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# ESTIMATION OF SCALE PARAMETER IN GAMMA POPULATION USING SAMPLES SELECTED BY SYSTEMATIC MANNER

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SUMMARY

The Systematic sampling method has been proven effective in finite population for estimation of the population characteristics. However, its application to an infinite population requires further exploration. This article aims to extend the systematic sampling method to infinite populations and presents several approximation techniques. The performance of each method is evaluated based on different characteristics of the population distribution. Specifically, we focus on important distributions and investigate the case of Gamma distribution here. Three approaches are considered, with the first method showing promising results. This method involves assigning Gamma probabilities to all possible samples, avoiding the need to truncate the infinite population into a finite one. Also, it is seen that estimates are closed to original one if all samples include mode or neighbourhood of mode. The findings of this study shed light on the applicability of systematic sampling for infinite populations and contribute towards the estimation of population characteristics by this kind of work.

*Keywords and phrases:* Systematic sampling; Risk function; Gamma distribution; Discretization of continuous distribution.

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# 1 Introduction

Systematic sampling first proposed by Madow and Madow (1944) is a probability-based sampling technique which needs the members of the underlying population to be serially numbered in so some order. The method of systematic sampling consists of choosing at first a population unit corresponding to a number r which is randomly selected from 1 to k and thereafter every k-th unit of the population are taken until a sample of desired size is obtained. The value of r actually determines the whole sample. Here r is called the random start and k is the sampling interval. If population and sample sizes are N and n respectively, then k is equal to $\frac{N}{n}$. Different aspects of this sampling method along with its different variants have been investigated in details by the researchers since its inception. Using Yates end-corrections (Yates, 1948), Subramani and Singh (2014) suggested an improved estimator with higher efficiency for population mean in presence of linear trend in population. A new sampling design, named as Modified Systematic Sampling with Multiple Random Starts was elaborately discussed by Gupta et al. (2018). Subramani (2019) studied the linear systematic sampling with unequal sampling intervals in the presence of linear trend among the population values. Subramani and Gupta (2014) introduced a more generalized form of modified linear systematic sampling namely generalized modified linear systematic sampling (GMLSS) scheme, which is applicable for any sample size, irrespective of the population size whether it is a multiple of sample size or not. Subramani (2000, 2009) explored different results of diagonal systematic sampling technique for finite population. Azeem (2021) discussed the estimation of finite population proportion of units which possess a particular qualitative characteristic under diagonal systematic sampling scheme.

Systematic sampling for sample selection and parameter estimation requires a finite underlying population. Ghosh et al. (2024) extended this technique to the Poisson distribution, a countably infinite population, by approximating it with a large finite population before drawing systematic samples of the required size. They proposed three different approximation methods and estimated the population mean using the corresponding sample mean. They computed the risk of the sample mean as a measure of precision Banerjee and Seal (2022) and conducted a comparative study of the sample mean's performance under each approximation method. In this paper, the authors want to study the modifications of the idea proposed by Ghosh et al. (2024) in case of continuous distribution using the technique of discretization.

The idea of drawing systematic samples from countably infinite population has been extended for a population governed by a continuous infinite distribution like gamma distribution. A two-parameter gamma distribution has been widely applied to model many physical situations in different applied fields namely climatology, hydrology, insurance etc. where the underlying data set is mainly right skewed or positive (Stephenson et al., 1999; Husak et al., 2007; Alam et al., 2012; Seal et al., 2024). The flexibility of this distribution in terms of decreasing, increasing, or constant failure rate depending on the value of shape parameter $p < 0$, $p > 0$, or $p = 0$, respectively (Pradhan and Kundu, 2011) is the salient factor behind its wide applicability in tackling real life problems.

Another variant of two-parameter continuous gamma distribution is two-parameter discrete gamma distribution that has attracted the attention of many researchers in recent past. In reality the researchers sometimes do face challenges to have samples from a continuous distribution. In most of the situations the values of a variable are measured only to finite number of places after decimal point thereby making the observed values discrete in nature. Even when the measurements are made using a continuous scale, the observations are so recorded that the discrete model appears to be more suitable. In some other cases due to the precision of the measurement scale and to save time and space, we divide the entire range of a continuous random variable into some non-overlapping classes and then record the values taken by the variable in terms of frequencies of different classes. Hu et al. (2024) applied the sequential sampling technique to derive the bounded variance point estimator (BVPE) of the different functions of the scale parameter of the gamma distribution. Ke et al. (2023) used transformed samples in estimating the gamma parameters. Hassan et al. (2022) discussed ranked set sampling (RSS), neoteric ranked set sampling (NRSS), percentile ranked set sampling (PRSS) and median ranked set sampling (MRSS) for working out the maximum likelihood estimator of the parameters of the gamma distribution. In an inspirational work in the area of sample survey, Rahman and Patwary (2015) advocated the uses of unequal probability sampling method in bootstrap sampling along with equal probability sampling.

Generally, a continuous random variable is discretized to derive the discrete counterpart in such a way that one or more characteristic features like probability density function, cumulative distribution function, survival function, moment generating function etc. of the parent continuous variable is preserved. By various methods, we can perform the discretization of a continuous random variable. One such method is to maintain the resemblance of the form of survival function of the parent continuous random variable and its discrete analogue.

The rest of this paper is organized as follows. In Section 2, we discuss the idea of replacing continuous population by discretization and then discuss the approximation of countably infinite population. In this regard, we discuss the discretized gamma distribution in Section 2. Different approximation methods are presented in the Section 3. A comparative risk analysis of the proposed methods through simulation is discussed in Section 4. Also, the behaviour of the scale parameter through systematic sample is obtained in Section 5. In Section 6 we illustrate the proposed method with the help of a real life data set coming. Finally, conclusion of the study is given in Section 7.

## 2 Discretized Gamma Distribution

In survival (reliability) analysis, the statistical models are usually functions of continuous random variable but the corresponding observations are found to be discrete in nature. For instance, the duration of stay of a patient in a hospital are measured in terms of days or reliability of a system depends on the number of break-down of a computer. So, the researchers feel it more appropriate to model a real-life situation using discrete random variable instead of using continuous random variable. Specially not only the discrete distributions but also the discrete analogues of continuous probability distribution are catching the attention of the survival analysts in recent years. Lai (2013) applied the technique of discretization of a continuous lifetime model to generate a discrete lifetime model corresponding to the continuous one. Roy (2003) suggested a discrete analogue of continuous normal distribution to model discrete data retaining the property of increasing failure rate of the discrete counterpart. A discrete Rayleigh distribution has been introduced by Roy (2004) for reliability modeling and providing the approximate values of probability integrals resulting from the corresponding continuous setting. Chakraborty and Chakravarty (2012) derived a discrete two-parameter gamma distribution corresponding to a continuous two-parameter gamma distribution and examined its important distributional and reliability properties and performed parameter estimation. An extensive review work done by Chakraborty (2015) regarding the methods and constructions of discrete counterpart of continuous probability distribution is worth mentioning. Here the discretization method is applied for a two-parameter continuous gamma distribution in the following fashion. Actually, in discretization of a continuous probability distribution, we add all masses belonging to an interval $(i, i+1)$ to $i$.

