Journal of Statistical Research *https://doi.org/10.3329/jsr.v59i2.88069*
2025, Vol. 59, No. 2, pp. 259-278 *ISSN 0256 - 422 X*

NONPARAMETRIC HAZARD ESTIMATION UNDER RANDOM CENSORING AND DEPENDENT DATA

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SUMMARY

This paper investigates the conditional hazard function of a scalar response variable given a predictor that takes values in a semi-metric space. We employ a local linear estimator for the conditional density and cumulative distribution function, and establish almost sure convergence rates under α-mixing dependence. The analysis is conducted under a set of regularity conditions for the proposed estimator. Furthermore, the practical relevance of the theoretical results is demonstrated through a simulation study.

Keywords and phrases: Functional data, local linear estimation, conditional hazard estimator, almost sure convergence, censored data, α-mixing dependeny.

AMS Classification: 62E20, 62G05, 62G07, 62G20, 62G30, 62N01, 62N02

1 Introduction

Estimating the hazard function plays a fundamental role in survival analysis, offering crucial insights into the instantaneous risk of event occurrence over time. However, the estimation process becomes substantially more complex when data are subject to censoring a common feature in survival studies where the exact time of failure is not observed for all subjects. The challenge is further amplified when the covariates are functional in nature, meaning each observation is a curve or a function defined over a continuum, such as time or space. This type of data arises frequently in longitudinal and spatial contexts, and is often characterized by intrinsic dependencies among observations.

Classical methods like the Kaplan-Meier estimator (Kaplan and Meier, 1958) and the Cox proportional hazards model (Cox, 1972) have served as foundational tools in survival analysis. While these methods are effective in handling scalar covariates, they typically assume independence between observations and do not directly accommodate the infinite-dimensional nature of functional data. To address this, nonparametric kernel-based approaches such as those proposed by Ramlau-Hansen (1983) and Tanner and Wong (1983) have been used to estimate hazard functions more flexibly. Nevertheless, kernel estimators often suffer from boundary bias and are sensitive to bandwidth selection.

A particularly relevant contribution in this context is the work of Quintela-Del-Rio (2008), who proposed a kernel-type estimator for the hazard function when covariates are functional and data exhibit strong mixing dependence. This estimator uses kernel smoothing in the presence of censoring and dependence, making it a pioneering effort in this area. However, like other kernel methods, it remains sensitive to boundary effects and bandwidth selection.

Local linear estimation emerged as a powerful alternative to overcome some of these limitations. By locally approximating the hazard function with a linear fit (Cleveland and Devlin, 1988; Fan and Gijbels, 1996), this approach achieves better bias control near boundaries and enhanced approximation accuracy, particularly when dealing with complex data structures.

Functional Data Analysis (FDA) has developed a suite of tools specifically designed to analyze data in the form of functions or curves (Ramsay and Silverman, 2005). Techniques such as Functional Principal Component Analysis (FPCA) and functional regression models (Ferraty and Vieu, 2006; Müller and Stadtmüller, 2005) have been widely adopted. However, the simultaneous presence of censoring, functional covariates, and dependency structures remains a relatively underexplored territory. While notable contributions have been made for example, by Gijbels and Wang (1997) on local polynomial estimation with censored data, and by Chiou and Müller (2009) and Hsing and Eubank (2015) on dependent functional data-few studies have tackled the joint complexity of all these features within a unified estimation framework.

This paper addresses this methodological gap by proposing a local linear estimator for the conditional hazard function in the presence of right-censored, dependent functional data. Our estimator is designed to account for the infinite-dimensional nature of functional predictors, the intricacies of dependent observations (modeled under α-mixing conditions), and the challenges introduced by censoring.

We rigorously establish almost sure consistency and convergence rates for the proposed estimator under a set of mild regularity conditions. To assess its finite-sample performance, we conduct a comprehensive simulation study in which functional covariates are generated from controlled stochastic processes. The local linear estimator is benchmarked against the K-nearest neighbors approach using Mean Squared Error (MSE) as the evaluation metric. The results demonstrate that our method achieves superior estimation accuracy, particularly in complex data settings.

The remainder of the paper is organized as follows. Section 2 introduces the proposed estimator for the conditional hazard function. Section 3 presents the main assumptions and theoretical results. Section 4 provides simulation results. Finally, Section 6 contains the technical proofs of the main theorems.

2 Construction of the Conditional Hazard Estimator Function

Let (X, Y) be a pair of random variables with values in $\mathcal{F} \times \mathbb{R}$, where $\mathcal{F}$ is a semi-metric space equipped with a semi-metric d. Consider a random sample of n pairs of random variables (X_i, Y_i) for $i = 1, \ldots, n$ drawn from (X, Y). In this paper, we address the problem of estimating the conditional distribution of Y given $X = x$ when the response variable (Y_i) for $i = 1, \ldots, n$ is subject to right censoring and the observations (X_i, Y_i) are strongly mixing.

Additionally, we denote the censoring random variables by (C_i) for $i = 1, \ldots, n$. These random variables are independent and identically distributed, with a common unknown continuous distribution function G. Consequently, we observe the triplets (X_i, T_i, δ_i) for $i = 1, \ldots, n$, where

$T_i = \min(Y_i, C_i)$ and $\delta_i = \mathbf{1}_{\{Y_i \le C_i\}}$ (where $\mathbf{1}$ denotes the indicator function of set A). We assume that the (Y_i) for $i = 1, \ldots, n$ and the (C_i) for $i = 1, \ldots, n$ are independent, ensuring the identifiability of the model.

