho}}}{1+\rho} \right) dt \\ &= \frac{\rho^2 e^{\frac{y}{\rho}} - \rho^2 - \rho y - y}{1+\rho - \rho e^{\frac{y}{\rho}}}. \end{aligned}$$

and for $y > 0$, MRLF of AALD is given by

$$\begin{aligned} m(y) &= \frac{1}{1-F(y)} \int_y^{\infty} (1-F(t)) dt \\ &= \frac{1}{\frac{e^{-y}}{1+\rho}} \int_y^{\infty} \frac{e^{-t}}{1+\rho} dt \\ &= 1. \end{aligned}$$

A1.8 Proof of Proposition 2.13

By definition,

$$\begin{aligned} E(Y_{k:n}^q) &= \int_{-\infty}^{\infty} y^q f_{Y_{k:n}}(y) dy \\ &= \frac{1}{B(k, n-k+1)} \frac{\rho^{k-1}}{(1+\rho)^k} \int_{-\infty}^0 y^q e^{\frac{ky}{\rho}} \left[1 - \frac{\rho e^{\frac{y}{\rho}}}{1+\rho} \right]^{n-k} dy \\ &\quad + \frac{1}{B(k, n-k+1)} \frac{1}{(1+\rho)^{n-k+1}} \int_0^{\infty} y^q e^{-(n-k+1)y} \left[1 - \frac{e^{-y}}{1+\rho} \right]^{k-1} dy. \end{aligned}$$

Using binomial expansion in the above equation leads to

$$\begin{aligned} E(Y_{k:n}^q) &= \frac{1}{B(k, n-k+1)} \frac{\rho^{k-1}}{(1+\rho)^k} \int_{-\infty}^0 y^q e^{\frac{ky}{\rho}} \sum_{j=0}^{n-k} \binom{n-k}{j} \left(\frac{\rho}{1+\rho} \right)^j (-1)^j e^{\frac{iy}{\rho}} dy \\ &\quad + \frac{1}{B(k, n-k+1)} \frac{1}{(1+\rho)^{n-k+1}} \int_0^{\infty} y^q e^{-(n-k+1)y} \sum_{i=0}^{k-1} \binom{k-1}{i} \left(\frac{1}{1+\rho} \right)^i (-1)^i e^{-iy} dy \\ &= \frac{1}{B(k, n-k+1)} \frac{\rho^{k-1}}{(1+\rho)^k} \sum_{j=0}^{n-k} \binom{n-k}{j} \left(\frac{\rho}{1+\rho} \right)^j (-1)^j \int_{-\infty}^0 y^q e^{\frac{(k+j)y}{\rho}} dy \\ &\quad + \frac{1}{B(k, n-k+1)} \frac{1}{(1+\rho)^{n-k+1}} \sum_{i=0}^{k-1} \binom{k-1}{i} \left(\frac{1}{1+\rho} \right)^i (-1)^i \int_0^{\infty} y^q e^{-(n-k+i+1)y} dy. \end{aligned}$$

Now, on integration we get,

$$\begin{aligned} E(Y_{k:n}^q) &= \frac{\Gamma(q+1)}{B(k, n-k+1)} \frac{\rho^{k-1}}{(1+\rho)^k} \sum_{j=0}^{n-k} \binom{n-k}{j} \left(\frac{\rho}{1+\rho} \right)^j (-1)^{j+q} \left(\frac{\rho}{j+k} \right)^{(q+1)} \\ &\quad + \frac{\Gamma(q+1)}{B(k, n-k+1)} \frac{1}{(1+\rho)^{n-k+1}} \sum_{i=0}^{k-1} \binom{k-1}{i} \left(\frac{1}{1+\rho} \right)^i (-1)^i \frac{1}{(n-k+i+1)^{q+1}}. \end{aligned}$$

A1.9 Data set 1 used in Section 4:

5.1, 1.2, 1.3, 0.6, 0.5, 2.4, 0.5, 1.1, 8.0, 0.8, 0.4, 0.6, 0.9, 0.4, 2.0, 0.5, 5.3, 3.2, 2.7, 2.9, 2.5, 2.3, 1.0, 0.2, 0.1, 0.1, 1.8, 0.9, 2.0, 4, 6.8, 1.2, 0.4, 0.2.

A1.10 Data set 2 used in Section 4:

1.485, 1.553, 1.099, 1.219, 1.271, 0.047, 0.115, 0.197, 0.203, 0.296, 0.334, 0.395, 0.458, 0.466, 0.501, 0.507, 0.529, 3.658, 3.743, 3.978, 0.26, 0.282, 0.534, 0.54, 0.57, 0.641, 0.644, 1.326, 1.447, 1.581, 1.589, 2.178, 0.121, 0.132, 0.164, 2.343, 2.416, 2.444, 2.825, 2.83, 3.578, 0.696, 0.841, 0.863, 4.003, 4.033.