If $S_X(x) = Pr(X > x)$ is the survival function of the parent continuous random variable $X$, then the random variable $Y = [X]$, the largest integer not exceeding $X$ will have the following probability mass function

$$\begin{aligned} P(Y=y) &= P(y \leq X \leq y+1) = P(X \geq y) - P(X \geq y+1) \\ &= F_x(y+1) - F_x(y) \\ &= S_X(y) - S_X(y+1); y = 0, 1, 2, \ldots \end{aligned}$$

Here we have followed Kemp (2004) convention according to which the discrete survival function is given by $S_y(k) = P(Y \geq k)$. Then the survival function corresponding to continuous two-parameter gamma distribution $G(p, \alpha)$, where $p$ is the shape parameter and $\alpha$ scale parameter is given by

$$S_X(i) = \frac{\alpha^p}{\Gamma(p)} \int_i^\infty e^{-\alpha x} x^{p-1} dx = \frac{\Gamma(p, i\alpha)}{\Gamma(p)}.$$

So, the PMF of discrete Gamma distribution with parameters $p$ (shape) and $\alpha$ (rate) is given by

$$Q(X=i) = \frac{(\Gamma(p, i\alpha) - \Gamma(p, \overline{(i+1)}\alpha))}{\Gamma(p)}; i = 0, 1, 2, \ldots \text{ and } p > 0, \alpha > 0. \tag{2.1}$$

The expectation of the discrete Gamma distribution comes out to be Equation

$$E(X) = \frac{\sum_{i=0}^{\infty} \Gamma(p, \overline{i+1}\alpha)}{\Gamma(p)}, \text{ where } \Gamma(p, x) = \frac{1}{\Gamma(p)} \int_x^\infty e^{-t} t^{p-1} dx.$$

## 3 Approximating Countable Infinite Population

In this study, our objective is to select samples from a countably infinite population in a systematic manner. However, the underlying population must be limited to draw samples using the systematic type technique. As a result, we use several techniques to approximate a countably infinite population using a large finite population. So, using the PMF (2.1), the following approximating approaches are applied and appropriate solutions are obtained. We use the sample mean as an appropriate estimator of the population parameter, which is given as

$$\text{Sample mean} = X + \frac{(n-1)k}{2}; X = 0, 1, 2, \ldots, k–1.$$

It is noted that the performance of an estimator is measured by the risk of that estimator (Seal et al., 2023). So, the risk function of $\alpha$ is expressed as follows

$$Risk = E\Big[X + \frac{(n-1)k}{2} - \alpha\Big]^2. \tag{3.1}$$

## 3.1 Approximation methods

### 3.1.1 Method 1

A finite population defined on $\{0, 1, 2, \ldots, N-1\}$, where $N = nk$ for a given value of the sample size $n$ and the sampling interval $k$, approximates a countably infinite population defined on $0, 1, 2, \ldots, \infty$. The samples are given in the following manner

$$\begin{array}{l} \{0, k, 2k, \ldots\} \\ \{1, 1+k, 1+2k, \ldots\} \\ \{2, 2+k, 2+2k, \ldots\} \\ \cdots\cdots\cdots\cdots \\ \{k-1, 2k–1, 3k–1, \ldots\} \end{array}$$

Now, to calculate the risk, we need the following probabilities obtained from (2.1).

$$P(i) = Pr(\text{selecting a sample having } ith \text{ unit of the population});\ i = 0, 1, 2, \ldots, k–1.$$

Therefore,

$$\begin{aligned} P(0) &= Q(X=0) + Q(X=k) + Q(X=2k) + \cdots \\ &= \frac{1}{\Gamma(p)}\{\Gamma(p) + \Gamma(p, K\alpha) + \Gamma(p, 2k\alpha) + \cdots\} - \frac{1}{\Gamma(p)}\{\Gamma(p,\alpha) + \Gamma(p, \overline{k+1}\alpha) + \Gamma(p, \overline{2k+1}\alpha) + \cdots\} \\ P(1) &= Q(X=1) + Q(X=1+k) + Q(X=1+2k) + \cdots \\ &= \frac{1}{\Gamma(p)}\{\Gamma(p,\alpha) + \Gamma(p, \overline{1+k}\alpha) + \Gamma(p, \overline{1+2k}\alpha) + \cdots\} - \\ &\qquad \frac{1}{\Gamma(p)}\{\Gamma(p, 2\alpha) + \Gamma(p, \overline{2+k}\alpha) + \Gamma(p, \overline{2+2k}\alpha) + \cdots\} \\ P(2) &= Q(X=2) + Q(X=2+k) + Q(X=2+2k) + \cdots \\ &= \frac{1}{\Gamma(p)}\{\Gamma(p, 2\alpha) + \Gamma(p, \overline{2+k}\alpha) + \Gamma(p, \overline{2+2k}\alpha) + \cdots\} - \\ &\qquad \frac{1}{\Gamma(p)}\{\Gamma(p, 3\alpha) + \Gamma(p, \overline{3+k}\alpha) + \Gamma(p, \overline{3+2k}\alpha) + \cdots\} \\ & \cdots\cdots\cdots\cdots \\ P(k-1) &= Q(X=k-1) + Q(X=2k-1) + Q(X=3k-1) + \cdots \\ &= \frac{1}{\Gamma(p)}\{\Gamma(p, \overline{k-1}\alpha) + \Gamma(p, \overline{2k-1}\alpha) + \Gamma(p, \overline{3k-1}\alpha) \cdots\} - \\ &\qquad \frac{1}{\Gamma(p)}\{\Gamma(p, k\alpha) + \Gamma(p, 2k\alpha) + \Gamma(p, 3k\alpha) + \cdots\} \end{aligned}$$

### 3.1.2 Method 2

A finite population defined on $\{0, 1, 2, \ldots, N-1\}$ is being used to approximate a countably infinite population specified on $0, 1, 2, \ldots, \infty$ after combining population units greater than $N$–1 i.e. $N^{th}, (N+1)^{th}, (N+2)^{th}, \ldots\ldots$ units of the population with the $(N\text{–}1)^{th}$ unit of the population. Thus, the population is comprised up of $0, 1, 2, \ldots, N-1$ units, where $N = nk$. As a result, the following example compositions are

$$\begin{array}{l} \{0, k, 2k, \ldots, (n-1)k\} \\ \{1, 1+k, 1+2k, \ldots, 1+(n-1)k\} \\ \{2, 2+k, 2+2k, \ldots, 2+(n\text{–}1)k\} \\ \ldots\ldots\ldots\ldots \\ \{k-1, 2k\text{–}1, 3k\text{–}1, \ldots, nk\text{–}1\} \end{array}$$

Probability of choosing $(N-1)^{th}$ population unit is

$$\begin{aligned} Q(X = nk-1) &= \frac{1}{\Gamma(p)} \sum_{i=nk-1}^{\infty} \{\Gamma(p, i\alpha) - \Gamma(p, \overline{i+1}\alpha)\} \\ &= \frac{1}{\Gamma(p)} \Gamma(p, \overline{nk-1}\alpha) \text{where} N = nk \end{aligned}$$

$P(i) = Pr$(selecting a sample containing $i^{th}$ unit of the population in this method); $where\ i = 0, 1, 2, \ldots, k$–1.

Hence,

$$\begin{aligned} P(0) &= Q(X=0) + Q(X=k) + Q(X=2k) + \cdots + Q(X=\overline{n-1}k) \\ &= \frac{1}{\Gamma(p)} \{\Gamma(p) + \Gamma(p, k\alpha) + \Gamma(p, 2k\alpha) + \cdots + \Gamma(p, \overline{n-1}k\alpha)\} - \\ &\qquad \frac{1}{\Gamma(p)} [\Gamma(p, \alpha) + \Gamma(p, \overline{k+1}\alpha) + \Gamma(p, \overline{2k+1}\alpha) + \cdots + \Gamma(p, (1+\overline{n-1}k)\alpha)] \\ P(1) &= Q(X=1) + Q(X=1+k) + Q(X=1+2k) + \cdots + Q(X=1+\overline{n-1}k) \\ &= \frac{1}{\Gamma(p)} \left\{\Gamma(p, \alpha) + \Gamma(p, \overline{1+k}\alpha) + \Gamma(p, \overline{1+2k}\alpha) + \cdots + \Gamma(p, \overline{1+(n-1)k}\alpha)\right\} - \\ &\qquad \frac{1}{\Gamma(p)} \left\{\Gamma(p, 2\alpha) + \Gamma(p, \overline{2+k}\alpha) + \Gamma(p, \overline{2+2k}\alpha) + \cdots + \Gamma(p, \overline{2+(n-1)k}\alpha)\right\} \\ P(2) &= Q(X=2) + Q(X=2+k) + Q(X=2+2k) + \cdots + Q(X=2+\overline{n-1}k) \\ &= \frac{1}{\Gamma(p)} \left\{\Gamma(p, 2\alpha) + \Gamma(p, \overline{2+k}\alpha) + \Gamma(p, \overline{2+2k}\alpha) + \cdots + \Gamma(p, \overline{2+(n-1)k}\alpha)\right\} - \\ &\qquad \frac{1}{\Gamma(p)} \left\{\Gamma(p, 3\alpha) + \Gamma(p, \overline{3+k}\alpha) + \Gamma(p, \overline{3+2k}\alpha) + \cdots + \Gamma(p, \overline{3+(n-1)k}\alpha)\right\} \\ &\ldots\ldots\ldots\ldots \end{aligned}$$