Our objective in this paper is to estimate the conditional hazard function h^x using n dependent observations (X_i, Y_i) drawn from random variables with the same distribution as (X, Y), where the regular version F^x of the conditional distribution function of Y given $X = x$ exists for any $x \in \mathcal{F}$. Moreover, we suppose that F^x has a continuous density f^x with respect to Lebesgue's measure over $\mathbb{R}$. We adopt a functional local linear estimator for the conditional distribution function $\widehat{F}^x$, estimated by $\widehat{a}$, where the couple $(\widehat{a}, \widehat{b})$ is obtained by minimizing the following quantity:

$$\min_{(a,b)\in\mathbb{R}^2} \sum_{i=1}^{n} \left(\delta_i \overline{G}_n^{-1}(T_i) H(h^{-1}(y - T_i)) - a - b\beta(X_i, x)\right)^2 K(h_K^{-1}\delta(x, X_i)),$$

where $\beta(.,.)$ and $\delta(.,.)$ are known bi-functional operators defined from $\mathcal{F}^2$ into $\mathbb{R}$, such that

$$\forall \xi \in \mathcal{F}, \;\; \beta(\xi, \xi) = 0 \text{ and } d(\cdot, \cdot) = |\delta(\cdot, \cdot)|,$$

with K being a kernel function, H a distribution function, and $h_K = h_{K,n}$ (respectively, $h_H = h_{H,n}$) chosen as a sequence of positive real numbers. The estimator $\widehat{a}$, defined by the above minimization problem, can be explicitly given by

$$\widehat{F}^x(y) = \frac{\sum_{i,j=1}^{n} \delta_i \overline{G}_n^{-1}(T_j) W_{ij}(x) H(h_H^{-1}(y - T_j))}{\sum_{i,j=1}^{n} W_{ij}(x)}, \tag{2.1}$$

where

$$W_{ij}(x) = \beta(X_i, x)(\beta(X_i, x) - \beta(X_j, x)) K(h_K^{-1}\delta(x, X_i)) K(h_K^{-1}\delta(x, X_j)),$$

with the convention $0/0 = 0$.

Furthermore, the density function estimator $\widehat{f}^x$ of f^x can be obtained from the distribution function and is defined by

$$\widehat{f}^x(y) = \frac{\sum_{i,j=1}^{n} \delta_i \overline{G}_n^{-1}(T_j) W_{ij}(x) H'(h_H^{-1}(y - T_j))}{h_H \sum_{i,j=1}^{n} W_{ij}(x)}, \tag{2.2}$$

where H' is the derivative of H.

The Kaplan-Meier (1958) estimator is used to estimate the cumulative distribution function G of the censoring random variables, defined as

$$\overline{G}_n(y) = \begin{cases} \prod \left(1 - \frac{1-\delta_{(i)}}{n-i+1}\right)^{\mathbf{1}_{\{T_{(i)} \le y\}}} & \text{if } y \le T_{(n)}, \\ 0 & \text{otherwise}, \end{cases}$$

where $T_{(1)} < T_{(2)} < \ldots < T_{(n)}$ are the order statistics of T_i and $\delta_{(i)}$ is the concomitant of T_i.

Finally, we estimate the conditional hazard function h^x by

$$\widehat{h}^x(y) = \frac{\widehat{f}^x(y)}{1 - \widehat{F}^x(y)}, \quad \forall y \in \mathbb{R}.$$

3 Assumptions and Main Results

In this section, we investigate the almost sure consistency of the estimator $\widehat{h}^x$, utilizing the following notations to define the strong mixing property.

Let $\mathcal{F}_1^k$ denote the σ-algebra generated by $(X_1, Y_1), \ldots, (X_k, Y_k)$ and $\mathcal{F}_{k+n}^\infty$ denote the σ-algebra generated by $(X_{k+n}, Y_{k+n}), \ldots$. For any $n \geq 1$, define

$$\alpha(n) = \sup_{n \leq k \leq 1} \sup\{|\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)| : A \in \mathcal{F}_1^k, \quad B \in \mathcal{F}_{k+n}^\infty\}.$$

The process $(X_i, Y_i)_{i \geq 1}$ is termed α-mixing or strongly mixing if

$$\lim_{n \longrightarrow \infty} \alpha(n) = 0.$$

Rosenblatt first introduced the strong mixing condition in 1956. This condition, although relatively weak, has numerous practical applications, as elaborated in **?**.

We also propose some conditions necessary to state our main asymptotic results. Throughout this paper, for any distribution function L, let $\tau_L := \sup y : L(y) < 1$ be its support?s right endpoint, and $\tau < \min(\tau_G, \tau_L)$.

Hereafter, x (resp. y) will denote a fixed point in $\mathcal{F}$ (resp. $\mathbb{R}$), $\aleph_x$ (resp. $\aleph_y$) will denote a fixed neighborhood of a fixed point x (resp. y), and $\phi_x(r_1, r_2) = \mathbb{P}(r_2 < \delta(X, x) < r_1)$. We assume that our nonparametric model satisfies the following conditions:

(H1) For any $r > 0$, $\phi_x(r) := \phi_x(-r, r) > 0$.

(H2) The conditional distribution function F^x (resp. density function f^x) is such that: there exist some positive constants b_1 and b_2, $\forall (y_1, y_2) \in \aleph_y \times \aleph_y$ and $\forall (x_1, x_2) \in \aleph_x \times \aleph_x$:

$$|F^{x_1^{(j)}}(y_1) - F^{x_2^{(j)}}(y_2)| \leq C_x \left(\delta^{b_1}|(x_1, x_2)^+|y_1 - y_2|^{b_2}\right), \quad j = 0, 1$$

where C_x is a positive constant depending on x.

(H3) The function $\beta(\cdot, \cdot)$ is such that

$$\forall x, x' \in \mathcal{F}, \ \ C_1\delta(x, x') \leq |\beta(x, x')| \leq C_2\delta(x, x')|, \quad \text{where} C_1 > 0, C_2 > 0.$$

(H4) The sequence $(X_i, Y_i)_{i \in \mathbb{N}}$ satisfies: $\exists a > 0, \quad \exists c > 0, \quad \forall n \in \mathbb{N} \alpha(n) < cn^{-a}$ and

$$\max_{i \neq j} \mathbb{P}((X_i, X_j) \in B(x, h) \times B(x, h)) = \varphi(h) > 0.$$

(H5) The conditional distribution (resp. density) of (Y_i, Y_j) given (X_i, X_j) exists and is bounded.

(H6) The kernel K is a Positive, differentiable function which is supported within $(-1, 1)$.