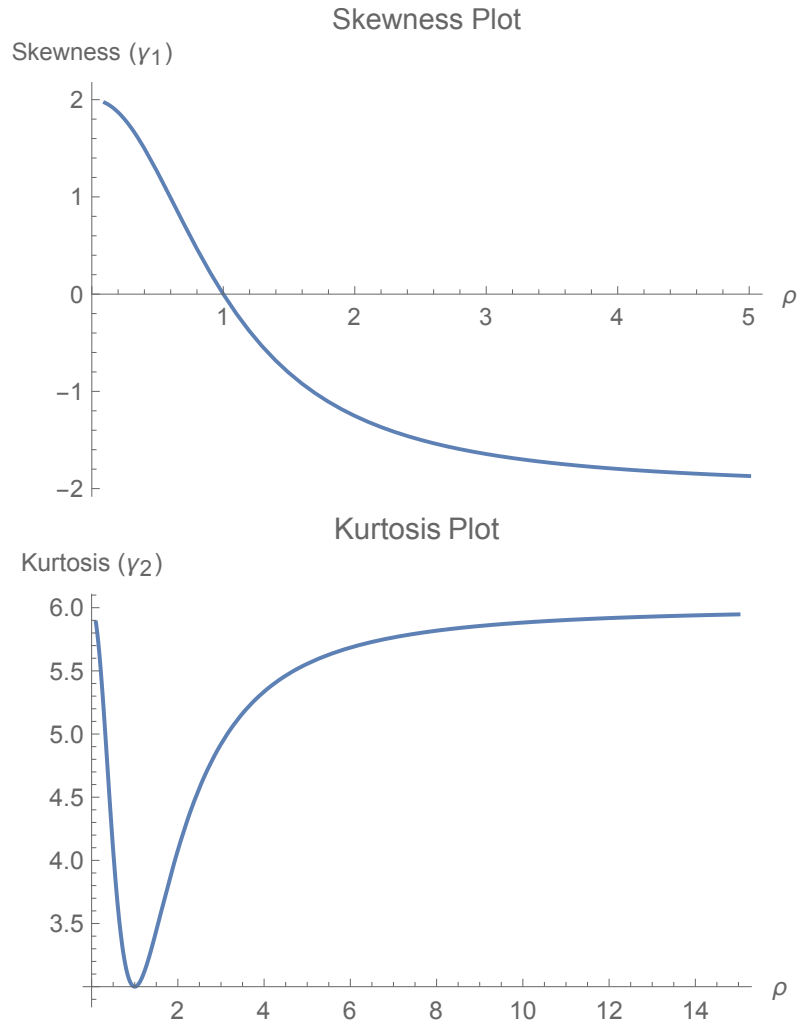


Figure 4: The skewness and kurtosis plots of the AALD(ρ).

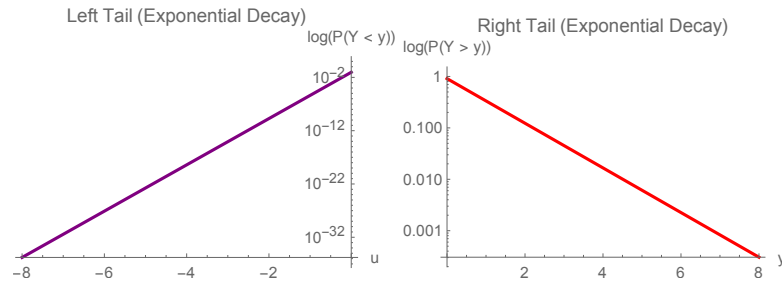


Figure 5: The left and right tail probability of the AALD(ρ) at $\rho = 0.1$.

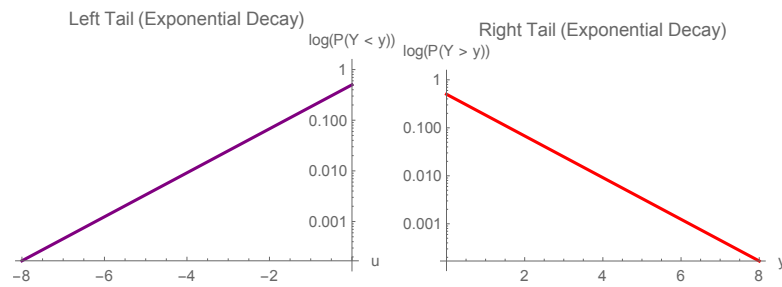


Figure 6: The left and right tail probability of the AALD(ρ) at $\rho = 1$.