$$
\begin{aligned}
P(k-1) &= Q(X=k-1)+Q(X=2k-1)+Q(X=3k-1)+\cdots+Q(X=nk-1)\\
&= \frac{1}{\Gamma(p)}\left\{\Gamma(p,\overline{k-1}\alpha)+\Gamma(p,\overline{2k-1}\alpha)+\Gamma(p,\overline{3k-1}\alpha)+\cdots+\Gamma(p,\overline{nk-1}\alpha)\right\}-\\
&\quad \frac{1}{\Gamma(p)}\left\{\Gamma(p,k\alpha)+\Gamma(p,2k\alpha)+\Gamma(p,3k\alpha)+\cdots+(p,\overline{n-1}k\alpha)\right\}
\end{aligned}
$$

### 3.1.3 Method 3

The finite population on $0,1,2,\ldots,N-1$ approximates a countably infinite population on $0,1,2,\ldots,\infty$ after truncating the population beyond $N$–1. This is done by eliminating the $N^{th}$, $(N+1)^{th}$, $(N+2)^{th},\ldots$ units of the population.

The PMF of truncated gamma distribution is given by

$$
\begin{aligned}
g(X=i) &= \frac{\frac{1}{\Gamma(p)}\left\{\Gamma(p,i\alpha)-\Gamma(p,\overline{i+1}\alpha)\right\}}{\sum_{i=0}^{N-1}\frac{1}{\Gamma(p)}\left\{\Gamma(p,i\alpha)-\Gamma(p,\overline{i+1}\alpha)\right\}};\ \text{where } N=nk \text{ and } i=0,1,2,\ldots,N-1\\
&= \frac{1}{L}\left\{\Gamma(p,i\alpha)-\Gamma(p,\overline{i+1}\alpha)\right\};\ \text{where } L=\Gamma(p)-\Gamma(p,N\alpha)
\end{aligned}
$$

$P(i)=Pr($selecting a sample having $i^{th}$ unit of the population in this method$)$; $where\ i=0,1,2,\ldots,k$–1.

Now,

$$
\begin{aligned}
P(0) &= g(X=0)+g(X=k)+g(X=2k)+\cdots+g(X=\overline{n-1}k)\\
&= \frac{1}{L}\left\{\Gamma(p)+\Gamma(p,k\alpha)+\Gamma(p,2k\alpha)+\cdots+\Gamma(p,\overline{n-1}k\alpha)\right\}-\\
&\qquad \frac{1}{L}\left[\Gamma(p,\alpha)+\Gamma(p,\overline{k+1}\alpha)+\Gamma(p,\overline{2k+1}\alpha)+\cdots+\Gamma\left\{p,(1+\overline{n-1}k)\alpha\right\}\right]\\
P(1) &= g(X=1)+g(X=1+k)+g(X=1+2k)+\cdots+g(X=1+\overline{n-1}k)\\
&= \frac{1}{L}\left\{\Gamma(p,\alpha)+\Gamma(p,\overline{1+k}\alpha)+\Gamma(p,\overline{1+2k}\alpha)+\cdots+\Gamma(p,\overline{1+(n-1)k}\alpha)\right\}-\\
&\qquad \frac{1}{L}\left\{\Gamma(p,2\alpha)+\Gamma(p,\overline{2+k}\alpha)+\Gamma(p,\overline{2+2k}\alpha)+\cdots+\Gamma(p,\overline{2+(n-1)k}\alpha)\right\}\\
P(2) &= g(X=2)+g(X=2+k)+g(X=2+2k)+\cdots+g(X=2+\overline{n-1}k)\\
&= \frac{1}{L}\left\{\Gamma(p,2\alpha)+\Gamma(p,\overline{2+k}\alpha)+\Gamma(p,\overline{2+2k}\alpha)+\cdots+\Gamma(p,\overline{2+(n-1)k}\alpha)\right\}-\\
&\qquad \frac{1}{L}\left\{\Gamma(p,3\alpha)+\Gamma(p,\overline{3+k}\alpha)+\Gamma(p,\overline{3+2k}\alpha)+\cdots+\Gamma(p,\overline{3+(n-1)k}\alpha)\right\}\\
&\quad \cdots\cdots\cdots\cdots\\
P(k-1) &= g(X=k-1)+g(X=2k-1)+g(X=3k-1)+\cdots+g(X=nk-1)\\
&= \frac{1}{L}\left\{\Gamma(p,\overline{k-1}\alpha)+\Gamma(p,\overline{2k-1}\alpha)+\Gamma(p,\overline{3k-1}\alpha)+\cdots+\Gamma(p,\overline{nk-1}\alpha)\right\}-\\
&\quad \frac{1}{L}\left\{\Gamma(p,k\alpha)+\Gamma(p,2k\alpha)+\gamma(p,3k\alpha)+\cdots+\Gamma(p,\overline{n-1}k\alpha)\right\}
\end{aligned}
$$

# 4 Risk Calculation Through Simulation

Here the performances of the estimators will be studied by looking at the risk which evaluates the difference of actual from its estimated value on an average. In this section, we present a simulation study conducted to evaluate the performance of the proposed methodology described in the previous section. This comprehensive simulation study aims to provide insights into the performance of our proposed methodology across different scenarios, thereby demonstrating its robustness and reliability. Specifically, we compute the risk of the estimator $\hat{\alpha}$ numerically for various combinations of parameters $(p, \alpha)$. The sample sizes considered are $n = 10, 15$ and the sampling intervals are taken as $k = 2, 3, 4$.

For each combination of $(n, k)$, the risk values of $\hat{\alpha}$ are calculated using the risk expression (3.1) provided earlier. These risk calculations are performed under three different approximation methods. The results of these calculations are then presented in Tables (1) - (4). Each table comprises of different approximation method, illustrating the risk values associated with each combination of sample size and sampling interval.

From the above computations, we observe that Method 1 performs consistently better than Method 2, and Method 2, in turn, performs better than Method 3. This ranking holds true in most cases of the risk values. This is also expected from our intuition that this kind of approximation reflects the form of the underlying unimodal distribution as close as possible. Additionally, as the sampling interval $k$ increases, the precision of the estimates appears to decrease. This decrease in precision is likely due to the fact that larger sampling intervals mean that many systematic samples fail to accurately reflect the underlying distribution. Specifically, some samples will start after neighbours of mode and fail to capture the original distribution's characteristics.

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Therefore, it is important to choose the sampling interval $k$ carefully, taking into consideration the parameters of the distribution. This phenomenon can be observed in each table, where the risk values tend to decrease within each column as the mode of the distribution increases. This trend underscores the need for an appropriate choice of $k$ to ensure the accuracy and reliability of the estimates.