(H7) The kernel H is a differentiable function which has a bounded first derivative such that

$$\int |t|^{b_2} H^{(j)}(t)dt < \infty \text{ and } \int {H^{(j)}}^2(t)dt < \infty. \quad \text{for } j \in \{0,1\}$$

$$\forall (y_1, y_2) \in \mathbb{R}^2, |H'(y_1) - H'(y_2)| \leq C|y_1 - y_2|,$$

H' is a bounded function.

(H8) The bandwidth h_K satisfies: that there exists a positive integer n_0, such that

$$\forall n > n_0, -\frac{1}{\phi_x(h_K)} \int_{-1}^{1} \phi(zh_K, h_K) \frac{d}{dz}(z^2 K(z))dz > C_3 > 0$$

and

$$h_K \int_{B(x,h_K)} \beta(u,x)dP(u) = o\bigg(\int_{B(x,h_K)} \beta^2(u,x)dP(u) \bigg),$$

where $B(x,r) = \{x' \in \mathcal{F}/d(x,x') \leq r\}$ and $dP(x)$ is the cumulative distribution of X.

(H9) The bandwidth h_H satisfies

$$\lim_{n\to\infty} n^\gamma h_H = \infty \text{ for some } \gamma > 0 \text{ and } \lim_{n\to\infty} h_H = 0.$$

(H10) (i) $\lim_{n\to\infty} h_K = 0, \ \lim_{n\to\infty} \frac{\psi_x^{(1/2)} h_K \log(n)}{n h_H^{(j)} \phi_x^2(h_K)} = 0, \quad \text{for } j \in \{0,1\}$

(ii) $Cn^{\frac{3-a}{a+1} + \frac{3\beta_1+1}{a+1}} \log n [\log_2 n]^{6/(a+1)} \leq \psi_x^{1/2}(h_K)$, where $\psi_x(h) = \max(\phi_x^2(h), \varphi_x(h))$.

It is important to note that conditions (H1), (H3), and (H8) are consistent with those used in **?** and **?**. Assumption (H2), a regularity condition, characterizes the functional space of our model and is necessary for evaluating the bias term in the asymptotic results of this paper. Additionally, conditions (H4), (H5), and (H10) are technical assumptions, as discussed in **?**.

Theorem 1. *Under assumptions (H1)-(H10), we obtain*

$$\sup_{y\in\Omega} |\widehat{h}^x(y) - h^x(y)| = O\left(h_K^{b_1} + h_H^{b_2}\right) + O\left(\sqrt{\frac{\psi_x^{(1/2)}(h_K)\log n}{n h_H \phi_x^2(h_K)}} \right), \qquad a.s.$$

Proof. The proof of theorem 1 is a direct consequence of the following decomposition

$$\widehat{h}^x(y) - h^x(y) = \frac{1}{1 - \widehat{F}^x(y)} \left[\widehat{f}^x(y) - f^x(y)\right] + \frac{h^x(y)}{1 - \widehat{F}^x(y)} \left[\widehat{F}^x(y) - F^x(y)\right].$$

And it's based on following results

$$\sup_{y\in\Omega} |\widehat{f}^x(y) - f^x(y)| = O\left(h_K^{b_1} + h_H^{b_2}\right) + O\left(\sqrt{\frac{\psi_x^{(1/2)}(h_K)\log n}{n h_H \phi_x^2(h_K)}} \right) \qquad a.s.$$

$$\sup_{y\in\Omega}|\widehat{F}^x(y)-F^x(y)| = O\left(h_K^{b_1}+h_H^{b_2}\right)+O\left(\sqrt{\frac{\psi_x^{(1/2)}(h_K)\log n}{n\phi_x^2(h_K)}}\right) \qquad a.s.$$

$$\exists\eta>0 \text{ such that } \sum_{n=1}^{\infty}\mathbb{P}\left\{|1-\widehat{F}^x(y)|<\eta\right\}<\infty.$$

Whose demonstrations are based, respectively, on the decompositions

$$\begin{aligned}\widehat{f}^x(y)-f^x(y) &= \frac{1}{\widehat{f}_D^x}\left\{(\widehat{f}_N^x(y)-\widetilde{f}_N^x(y))+(\widetilde{f}_N^x(y)-\mathbb{E}[\widetilde{f}_N^x(y)])+(\mathbb{E}[\widetilde{f}_N^x(y)]-f^x(y))\right\}\\ &\quad +\frac{f^x(y)}{\widehat{f}_D^x}\left(1-\widehat{f}_D^x\right),\end{aligned}$$

and

$$\begin{aligned}\widehat{F}^x(y)-F^x(y) &= \frac{1}{\widehat{f}_D^x}\left\{(\widehat{F}_N^x(y)-\widetilde{F}_N^x(y))+(\widetilde{F}_N^x(y)-\mathbb{E}[\widetilde{F}_N^x(y)])+(\mathbb{E}[\widetilde{F}_N^x(y)]-F^x(y))\right\}\\ &\quad +\frac{F^x(y)}{\widehat{f}_D^x}\left(1-\widehat{f}_D^x\right).\end{aligned}$$

Where

$$\widehat{f}_D^x=\frac{1}{n(n-1)\mathbb{E}\left[W_{12}(x)\right]}\sum_{i\neq j}W_{ij}(x),$$

and

$$\widehat{f}_N^x(y)=\frac{1}{n(n-1)h_H\mathbb{E}\left[W_{12}(x)\right]}\sum_{i\neq j}\delta_j\overline{G}_n^{-1}(T_j)W_{ij}(x)H(h_H^{-1}(y-T_j)),$$

$$\widetilde{f}_N^x(y)=\frac{1}{n(n-1)h_H\mathbb{E}\left[W_{12}(x)\right]}\sum_{i\neq j}\delta_j\overline{G}^{-1}(T_j)W_{ij}(x)H(h_H^{-1}(y-T_j)),$$

also

$$\widehat{F}_N^x(y)=\frac{1}{n(n-1)\mathbb{E}\left[W_{12}(x)\right]}\sum_{i\neq j}\delta_j\overline{G}_n^{-1}(T_j)W_{ij}(x)H(h_H^{-1}(y-T_j)),$$

$$\widetilde{F}_N^x(y)=\frac{1}{n(n-1)\mathbb{E}\left[W_{12}(x)\right]}\sum_{i\neq j}\delta_j\overline{G}^{-1}(T_j)W_{ij}(x)H(h_H^{-1}(y-T_j)),$$