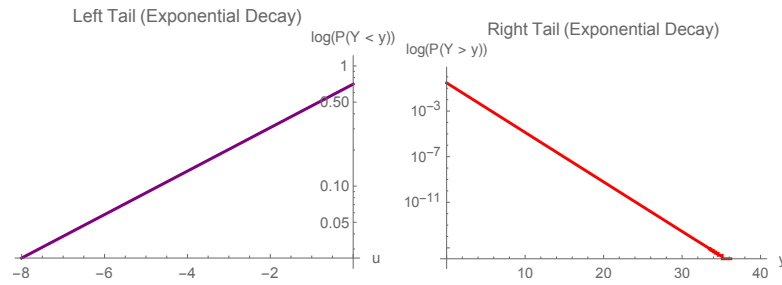


Figure 7: The left and right tail probability of the AALD(ρ) at $\rho = 2.4$.

Table A1: The values of mean, variance, skewness and kurtosis of AALD(ρ) for particular choices of the parameter ρ mentioned in Proposition 2.3.

ρ	μ'_1	μ_2	γ_1	γ_2
0.100	0.900	1.010	1.968	5.882
0.250	0.750	1.062	1.798	5.336
0.300	0.700	1.090	1.710	5.091
0.500	0.500	1.200	1.252	4.080
0.750	0.250	1.562	0.592	3.235
0.900	0.100	1.810	0.223	3.033
1.000	0.000	2.000	0.000	3.000
1.500	-0.500	3.250	-0.811	3.444
3.000	-2.000	10.000	-1.040	4.920
5.000	-4.000	26.000	-0.734	5.556
7.500	-6.500	57.250	-1.943	5.794
10.000	-9.000	101.000	-1.968	5.882
20.000	-19.000	401.000	-1.993	5.970
50.000	-49.000	2501.000	-1.999	5.995
100.000	-99.000	10001.000	-1.999	5.999

Table A2: Descriptive statistics of the data sets of Section 4.

Data set	Min	Q_1	Q_2	Q_3	Max	Mean	S.D.	C.V.
1	0.100	0.500	1.150	2.500	8.000	1.879	1.953	1.039
2	0.047	0.395	0.768	2.178	4.033	1.325	1.238	0.934

Table A3: Probability of betterment for the models of Section 4

Dataset	Model	w_j	% chance of being the best
1	$ALD^1(\mu, \sigma)$	7.322×10^{-4}	0.073
	$ALD^2(\gamma, \mu, \sigma)$	5.910×10^{-19}	0.000
	$ASLD(\alpha, \mu, \sigma)$	1.661×10^{-7}	0.000
	$SND(\lambda, \mu, \sigma)$	0.172	1.725
	$StD(a, b, \mu)$	2.363×10^{-4}	0.027
	$SlD(\lambda, \mu, \sigma)$	$4.748 \times 10^{-25209685}$	0.000
	$MSlD(\beta, \delta, \mu, \sigma)$	3.183×10^{-7}	0.000
	$EADLD(\theta, k, \mu, \sigma)$	$1.568 \times 10^{-100510919}$	0.000
	$EAALD(\rho)$	0.982	98.178
2	$ALD^1(\mu, \sigma)$	2.197×10^{-4}	0.022
	$ALD^2(\gamma, \mu, \sigma)$	9.018×10^{-16}	0.000
	$ASLD(\alpha, \mu, \sigma)$	2.622×10^{-17}	0.000
	$SND(\lambda, \mu, \sigma)$	0.088	8.776
	$StD(a, b, \mu)$	6.613×10^{-7}	0.000
	$SlD(\lambda, \mu, \sigma)$	$1.365 \times 10^{-12204545}$	0.000
	$MSlD(\beta, \delta, \mu, \sigma)$	1.357×10^{-7}	0.000
	$EADLD(\theta, k, \mu, \sigma)$	5.906×10^{-13}	0.000
	$EAALD(\rho)$	0.912	91.202

Table A4: Testing the significance of parameter of the EAALD using GLRT of Section 4.

Dataset	$L(\hat{\Theta}^* z)$	$L(\hat{\Theta} z)$	LRT	P-value
1	4.976×10^{-30}	4.787×10^{-24}	27.554	< 0.0001
2	1.633×10^{-33}	1.227×10^{-25}	36.270	< 0.0001

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