Table 1: Risk values of $\hat{\alpha}$ for different combinations of $(p, \alpha)$ under different approximation methods $(1, 2, 3)$ for sample of size $n = 10$.

| Parameters | | | $K = 2$ | | | $K = 3$ | | | $K = 4$ | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $p$ | $\alpha$ | Mode | 1 | 2 | 3 | 1 | 2 | 3 | 1 | 2 | 3 |
| 5.5 | 5.8 | 0.78 | 80.64 | 80.64 | 80.64 | 182.503 | 182.503 | 182.503 | 324.276 | 324.276 | 324.276 |
| 5.6 | 5.7 | 0.81 | 80.498 | 80.498 | 80.498 | 182.507 | 182.507 | 182.507 | 324.29 | 324.29 | 324.29 |
| 5.7 | 5.6 | 0.84 | 80.323 | 80.323 | 80.323 | 182.506 | 182.506 | 182.506 | 324.302 | 324.302 | 324.302 |
| 5.8 | 5.5 | 0.87 | 80.11 | 80.11 | 80.11 | 182.5 | 182.5 | 182.5 | 324.314 | 324.314 | 324.314 |
| 5.9 | 5.4 | 0.91 | 79.853 | 79.853 | 79.853 | 182.485 | 182.485 | 182.485 | 324.325 | 324.325 | 324.325 |
| 6 | 5.2 | 0.96 | 79.337 | 79.337 | 79.337 | 182.437 | 182.437 | 182.437 | 324.34 | 324.34 | 324.34 |
| 6.1 | 5.1 | 1 | 78.938 | 78.938 | 78.938 | 182.389 | 182.389 | 182.389 | 324.347 | 324.347 | 324.347 |
| 6.2 | 5 | 1.04 | 78.474 | 78.474 | 78.474 | 182.32 | 182.32 | 182.32 | 324.352 | 324.352 | 324.352 |
| 6.3 | 4.9 | 1.08 | 77.938 | 77.938 | 77.938 | 182.226 | 182.226 | 182.226 | 324.355 | 324.355 | 324.355 |
| 6.4 | 4.8 | 1.13 | 77.325 | 77.325 | 77.325 | 182.098 | 182.098 | 182.098 | 324.355 | 324.355 | 324.355 |
| 6.5 | 4.7 | 1.17 | 76.631 | 76.631 | 76.631 | 181.929 | 181.929 | 181.929 | 324.349 | 324.349 | 324.349 |
| 6.6 | 4.5 | 1.24 | 75.316 | 75.316 | 75.316 | 181.524 | 181.524 | 181.524 | 324.323 | 324.323 | 324.323 |
| 6.7 | 4.4 | 1.3 | 74.385 | 74.385 | 74.385 | 181.183 | 181.183 | 181.183 | 324.29 | 324.29 | 324.29 |
| 6.8 | 4.3 | 1.35 | 73.362 | 73.362 | 73.362 | 180.75 | 180.75 | 180.75 | 324.237 | 324.237 | 324.237 |
| 6.9 | 4.2 | 1.4 | 72.247 | 72.247 | 72.247 | 180.205 | 180.205 | 180.205 | 324.156 | 324.156 | 324.156 |
| 7 | 4.1 | 1.46 | 71.042 | 71.042 | 71.042 | 179.525 | 179.525 | 179.525 | 324.035 | 324.035 | 324.035 |
| 7.1 | 3.9 | 1.56 | 68.861 | 68.861 | 68.861 | 177.996 | 177.996 | 177.996 | 323.689 | 323.689 | 323.689 |
| 7.2 | 3.8 | 1.63 | 67.431 | 67.431 | 67.431 | 176.823 | 176.823 | 176.823 | 323.369 | 323.369 | 323.369 |
| 7.3 | 3.7 | 1.7 | 65.929 | 65.929 | 65.929 | 175.419 | 175.419 | 175.419 | 322.924 | 322.924 | 322.924 |
| 7.4 | 3.6 | 1.78 | 64.364 | 64.364 | 64.364 | 173.757 | 173.757 | 173.757 | 322.317 | 322.317 | 322.317 |
| 7.5 | 3.5 | 1.86 | 62.742 | 62.742 | 62.742 | 171.812 | 171.812 | 171.812 | 321.501 | 321.501 | 321.501 |
| 7.6 | 3.4 | 1.94 | 61.07 | 61.07 | 61.07 | 169.565 | 169.565 | 169.565 | 320.42 | 320.42 | 320.42 |
| 7.7 | 3.2 | 2.09 | 58.123 | 58.123 | 58.123 | 164.939 | 164.939 | 164.939 | 317.681 | 317.681 | 317.681 |
| 7.8 | 3.1 | 2.19 | 56.316 | 56.316 | 56.316 | 161.803 | 161.803 | 161.803 | 315.498 | 315.498 | 315.498 |
| 7.9 | 3 | 2.3 | 54.463 | 54.463 | 54.463 | 158.349 | 158.349 | 158.349 | 312.763 | 312.763 | 312.763 |
| 8 | 2.9 | 2.41 | 52.556 | 52.556 | 52.556 | 154.595 | 154.595 | 154.595 | 309.392 | 309.392 | 309.392 |
| 8.1 | 2.8 | 2.54 | 50.589 | 50.589 | 50.589 | 150.568 | 150.568 | 150.568 | 305.307 | 305.307 | 305.307 |
| 8.2 | 2.6 | 2.77 | 46.946 | 46.946 | 46.946 | 142.925 | 142.925 | 142.925 | 296.092 | 296.092 | 296.092 |
| 8.3 | 2.5 | 2.92 | 44.719 | 44.719 | 44.719 | 138.278 | 138.278 | 138.278 | 289.721 | 289.721 | 289.721 |
| 8.4 | 2.4 | 3.08 | 42.377 | 42.377 | 42.377 | 133.466 | 133.466 | 133.466 | 282.514 | 282.514 | 282.514 |
| 8.5 | 2.3 | 3.26 | 39.905 | 39.905 | 39.905 | 128.499 | 128.499 | 128.499 | 274.517 | 274.517 | 274.517 |
| 8.6 | 2.2 | 3.45 | 37.29 | 37.29 | 37.29 | 123.369 | 123.369 | 123.369 | 265.811 | 265.811 | 265.811 |
| 8.7 | 2 | 3.85 | 32.154 | 32.154 | 32.154 | 113.557 | 113.557 | 113.557 | 248.586 | 248.586 | 248.586 |
| 8.8 | 1.9 | 4.11 | 29.066 | 29.066 | 29.066 | 107.669 | 107.669 | 107.669 | 238.356 | 238.356 | 238.356 |
| 8.9 | 1.8 | 4.39 | 25.817 | 25.817 | 25.817 | 101.387 | 101.387 | 101.387 | 227.733 | 227.733 | 227.733 |
| 9 | 1.7 | 4.71 | 22.42 | 22.42 | 22.42 | 94.63 | 94.63 | 94.63 | 216.691 | 216.691 | 216.691 |
| 9.1 | 1.6 | 5.06 | 18.901 | 18.901 | 18.901 | 87.331 | 87.331 | 87.331 | 205.126 | 205.126 | 205.126 |
| 9.2 | 1.5 | 5.47 | 15.31 | 15.31 | 15.31 | 79.453 | 79.453 | 79.453 | 192.869 | 192.869 | 192.869 |
| 9.3 | 1.3 | 6.38 | 8.725 | 8.725 | 8.725 | 63.253 | 63.253 | 63.253 | 167.47 | 167.47 | 167.47 |
| 9.4 | 1.2 | 7 | 5.567 | 5.566 | 5.567 | 54.05 | 54.05 | 54.05 | 152.472 | 152.472 | 152.472 |
| 9.5 | 1.1 | 7.73 | 2.971 | 2.97 | 2.972 | 44.893 | 44.893 | 44.893 | 136.916 | 136.916 | 136.916 |
| 9.6 | 1 | 8.6 | 1.178 | 1.176 | 1.18 | 36.239 | 36.239 | 36.239 | 121.47 | 121.47 | 121.47 |
| 9.7 | 0.9 | 9.67 | 0.33 | 0.328 | 0.332 | 28.603 | 28.602 | 28.603 | 107.059 | 107.059 | 107.059 |
| 9.8 | 0.7 | 12.57 | 0.845 | 0.797 | 0.884 | 18.256 | 18.229 | 18.271 | 85.803 | 85.802 | 85.804 |
| 9.9 | 0.6 | 14.83 | 1.369 | 1.219 | 1.588 | 15.237 | 15.112 | 15.331 | 79.148 | 79.126 | 79.157 |
| 10 | 0.5 | 18 | 1.391 | 1.131 | 2.087 | 13.316 | 12.857 | 13.822 | 75.374 | 75.132 | 75.509 |