Lemma 3.1. *Under assumptions (H1), (H3), (H4), (H6), (H8) and (h10) we have that*

$$1-\widehat{f}_D^x=O\left(\sqrt{\frac{\psi_x^{(1/2)}(h_K)\log n}{n\phi_x^2(h_K)}}\right),\ a.co.,$$

and

$$\exists\delta>0,\ \textit{such that}\ \sum_n\mathbb{P}\left(\widehat{f}_D^x<\delta\right)<\infty.$$

Lemma 3.2. *Under assumptions (H1), (H2) and (H7), we obtain*

$$\sup_{y\in\Omega} |\mathbb{E}[\widetilde{f}_N^x(y)] - f^x(y)| = O\left(h_K^{b_1} + h_H^{b_2}\right), \ \ a.s.$$

Lemma 3.3. *Under assumptions (H1), (H2) and (H7), we obtain*

$$\sup_{y\in\Omega} |\mathbb{E}[\widetilde{F}_N^x(y)] - F^x(y)| = O\left(h_K^{b_1} + h_H^{b_2}\right), \ \ a.s.$$

Lemma 3.4. *Under assumptions of Theorem 1, we get*

$$\sup_{y\in\Omega} |\widetilde{f}_N^x(y) - \mathbb{E}[\widetilde{f}_N^x(y)]| = O\left(\sqrt{\frac{\psi_x^{(1/2)}(h_K)\log n}{nh_H\phi_x^2(h_K)}}\right), \ \ a.co.$$

Lemma 3.5. *Under assumptions of Theorem 1, we get*

$$\sup_{y\in\Omega} |\widetilde{F}_N^x(y) - \mathbb{E}[\widetilde{F}_N^x(y)]| = O\left(\sqrt{\frac{\psi_x^{(1/2)}(h_K)\log n}{n\,\phi_x^2(h_K)}}\right), \ \ a.co.$$

Lemma 3.6. *Under assumptions (H1), (H6), (H8), (H9) and (H10) we obtain*

$$\sup_{y\in\Omega} |\widehat{f}_N^x(y) - \widetilde{f}_N^x(y)]| = O\left(\sqrt{\frac{\log(\log n)}{n}}\right), \ \ a.s.$$

Lemma 3.7. *Under assumptions (H1), (H6), (H8), (H9) and (H10) we obtain*

$$\sup_{y\in\Omega} |\widehat{F}_N^x(y) - \widetilde{F}_N^x(y)]| = O\left(\sqrt{\frac{\log(\log n)}{n}}\right), \ \ a.s.$$

4 Simulation Study

To assess the performance of the proposed local linear estimator for the conditional hazard function under censoring and dependence, we conduct a simulation study using α-mixing functional data generated via a functional autoregressive process. The simulation follows a structured sequence of steps, from data generation to prediction evaluation.

Step 1: Generating dependent functional data We simulate $n = 300$ functional curves $X_i(t)$ on the interval $[0, 1]$, generated via a functional autoregressive process of order 1 (FAR(1))

$$X_i(t) = \rho X_{i-1}(t) + \varepsilon_i(t), \quad i = 2, \ldots, n,$$

with $\rho = 0.8$. The initial curve is defined as

$$X_1(t) = \sin(4\pi(b_1 - t)) + a_1 t^2, \quad a_1 \sim \mathcal{N}(4, 3), \quad b_1 \sim \mathcal{N}(0, 1),$$

and $\varepsilon_i(t)$ is i.i.d. Gaussian noise. All functions are evaluated on 300 equally spaced points in $[0, 1]$. Figure 1 shows a sample of 200 such curves.

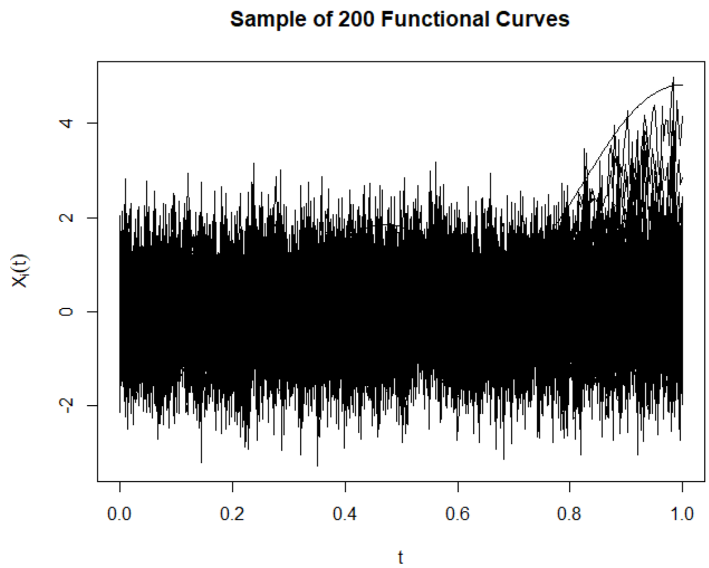

Figure 1: Sample of 200 curves $X_i(t)$.

Step 2: Generating the response and censoring Each scalar response Y_i is generated by

$$Y_i = r(X_i) + \varepsilon_i, \quad \varepsilon_i \sim \text{Exp}(0.5),\ with\ r(X_i) = \exp\left(-\int_0^1 \frac{1}{1+X_i^2(t)}\,dt\right).$$

Censoring is introduced by sampling $C_i \sim \text{Exp}(\lambda)$, and we observe

$$T_i = \min(Y_i, C_i), \quad \delta_i = \mathbf{1}_{Y_i \le C_i}.$$

Step 3: Estimators and semi-metric definitions We compare two nonparametric estimators

- **Local Linear Estimator:** Based on a quadratic kernel

$$K(u) = \frac{3}{4}(1-u^2)\mathbf{1}_{[-1,1]}(u),$$

  and a weighted least squares formulation using a second-derivative-based semi-metric

$$d(x_1, x_2) = \left(\int_0^1 \left(x_1^{(2)}(t) - x_2^{(2)}(t)\right)^2 dt\right)^{1/2}.$$

- **KNN Estimator:** Predicts using the average response of the k-nearest neighbors of x based on the same semi-metric.

A localization operator is further used in the local linear estimation:

$$\beta(x_1, x_2) = \int_0^1 \theta(t) \left(x_1^{(2)}(t) - x_2^{(2)}(t) \right) dt,$$

where $\theta(t)$ is the second eigenfunction of the empirical covariance operator. Figure 2 shows the first eight eigenfunctions.

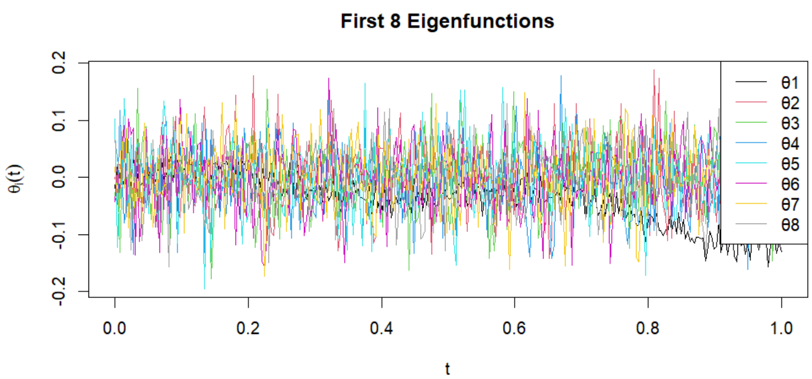

Figure 2: The curves $\theta_i(t_j)$, $t_j \in [0, 1]$, , for $i \in 1, \ldots, 8$.

Step 4: Prediction using training and testing samples The sample is split into

- Training set: $E_1 = \{1, \ldots, 200\}$
- Testing set: $E_2 = \{201, \ldots, 300\}$

Predictions are made for the testing sample using both estimators, and the predicted responses are compared to the true values. Figure 3 shows the predicted versus actual values.

Step 5: Evaluation under censoring To assess robustness, the experiment is repeated for censoring levels of approximately 0%, 20%, 40%, and 60%. At each level, the prediction Mean Squared Error (MSE) is computed across 10 replications

$$\text{MSE} = \frac{1}{100} \sum_{j \in E_2} (Y_j - \hat{Y}_j)^2.$$

Figure 4 shows the boxplots of MSEs.

Our local linear estimator demonstrates robust performance and appears to surpass the K nearest neighbors (Knn) estimator, even when applied to censored data. This is corroborated by the mean squared error (MSE), which indicates the superior efficacy of our approach.

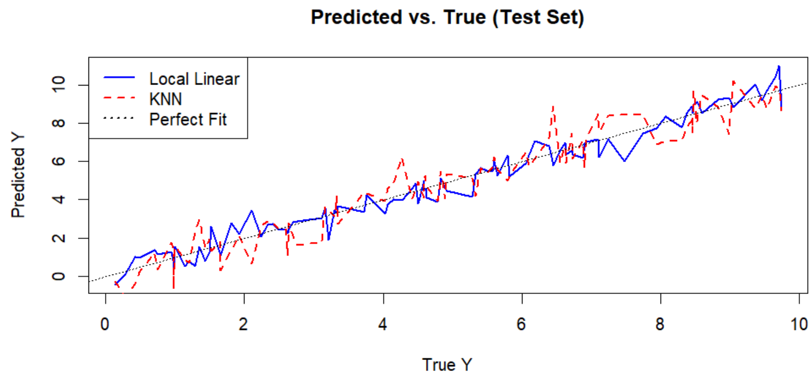

Figure 3: Testing sample.

4.1 Real data example

In this section, we illustrate the application of our proposed censored functional hazard estimators on a real-world dataset. We use the historical daily spot exchange rate between the U.S. Dollar (USD) and the Euro (EUR), published by the Federal Reserve Bank of St. Louis through the FRED database https://fred.stlouisfed.org/series/EXUSEU.

We extracted daily exchange rates over the period from January 2022 to December 2023. These data were aggregated into monthly averages, resulting in 12-dimensional functional covariates for each observation. The final dataset consists of $n = 110$ trajectories, each defined as a curve $X_i(t)$ for $t = 1, \ldots, 12$ months.

To evaluate the estimators' performance, we constructed a synthetic time-to-event variable Y_i for each trajectory using a nonlinear transformation of $X_i(t)$ and an additive noise term. Specifically, the true failure times were generated using the model

$$Y_i = h(X_i) + \varepsilon_i,$$

where h is a nonlinear functional operator and $\varepsilon_i \sim \mathcal{N}(0, \sigma^2)$. Right-censoring times C_i were independently sampled from a uniform distribution $C_i \sim \mathcal{U}(10, 30)$ and were drawn independently of Y_i. The observed time was defined as $T_i = \min(Y_i, C_i)$, and the censoring indicator as $\delta_i = \mathbf{1}_{\{Y_i \leq C_i\}}$.

We applied both the local linear estimator $\widehat{Y}_i^{LL}(t \mid X_0)$ and the KNN estimator $\widehat{Y}_i^{KNN}(t \mid X_0)$ to a fixed reference curve X_0. For each estimator, we computed the Mean Squared Error (MSE) over a regular grid $\{t_1, \ldots, t_N\}$

$$\text{MSE} = \frac{1}{N} \sum_{j=1}^{N} \left(\widehat{Y}_i(t_j \mid X_0) - Y_i(t_j \mid X_0) \right)^2 .$$