Table 2: Risk values of $\hat{\alpha}$ for different combinations of $(p, \alpha)$ under different approximation methods $(1, 2, 3)$ for sample of size $n = 15$.

| Parameters | | | $K = 2$ | | | $K = 3$ | | | $K = 4$ | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $p$ | $\alpha$ | Mode | 1 | 2 | 3 | 1 | 2 | 3 | 1 | 2 | 3 |
| 8.5 | 3 | 2.5 | 147.909 | 147.909 | 147.909 | 393.51 | 393.51 | 393.51 | 758.32 | 758.32 | 758.32 |
| 8.6 | 3 | 2.53 | 147.072 | 147.072 | 147.072 | 391.889 | 391.889 | 391.889 | 756.867 | 756.867 | 756.867 |
| 8.7 | 2.9 | 2.66 | 143.844 | 143.844 | 143.844 | 385.335 | 385.335 | 385.335 | 749.873 | 749.873 | 749.873 |
| 8.8 | 2.8 | 2.79 | 140.465 | 140.465 | 140.465 | 378.448 | 378.448 | 378.448 | 741.656 | 741.656 | 741.656 |
| 8.9 | 2.7 | 2.93 | 136.904 | 136.904 | 136.904 | 371.277 | 371.277 | 371.277 | 732.17 | 732.17 | 732.17 |
| 9 | 2.6 | 3.08 | 133.132 | 133.132 | 133.132 | 363.86 | 363.86 | 363.86 | 721.414 | 721.414 | 721.414 |
| 9.1 | 2.5 | 3.24 | 129.116 | 129.116 | 129.116 | 356.216 | 356.216 | 356.216 | 709.438 | 709.438 | 709.438 |
| 9.2 | 2.4 | 3.42 | 124.826 | 124.826 | 124.826 | 348.334 | 348.334 | 348.334 | 696.345 | 696.345 | 696.345 |
| 9.3 | 2.4 | 3.46 | 123.911 | 123.911 | 123.911 | 346.633 | 346.633 | 346.633 | 693.561 | 693.561 | 693.561 |
| 9.4 | 2.3 | 3.65 | 119.29 | 119.29 | 119.29 | 338.482 | 338.482 | 338.482 | 679.394 | 679.394 | 679.394 |
| 9.5 | 2.2 | 3.86 | 114.332 | 114.332 | 114.332 | 329.959 | 329.959 | 329.959 | 664.467 | 664.467 | 664.467 |
| 9.6 | 2.1 | 4.1 | 109.009 | 109.009 | 109.009 | 320.941 | 320.941 | 320.941 | 648.927 | 648.927 | 648.927 |
| 9.7 | 2 | 4.35 | 103.291 | 103.291 | 103.291 | 311.279 | 311.279 | 311.279 | 632.858 | 632.858 | 632.858 |
| 9.8 | 1.9 | 4.63 | 97.15 | 97.15 | 97.15 | 300.813 | 300.813 | 300.813 | 616.245 | 616.245 | 616.245 |
| 9.9 | 1.9 | 4.68 | 96.119 | 96.119 | 96.119 | 299.038 | 299.038 | 299.038 | 613.411 | 613.411 | 613.411 |
| 10 | 1.8 | 5 | 89.511 | 89.511 | 89.511 | 287.546 | 287.546 | 287.546 | 596.135 | 596.135 | 596.135 |
| 10.1 | 1.7 | 5.35 | 82.439 | 82.439 | 82.439 | 274.948 | 274.948 | 274.948 | 577.855 | 577.855 | 577.855 |
| 10.2 | 1.6 | 5.75 | 74.905 | 74.905 | 74.905 | 261.156 | 261.156 | 261.156 | 558.19 | 558.19 | 558.19 |
| 10.3 | 1.5 | 6.2 | 66.947 | 66.947 | 66.947 | 246.151 | 246.151 | 246.151 | 536.779 | 536.779 | 536.779 |
| 10.4 | 1.4 | 6.71 | 58.661 | 58.662 | 58.662 | 230.013 | 230.013 | 230.013 | 513.42 | 513.42 | 513.42 |
| 10.5 | 1.3 | 7.31 | 50.23 | 50.231 | 50.231 | 212.976 | 212.976 | 212.976 | 488.219 | 488.219 | 488.219 |
| 10.6 | 1.3 | 7.38 | 49.279 | 49.281 | 49.281 | 211.011 | 211.011 | 211.011 | 485.288 | 485.288 | 485.288 |
| 10.7 | 1.2 | 8.08 | 41.078 | 41.085 | 41.085 | 193.619 | 193.619 | 193.619 | 458.888 | 458.888 | 458.888 |
| 10.8 | 1.1 | 8.91 | 33.414 | 33.44 | 33.44 | 176.527 | 176.527 | 176.527 | 432.375 | 432.375 | 432.375 |
| 10.9 | 1 | 9.9 | 26.667 | 26.749 | 26.749 | 160.632 | 160.633 | 160.633 | 407.24 | 407.24 | 407.24 |
| 11 | 0.9 | 11.11 | 21.1 | 21.341 | 21.344 | 146.899 | 146.902 | 146.902 | 385.148 | 385.148 | 385.148 |
| 11.1 | 0.8 | 12.63 | 16.731 | 17.363 | 17.384 | 136.103 | 136.137 | 136.137 | 367.561 | 367.561 | 367.561 |
| 11.2 | 0.8 | 12.75 | 16.445 | 17.106 | 17.128 | 135.378 | 135.416 | 135.416 | 366.373 | 366.374 | 366.374 |
| 11.3 | 0.7 | 14.71 | 13.026 | 14.559 | 14.674 | 127.998 | 128.29 | 128.292 | 354.575 | 354.581 | 354.581 |
| 11.4 | 0.6 | 17.33 | 9.784 | 12.82 | 13.338 | 122.302 | 124.178 | 124.208 | 347.646 | 347.763 | 347.763 |
| 11.5 | 0.5 | 21 | 6.211 | 10.874 | 12.784 | 112.705 | 121.9 | 122.387 | 342.936 | 344.67 | 344.683 |
| 11.6 | 0.4 | 26.5 | 2.731 | 7.409 | 12.672 | 86.734 | 116.577 | 121.911 | 326.065 | 343.096 | 343.721 |

Table 3: Risk values of $\hat{\alpha}$ for different combinations of $(p, \alpha)$ under different approximation methods $(1, 2, 3)$, for sample of size $n = 20$.