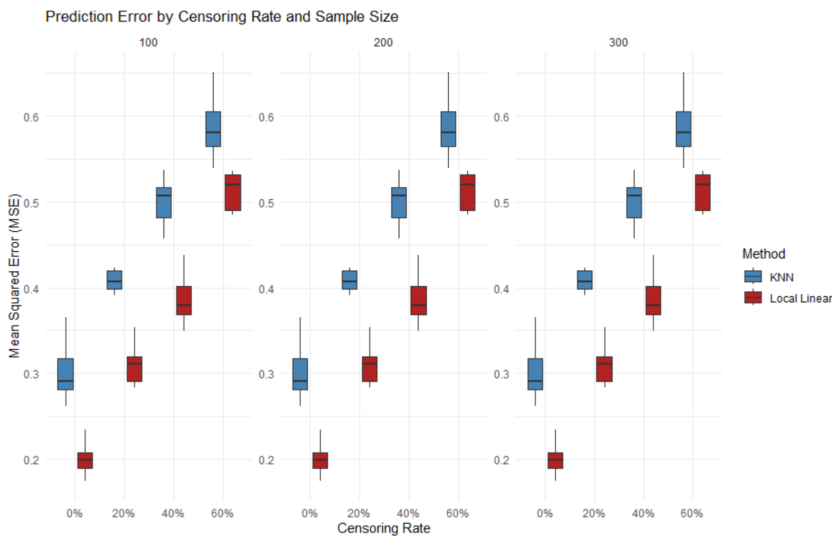

Figure 4: Distribution of MSE for Different Censoring Rates.

This procedure was repeated 50 times using random training/test splits (80% training, 20% test). The distribution of MSE values for both estimators is summarized in Figure 5, which shows that the local linear estimator consistently outperforms the KNN approach in terms of both accuracy and stability.

5 Conclusion

This paper has addressed the problem of nonparametric estimation of the conditional hazard function in the context of randomly right-censored and dependent functional data. We proposed a local linear estimator that leverages the flexibility of kernel smoothing and the robustness of local polynomial methods, adapted to handle α-mixing dependent structures. The theoretical properties of the estimator were established under suitable regularity conditions, including almost sure consistency with explicit convergence rates.

Extensive simulation studies were conducted to evaluate the finite-sample performance of the proposed estimator, particularly in comparison with the K-nearest neighbors (KNN) approach. The results demonstrated that our method yields lower mean squared errors, especially in the presence of moderate to strong dependence and censoring. These findings highlight the efficiency and adaptability of the estimator in complex data settings.

To illustrate its practical utility, we applied the methodology to real-world financial time series data, specifically the Euro to U.S. dollar exchange rate. The results confirmed the model's capacity to capture the underlying hazard dynamics effectively, even when the data exhibit dependency and censoring.

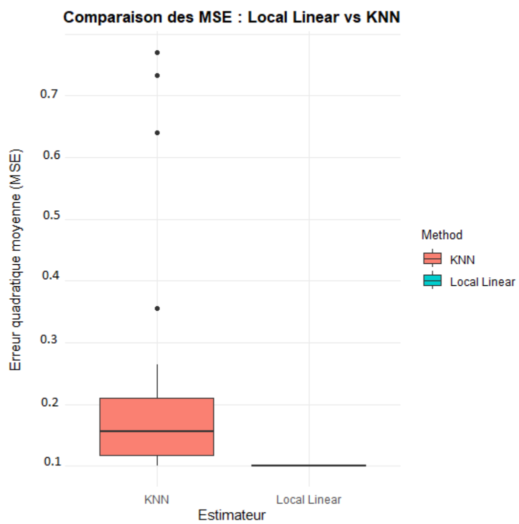

Figure 5: Comparison of MSE values for the local linear and KNN estimators over 50 repetitions using real exchange rate data.

This work contributes to the ongoing development of functional data analysis under censoring and dependence. Future research may consider extending the approach to multivariate or spatially correlated functional covariates, integrating data-driven bandwidth selection techniques, or investigating the estimator's performance in high-dimensional settings.

Acknowledgements

The author would like to think the Editor and the anonymous reviewers for their comments and criticims, which improved substantially the quality of this paper

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Received: April 25, 2025

Accepted: July 25, 2025

A Appendix

In what follows, we will denote by C and C' some strictly positive generic constants. Moreover, we define the quantities, for any $x \in \mathcal{F}$, and for all $i = 1, \dots, n$:

$$K_i(x) = K(h_K^{-1}\delta(x, X_i)), \ \ \beta_i(x) = \beta(X_i, x) \text{and } H_j'(y) = H'(h_H^{-1}(y - Y_j)).$$

Proposition A.1. *(?, page 237) Assume that $U_i, i \geq 1$ are identically distributed, with strong mixing coefficient $\alpha(n) = O(n^{-a})$, $a > 1$, such that $|U_1|$ is bounded. Then, for each $r > 1$ and $\varepsilon > 0$:*

$$\mathbb{P}\left\{\left|\sum_{i=1}^{n} U_i\right| > \varepsilon\right\} \leq C\left(1 + \frac{\varepsilon^2}{rS_n^2}\right)^{-r/2} + Cnr^{-1}\left(\frac{2r}{\varepsilon}\right)^{a+1},$$

where

$$S_n^2 = \sum_{1 \leq i,j \leq n} |Cov(U_i, U_j)|.$$

Proof of Lemma 3.2. The bias term remains unaffected by the dependence condition of (X_i, Y_i). Consequently, due to the equiprobability of the pairs (X_i, Y_i), we have

$$\forall y \in \mathcal{N}_y,\ \mathbb{E}[\widetilde{f}_N^x(y)] = \frac{1}{h_H \mathbb{E}[W_{12}(x)]}\mathbb{E}[\delta_1 \overline{G}^{-1}(T_1) W_{12}(x) H_i'(y)].$$

By applying the properties of conditional expectation and given that

$$\mathbf{1}_{\{Y_1 \leq C_1\}}\varphi(T_1) = \mathbf{1}_{\{Y_1 \leq C_1\}}\varphi(Y_1) \tag{A.1}$$

we get

$$\mathbb{E}[\widetilde{f}_N^x(y)] = \frac{1}{h_H \mathbb{E}[W_{12}(x)]}\mathbb{E}\left[W_{12}\mathbb{E}\left[H_i(y)\mathbf{1}\{Y_1 \leq C_1\}\overline{G}^{-1}(T_1)\middle|X_1\right]\right].$$