| Parameters | | | $K = 2$ | | | $K = 3$ | | | $K = 4$ | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $p$ | $\alpha$ | Mode | 1 | 2 | 3 | 1 | 2 | 3 | 1 | 2 | 3 |
| 5 | 0.1 | 40 | 67.293 | 345.069 | 929.697 | 99.431 | 300.566 | 420.405 | 49.539 | 100.241 | 111.333 |
| 5.1 | 0.1 | 41 | 64.936 | 350.394 | 991.64 | 102.511 | 323.495 | 462.353 | 56.904 | 119.011 | 133.305 |
| 5.2 | 0.1 | 42 | 62.44 | 354.647 | 1055.58 | 105.088 | 346.383 | 506.298 | 64.326 | 139.16 | 157.274 |
| 5.3 | 0.1 | 43 | 59.836 | 357.833 | 1121.52 | 107.161 | 369.119 | 552.24 | 71.723 | 160.615 | 183.241 |
| 5.4 | 0.2 | 22 | 21.15 | 48.982 | 56.439 | 5.157 | 6.842 | 6.922 | 144.433 | 157.397 | 157.503 |
| 5.5 | 0.2 | 22.5 | 22.96 | 55.123 | 64.181 | 3.425 | 4.612 | 4.671 | 132.289 | 145.142 | 145.253 |
| 5.6 | 0.2 | 23 | 24.705 | 61.524 | 72.422 | 2.109 | 2.879 | 2.92 | 120.712 | 133.388 | 133.503 |
| 5.7 | 0.2 | 23.5 | 26.369 | 68.163 | 81.163 | 1.186 | 1.644 | 1.669 | 109.702 | 122.135 | 122.254 |
| 5.8 | 0.2 | 24 | 27.942 | 75.017 | 90.403 | 0.637 | 0.903 | 0.918 | 99.258 | 111.382 | 111.504 |
| 5.9 | 0.2 | 24.5 | 29.413 | 82.061 | 100.142 | 0.44 | 0.654 | 0.667 | 89.38 | 101.129 | 101.254 |
| 6 | 0.2 | 25 | 30.772 | 89.272 | 110.381 | 0.572 | 0.896 | 0.915 | 80.066 | 91.378 | 91.505 |
| 6.1 | 0.3 | 17 | 0.577 | 0.922 | 0.943 | 78.387 | 84.664 | 84.695 | 364.882 | 368.608 | 368.61 |
| 6.2 | 0.3 | 17.33 | 0.964 | 1.569 | 1.608 | 72.407 | 78.663 | 78.695 | 351.986 | 355.942 | 355.943 |
| 6.3 | 0.3 | 17.67 | 1.464 | 2.43 | 2.496 | 66.677 | 72.884 | 72.918 | 339.307 | 343.498 | 343.499 |
| 6.4 | 0.3 | 18 | 2.065 | 3.503 | 3.606 | 61.198 | 67.327 | 67.362 | 326.848 | 331.275 | 331.277 |
| 6.5 | 0.3 | 18.33 | 2.757 | 4.785 | 4.938 | 55.969 | 61.992 | 62.029 | 314.607 | 319.275 | 319.277 |
| 6.6 | 0.3 | 18.67 | 3.529 | 6.273 | 6.492 | 50.99 | 56.88 | 56.918 | 302.587 | 307.498 | 307.5 |
| 6.7 | 0.3 | 19 | 4.372 | 7.967 | 8.269 | 46.262 | 51.99 | 52.029 | 290.788 | 295.942 | 295.944 |
| 6.8 | 0.4 | 14.5 | 5.221 | 6.48 | 6.501 | 153.936 | 156.915 | 156.917 | 506.954 | 507.498 | 507.498 |
| 6.9 | 0.4 | 14.75 | 4.215 | 5.294 | 5.314 | 147.617 | 150.728 | 150.729 | 495.718 | 496.311 | 496.311 |
| 7 | 0.4 | 15 | 3.33 | 4.234 | 4.251 | 141.424 | 144.665 | 144.667 | 484.603 | 485.249 | 485.249 |
| 7.1 | 0.4 | 15.25 | 2.563 | 3.299 | 3.314 | 135.358 | 138.727 | 138.729 | 473.61 | 474.312 | 474.312 |
| 7.2 | 0.4 | 15.5 | 1.91 | 2.489 | 2.501 | 129.42 | 132.915 | 132.917 | 462.738 | 463.5 | 463.5 |
| 7.3 | 0.4 | 15.75 | 1.367 | 1.804 | 1.813 | 123.611 | 127.227 | 127.229 | 451.988 | 452.813 | 452.813 |
| 7.4 | 0.5 | 12.8 | 20.132 | 22.331 | 22.34 | 215.858 | 216.757 | 216.757 | 611.282 | 611.339 | 611.339 |
| 7.5 | 0.5 | 13 | 18.349 | 20.491 | 20.5 | 209.955 | 210.917 | 210.917 | 601.436 | 601.5 | 601.5 |
| 7.6 | 0.5 | 13.2 | 16.655 | 18.731 | 18.74 | 204.13 | 205.157 | 205.157 | 591.67 | 591.741 | 591.741 |
| 7.7 | 0.5 | 13.4 | 15.051 | 17.051 | 17.06 | 198.383 | 199.477 | 199.477 | 581.983 | 582.062 | 582.062 |
| 7.8 | 0.5 | 13.6 | 13.535 | 15.451 | 15.46 | 192.712 | 193.877 | 193.877 | 572.376 | 572.462 | 572.462 |
| 7.9 | 0.5 | 13.8 | 12.107 | 13.931 | 13.94 | 187.12 | 188.357 | 188.357 | 562.847 | 562.942 | 562.942 |
| 8 | 0.5 | 14 | 10.767 | 12.491 | 12.5 | 181.605 | 182.917 | 182.917 | 553.397 | 553.503 | 553.503 |

Table 4: Risk values of $\hat{\alpha}$ for different combinations of $(p, \alpha)$ under different approximation methods $(1, 2, 3)$ for sample of size $n = 30$.

| Parameters | | | $K = 2$ | | | $K = 3$ | | | $K = 4$ | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $p$ | $\alpha$ | Mode | 1 | 2 | 3 | 1 | 2 | 3 | 1 | 2 | 3 |
| 25.2 | 0.3 | 80.67 | 0 | 186.459 | 2966.48 | 0.051 | 1031.38 | 1559.25 | 1.232 | 586.574 | 601.33 |
| 25.3 | 0.4 | 60.75 | 0.005 | 480.416 | 1138.02 | 0.674 | 343.104 | 352.133 | 0.657 | 15.31 | 15.312 |
| 25.4 | 0.5 | 48.8 | 0.063 | 373.489 | 453.642 | 0.944 | 40.332 | 40.356 | 20.854 | 76.94 | 76.94 |
| 25.5 | 0.6 | 40.83 | 0.282 | 164.477 | 169.209 | 0.653 | 4.667 | 4.667 | 174.796 | 290.25 | 290.25 |
| 25.6 | 0.7 | 35.14 | 0.516 | 50.121 | 50.252 | 21.423 | 63.529 | 63.529 | 448.805 | 526.969 | 526.969 |
| 25.7 | 0.8 | 30.88 | 0.27 | 7.139 | 7.141 | 90.247 | 153.807 | 153.807 | 721.113 | 750.641 | 750.641 |
| 25.8 | 0.9 | 27.56 | 0.122 | 0.944 | 0.944 | 198.886 | 251.361 | 251.361 | 944.501 | 951.944 | 951.944 |
| 25.9 | 1 | 24.9 | 3.145 | 13.21 | 13.21 | 316.552 | 346.627 | 346.627 | 1128.82 | 1130.21 | 1130.21 |
| 26 | 1.1 | 22.73 | 13.544 | 34.632 | 34.632 | 422.745 | 435.958 | 435.958 | 1287.24 | 1287.45 | 1287.45 |
| 26.1 | 1.2 | 20.92 | 33.485 | 60.312 | 60.313 | 513.517 | 518.229 | 518.229 | 1426.28 | 1426.31 | 1426.31 |
| 26.2 | 1.3 | 19.38 | 61.576 | 87.601 | 87.601 | 591.985 | 593.402 | 593.402 | 1549.37 | 1549.37 | 1549.37 |
| 26.3 | 1.4 | 18.07 | 94.101 | 115.046 | 115.046 | 661.521 | 661.891 | 661.891 | 1658.90 | 1658.90 | 1658.90 |
| 26.4 | 1.5 | 16.93 | 127.259 | 141.86 | 141.86 | 724.191 | 724.277 | 724.277 | 1756.86 | 1756.86 | 1756.86 |
| 26.5 | 1.6 | 15.94 | 158.575 | 167.629 | 167.629 | 781.153 | 781.171 | 781.171 | 1844.87 | 1844.87 | 1844.86 |
| 26.6 | 1.6 | 16 | 156.706 | 166.016 | 166.016 | 777.663 | 777.682 | 777.682 | 1839.50 | 1839.50 | 1839.50 |
| 26.7 | 1.7 | 15.12 | 185.266 | 190.528 | 190.528 | 829.764 | 829.768 | 829.768 | 1919.15 | 1919.15 | 1919.15 |
| 26.8 | 1.8 | 14.33 | 211.004 | 213.735 | 213.735 | 877.484 | 877.485 | 877.485 | 1991.38 | 1991.38 | 1991.38 |
| 26.9 | 1.9 | 13.63 | 234.315 | 235.63 | 235.63 | 921.31 | 921.311 | 921.311 | 2057.19 | 2057.19 | 2057.19 |
| 27 | 2 | 13 | 255.657 | 256.25 | 256.25 | 961.666 | 961.666 | 961.666 | 2117.39 | 2117.39 | 2117.39 |
| 27.1 | 2.1 | 12.43 | 275.4 | 275.652 | 275.652 | 998.922 | 998.922 | 998.922 | 2172.65 | 2172.65 | 2172.65 |
| 27.2 | 2.2 | 11.91 | 293.803 | 293.905 | 293.905 | 1033.40 | 1033.40 | 1033.40 | 2223.49 | 2223.49 | 2223.49 |
| 27.3 | 2.3 | 11.43 | 311.043 | 311.082 | 311.082 | 1065.4 | 1065.4 | 1065.4 | 2270.3 | 2270.3 | 2270.3 |