Since

$$\begin{aligned}
\mathbb{E}[H_i'(y)\mathbf{1}_{\{Y_1 \leq C_1\}}\overline{G}^{-1}(T_1)|X_1] &= \mathbb{E}[\mathbb{E}[H_i'(y)\mathbf{1}_{\{Y_1 \leq C_1\}}\overline{G}^{-1}(Y_1)|Y_1]|X_1] \\
&= \mathbb{E}[H_i'(y)\overline{G}^{-1}(Y_1)\mathbb{E}[\mathbf{1}_{\{Y_1 \leq C_1\}}|Y_1]|X_1] \\
&= \mathbb{E}[H_i'(y)|X_1],
\end{aligned}$$

and

$$\mathbb{E}[H_i'(y)|X_1] = \int_{\mathbb{R}} H'\left(\frac{y-z}{h_H}\right) f^X(z)dz = h_H \int_{\mathbb{R}} H'(t) f^X(y - h_H t)dt.$$

Therefore,

$$|\mathbb{E}[\widetilde{f}_N^x(y)] - f^x(y)| \leq \int_{\mathbb{R}} H'(t)|f^X(y - h_H t) - f^x(y)|dt.$$

Thus, based on assumptions (H2) and (H7), we have that

$$\forall y \in \mathcal{N}_y, \quad |\mathbb{E}[f_N^x(y)] - f^x(y)| \leq C\int_{\mathbb{R}} H'(t)(h_K^{b_1} + |t|^{b_2} h_H^{b_2})dt = O(h_K^{b_1} + h_H^{b_1}),$$

which finishes the proof. ∎

Proof of Lemma 3.6 $\mathcal{N}_y$ is a compact subset of $\mathbb{R}$, and it can be covered by a finite number s_n of intervals of length l_n at some points $(z_k)_{k=1,\ldots,s_n}$, i.e $\mathcal{N}_y \subset \cup_{k=1}^{s_n}(z_k - l_n, z_k + l_n)$ with $l_n = n^{-\frac{3\gamma}{2}-\frac{1}{2}}$ and $s_n = O(l_n^{-1})$.
Let $z_y = \arg\min_{z\in\{z_1,\ldots,z_{s_n}\}} |y - z|$ and consider the following decomposition

$$\sup_{y\in\Omega} |\widetilde{f}_N^x(y) - \mathbb{E}[\widetilde{f}_N^x(y)]| \leq \underbrace{\sup_{y\in\Omega} |\widetilde{f}_N^x(y) - \widetilde{f}_N^x(z_y)|}_{N_1} + \underbrace{\sup_{y\in\Omega} |\widetilde{f}_N^x(z_y) - \mathbb{E}[\widetilde{f}_N^x(z_y)]|}_{N_2} + \underbrace{\sup_{y\in\Omega} |\mathbb{E}[f_N^x(z_y)] - \mathbb{E}[f_N^x(y)]|}_{N_3}.$$

- Concerning (N_1) and (N_3): Under assumptions (H7), (H9), and Lemma 3.2, we obtain

$$\sup_{y\in\Omega} |\widetilde{f}_N^x(y) - \mathbb{E}[\widetilde{f}_N^x(z_y)]| = o_{a.co.}\left(\sqrt{\frac{\psi_x^(1/2)(h_K)\log n}{nh_H\phi_x^2(h_K)}}\right). \tag{A.2}$$

  Similarly to the previous method, we find

$$\sup_{y\in\Omega} |\mathbb{E}[\widetilde{f}_N^x(z_y)] - \mathbb{E}[\widetilde{f}_N^x(y)]| = o_{a.co.}\left(\sqrt{\frac{\psi_x^{(1/2)}(h_K)\log n}{nh_H\phi_x^2(h_K)}}\right). \tag{A.3}$$

- Concerning (N_2): For all $\eta > 0$, we can show

$$\mathbb{P}\left(\sup_{y\in\Omega} |\widetilde{f}_N^x(z_y) - \mathbb{E}[\widetilde{f}_N^x(z_y)]| > \eta\sqrt{\frac{\psi_x^{(1/2)}(h_K)\log n}{nh_H\phi_x^2(h_K)}}\right) \tag{A.4}$$

$$= \mathbb{P}\left(\max_{z_y\in\{z_1,\ldots,z_{s_n}\}} |\widetilde{f}_N^x(z_y) - \mathbb{E}[\widetilde{f}_N^x(z_y)]| > \eta\sqrt{\frac{\psi_x^{(1/2)}(h_K)\log n}{nh_H\phi_x^2(h_K)}}\right) \tag{A.5}$$

$$\leq s_n\mathbb{P}\left(|\widetilde{f}_N^x(z_y) - \mathbb{E}[\widetilde{f}_N^x(z_y)]| > \eta\sqrt{\frac{\psi_x^{(1/2)}(h_K)\log n}{nh_H\phi_x^2(h_K)}}\right). \tag{A.6}$$

  Next, we calculate the following quantity

$$s_n\mathbb{P}\left(|\widetilde{f}_N^x(z_y) - \mathbb{E}[\widetilde{f}_N^x(z_y)]| > \eta\sqrt{\frac{\psi_x^{(1/2)}(h_K)\log n}{nh_H\phi_x^2(h_K)}}\right), \text{for all } z_y \in \{z_1,\ldots,z_{s_n}\}.$$

  To achieve this, we consider the following decomposition

$$\begin{aligned}\widetilde{f}_N^x(z_y) &= \underbrace{\frac{n^2h_K^2\phi_x^2(h_K)}{n(n-1)\mathbb{E}[W_{12}]}}_{D_1}\left[\underbrace{\left(\frac{1}{n}\sum_{j=1}^{n}\frac{\delta_jK_j(x)H_j'(z_y)}{\overline{G}(T_j)h_H\phi_x(h_K)}\right)}_{D_2}\underbrace{\left(\frac{1}{n}\sum_{i=1}^{n}\frac{K_i(x)\beta_i^2(x)}{h_K^2\phi_x(h_K)}\right)}_{D_3}\right.\\ &\left.-\underbrace{\left(\frac{1}{n}\sum_{j=1}^{n}\frac{\delta_jK_j(x)\beta_j(x)H_j'(z_y)}{\overline{G}(T_j)h_Hh_K\phi_x(h_K)}\right)}_{D_4}\underbrace{\left(\frac{1}{n}\sum_{i=1}^{n}\frac{K_i(x)\beta_i(x)}{h_K\phi_x(h_K)}\right)}_{D_5}\right],\end{aligned}$$

which allows us to write that

$$\widetilde{f}_N^x(z_y)-\mathbb{E}[\widetilde{f}_N^x(z_y)] = D_1((D_2D_3-\mathbb{E}[D_2D_3])-(D_4D_5-\mathbb{E}[D_4D_5])).$$