## 5 Behaviour of the Estimate of Scale Parameter in Systematic and Original Population

The method of moments (MME) is a statistical technique where parameters of a population distribution are estimated by equating sample moments (such as the sample mean or variance) to the corresponding population moments. Here, we have applied the MME method to estimate the scale parameter of the gamma distribution by equating the population mean i.e. the mean of discrete gamma distribution to the systematic sample mean. The mean of discrete gamma distribution is

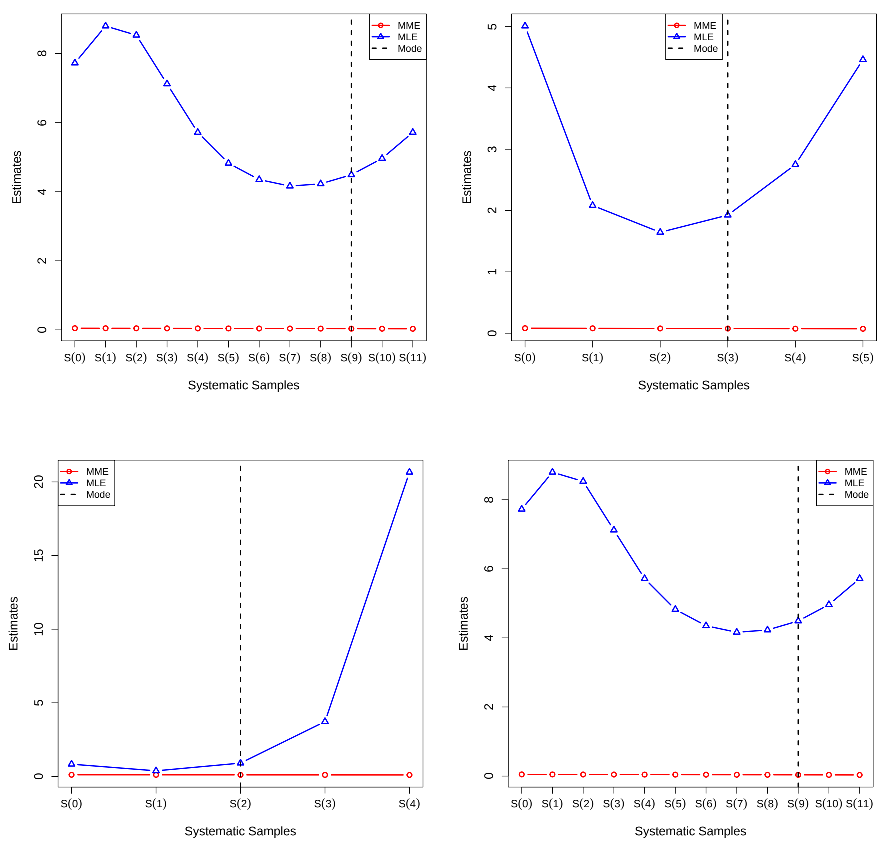


Figure 1: The MLE and MME estimate of the scale parameter $\alpha$. From upper-left to lower-right: When (a) p=5.0, $\alpha = 0.5$, (b) p = 3.5, $\alpha = 1.0$, (c) p = 4.0, $\alpha = 2.5$, (d) p = 5.0, $\alpha = 2.0$.

$$\frac{\sum_{i=0}^{\infty} \Gamma(p, \overline{i+1}\alpha)}{\Gamma(p)},$$

and the corresponding mean of systematic sample $X + \frac{(n-1)k}{2}$; where $X = 0, 1, 2, \ldots, k-1$.

Now equating these two means, we have the following,

$$\frac{\sum_{i=0}^{\infty}\Gamma(p,\overline{i+1}\alpha)}{\Gamma(p)} = X + \frac{(n-1)k}{2} \tag{5.1}$$

By solving (5.1) for different values of $X$, we get $k$ number of estimates of $\alpha$. For $X = i; i = 0, 1, 2, \ldots, k-1$, the corresponding estimate of $\alpha$ is denoted by $\hat{\alpha}_i$. For $X = i$, the corresponding systematic sample consists of $i, i+k, i+2k, \ldots, i+(n-1)k$. Based on this sample, we can find $\hat{\alpha}_i( i = 0, 1, 2, \ldots, k-1)$, the maximum likelihood estimates of $\alpha$.

We plot both estimators corresponding to each of the systematic samples for various parameter choices in order to compare the behavior of the MLE and MME estimators. Figure (1) illustrates that when the mode or its neighbours are included in the systematic sample, the moment approach described above produces estimates of $\alpha$ that are reasonably close to the original parameter. So, the classical moment estimates are close to our suggested estimator.

On the other hand, the MLE assuming the same samples if they were drawn at random would perform differently if it excluded the mode's neighbours but similarly close to the mode otherwise. This comparison highlights the sensitivity of both estimators to the sample's composition relative to the mode of the distribution. While the MME tends to be more stable around the mode, the MLE's accuracy is more contingent on the sample including or being close to the mode.

# 6 Real Data Application

In this section, With a view to demonstrate the methodology discussed in this paper, we consider a real-life dataset comprising average monthly rainfall observations given in the following table.

Table 5: Historical rainfall averages over last 56 years in State of São Paulo.

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.2 | 3.5 | 2.8 | 3.7 | 8.7 | 6.9 | 7.4 | 0.8 | 4.8 | 2.5 | 2.9 | 3.1 | 4.0 | 5.0 | 3.8 | 3.5 |
| 5.4 | 3.3 | 2.9 | 1.7 | 7.3 | 2.9 | 4.6 | 1.1 | 1.4 | 3.9 | 6.2 | 4.1 | 10.8 | 3.8 | 7.3 | 1.8 |
| 6.7 | 3.5 | 3.2 | 5.2 | 2.8 | 5.2 | 5.4 | 2.2 | 9.9 | 2.1 | 4.7 | 5.5 | 2.6 | 4.1 | 5.4 | 5.5 |
| 2.1 | 1.9 | 8.8 | 1.3 | 24.1 | 5.4 | 6.2 | 2.9 | | | | | | | | |

The above data set (Moala et al., 2013) obtained from the Information System for Management of Water Resources, represents the monthly average rainfall of the month of November for a period of 56 years from 1947 to 2003 of the State of Sao Paulo of the USA. At first, we check whether the given data set arises from a two-parameter Gamma distribution with the help of the Goodness-of-fit testing procedure which measures the gap between the available data and a statistical model (Banerjee et al., 2024; Bhunia and Banerjee, 2022). Pearson's $\chi^2$ statistic and Kolmogorov–Smirnov test are widely applied to examine how good a model is in fitting to a given data set. Here our focus is on Kolmogorov–Smirnov (K-S) distance between the fitted distribution and the empirical distribution due to the fact that for a data set with limited number sample observations, the performance of the Pearson's $\chi^2$ test may not be satisfactory. The K-S distance value for the given data set is found

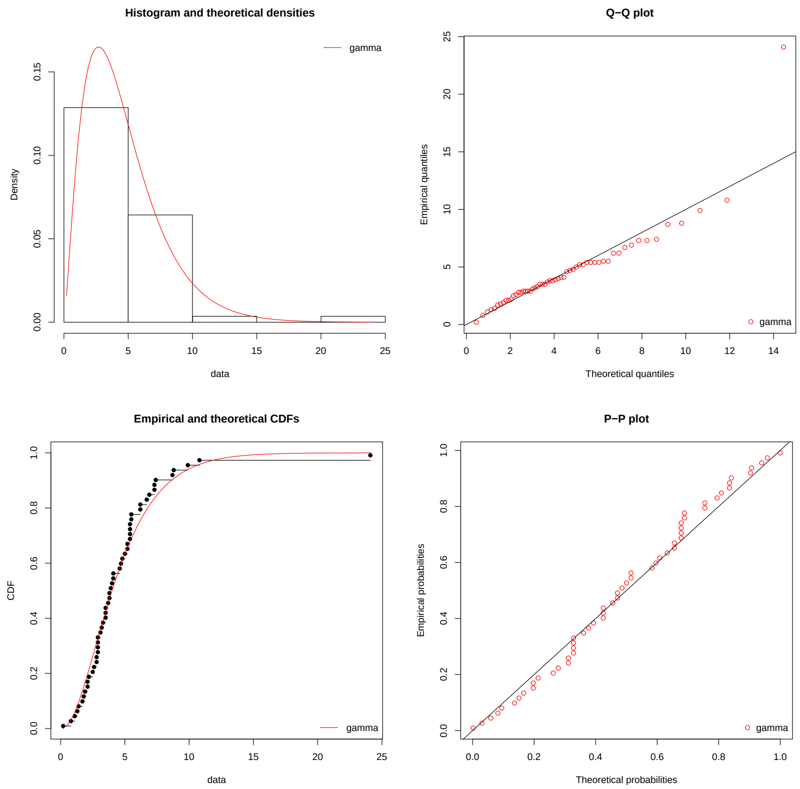


Figure 2: Diagnostic plots of fitted Gamma distribution for Rainfall data.

to be $0.0978$. The fitting of the Gamma distribution to the given data set has also been examined employing different diagnostics plots given in Figure (2).

For the data set, the maximum likelihood estimate of the parameters $\alpha$ and $p$ come out to be $\hat{\alpha} = 0.518$ and $\hat{p} = 2.396$ and hence the estimated value of the mean and the mode of the gamma distribution are $4.625$ and $2.695$ respectively. Corresponding to this continuous Gamma distribution $G(2.396, 0.518)$, the probability mass function of discrete Gamma distribution with same values of the parameters is given by,

$$Q(X = i) = \frac{\Gamma(2.396, 0.518i) - \Gamma(2.396, 0.518(i+1))}{\Gamma(2.396)}; i = 0, 1, 2, \ldots$$

The mean of this discrete Gamma distribution is $4.125$.

From the above table and plots, it is clear that for the discrete Gamma distribution $\sum_{i=0}^{11} Q(X = i)$ or $Q(X \leq 11)$ is nearly 1 and mode is approximately equal to 3. For drawing systematic samples, we take population size $N = 12$. If we take the sampling interval $k = 3$ which is greater than the modal value of the parent distribution then sample size $n(= \frac{N}{K})$ becomes $4$. Using the expression of the sample mean given in Equation 3.1, the value of the systematic sample mean in this case is

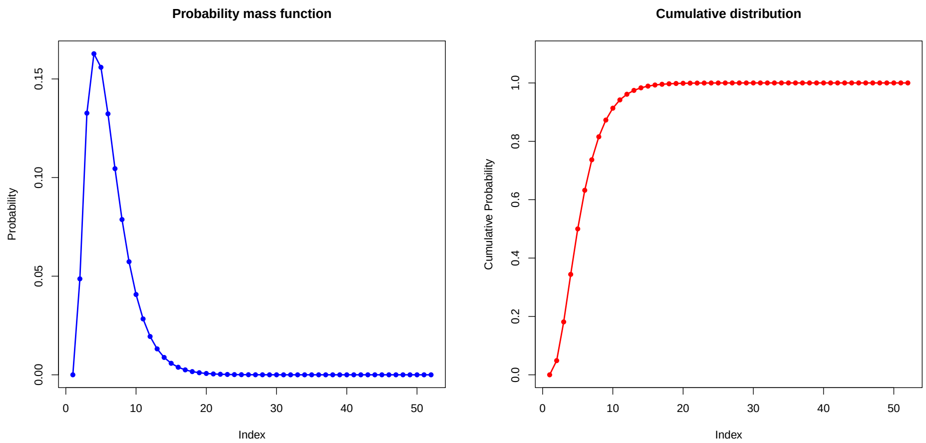


Figure 3: PMF and CDF of discretized gamma.

Table 6: Cumulative probability distribution table

| i | $Q(X = i)$ | $Q(X \leq i)$ |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 0.04865 | 0.04865 |
| 2 | 0.13265 | 0.18130 |
| 3 | 0.16276 | 0.34406 |
| 4 | 0.15589 | 0.49995 |
| 5 | 0.13230 | 0.63225 |
| 6 | 0.10451 | 0.73676 |
| 7 | 0.07872 | 0.81548 |
| 8 | 0.05733 | 0.88281 |
| 9 | 0.04071 | 0.94352 |
| 10 | 0.02834 | 0.97186 |
| 11 | 0.01943 | 0.99029 |

either 4.5, 5.5 or 6.5. The range of estimates includes the mean of the original continuous gamma distribution as well as that of its discretized version. So it seems that our proposed technique is quite useful at least in this case.

# 7 Conclusion

In this article, we explore the concept of approximating a countably infinite population having discretized gamma distribution. This problem is approached from three different perspectives, and we find that the first method performs best in terms of risk. Intuitively, the first method is preferred because it assigns probability masses at regular intervals which gives all probability masses (may be negligible) to the finite units. As a result, it provides the most accurate approximation of the original distribution whereas the second method heaps the tail probability to the last unit of finite sample and the overall shape of the distribution will not change compared to the first one. But according to the third method, the masses are being divided by some factor to get truncated probability and the shape of the distribution will change and consequently role of mode and choice of the sampling interval will not align with our goal of selecting samples that include observations near the mode, thereby failing to accurately reflect the original distribution.

It should be important to note that as the sampling interval increases, the risk values also increase for all the methods considered. It occurs because many systematic samples fail to accurately capture the characteristics of the original distribution. Consequently, it is crucial to select the sampling interval carefully, ensuring that the parameters of the distribution are taken into account to maintain the accuracy and reliability of the estimates. Actually this article is a demonstration of the novel approach of using systematic sampling technique to estimate the mean parameter where we have distributional model in regulating the population. Additionally, it can be inferred that our approach will be helpful for asymmetric unimodal distribution. However, this approach will be modified for symmetric unimodal distribution in future.

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