Furthermore, note that

$$\begin{aligned}D_2D_3-\mathbb{E}[D_2D_3] &= (D_2-\mathbb{E}[D_2])(D_3-\mathbb{E}[D_3])+(D_3-\mathbb{E}[D_3])\mathbb{E}[D_2]\\ &+(D_2-\mathbb{E}[D_2])\mathbb{E}[D_3]+\mathbb{E}[D_2]\mathbb{E}[D_3]-\mathbb{E}[D_2D_3]\end{aligned}$$

also

$$\begin{aligned}D_4D_5-\mathbb{E}[D_4D_5] &= (D_4-\mathbb{E}[D_4])(D_5-\mathbb{E}[D_5])+(D_5-\mathbb{E}[D_5])\mathbb{E}[D_4]\\ &+(D_4-\mathbb{E}[D_4])\mathbb{E}[D_5]+\mathbb{E}[D_4]\mathbb{E}[D_5]-\mathbb{E}[D_4D_5].\end{aligned}$$

Consequently, our claimed result directly follows from the following assertions

$$s_n\mathbb{P}\left(|D_i-\mathbb{E}[D_i]|>\eta\sqrt{\frac{\psi_x^{(1/2)}(h_K)\log n}{nh_H\phi_x^2(h_K)}}\right)<\infty,\quad \text{for } i=2,3,4,5, \tag{A.7}$$

$$D_1=O(1) \text{ and } \mathbb{E}[D_i]=O(1) \text{ for } i=2,3,4,5, \tag{A.8}$$

$$Cov(D_2,D_3)=o_{a.co.}\left(\sqrt{\frac{\psi_x^{(1/2)}(h_K)\log n}{nh_H\phi_x^2(h_K)}}\right), \tag{A.9}$$

and almost completely

$$Cov(D_4,D_5)=o_{a.co.}\left(\sqrt{\frac{\psi_x^{(1/2)}(h_K)\log n}{nh_H\phi_x^2(h_K)}}\right). \tag{A.10}$$

Proof of A.7. Note that for $i=3,5$, the result has already been established in Lemma 3.1. Therefore, we will concentrate solely on the cases where $i=2,4$. To address these, we employ Proposition A.1. For $1\leq i\leq n$ and $k=0,1$, let

$$Z_i^k=\frac{1}{h_K^k}\left(\delta_i\overline{G}^{-1}(T_i)K_i(x)H_i'(z_y)\beta_i^k(x)-\mathbb{E}[\delta_i\overline{G}^{-1}(T_i)K_i(x)H_i'(z_y)\beta_i^k(x)]\right). \tag{A.11}$$

Consequently, we can observe that

$$D_{2(k+1)} - \mathbb{E}[D_{2(k+1)}] = \frac{1}{nh_H\phi_x(h_K)}\sum_{i=1}^{n} Z_i^k, \text{ for } k=0,1.$$

Under assumptions (H1), (H3), (H5), and (H7), we have

$$\begin{aligned}
\frac{1}{h_K^k}\delta_i\overline{G}^{-1}(T_i)K_i(x)H_i(z_y)\beta_i^k(x) &\leq \frac{C}{h_K^k}\overline{G}^{-1}(\tau)K_i(x)|\delta(X_i,x)|^k \\
&\leq \frac{C}{h_K^k}\overline{G}^{-1}(\tau)K_i(x)|\delta(X_i,x)|^k\mathbf{1}_{]-1,1[}(h_K^{-1}\delta(x,X_i)) \\
&\leq C\overline{G}^{-1}(\tau)K_i(x)\mathbf{1}_{B(x,h_K)}(X_i) \\
&\leq C'\mathbf{1}_{B(x,h_K)}(X_i). \qquad \text{(A.12)}
\end{aligned}$$

Let us now calculate

$$S_n^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} Cov(Z_i^k, Z_j^k) = S_n^{2*} + nVar[Z_1^k],$$

where

$$S_n^{2*} = \sum_{i=1}^{n}\sum_{i\neq j} Cov(Z_i^k, Z_j^k).$$

By using Equations (A.1) and (A.1), along with the properties of conditional expectation, we obtain

$$\begin{aligned}
Var[Z_1^k] &= Var\left[\frac{1}{h_K^k}\delta_i\overline{G}^{-1}(T_i)K_i(x)H_i'(z_y)\beta_i^k(x)\right] \\
&\leq C'\mathbb{E}[\mathbf{1}_{B(x,h_K)}(X_i)\mathbb{E}[H_i'^2(z_y)|X_1]] + C\mathbb{E}[\mathbf{1}_{B(x,h_K)}(X_i)\mathbb{E}^2[H_i'(z_y)|X_1]] \\
&= C'\phi_x(h_K)\mathbb{E}[H_i'^2(z_y)|X_1] + C\phi_x^2(h_K)\mathbb{E}^2[H_i'(z_y)|X_1] \\
&\leq C'\psi_x^{(1/2)}(h_K)\mathbb{E}[H_i'^2(z_y)|X_1] + C\psi_x(h_K)\mathbb{E}^2[H_i'(z_y)|X_1].
\end{aligned}$$

Finally, by considering the fact that

$$\mathbb{E}[H_i'^2(z_y)|X_1] = O(h_H) \text{ and } \mathbb{E}[H_i'(z_y)|X_1] = O(h_H),$$

we get

$$nVar[Z_1^k] = O(nh_H\psi^{(1/2)}(h_K)).$$

Subsequently, from Equation (A.11), by applying conditional expectation once more, and considering the fact that

$$\mathbb{E}[\delta_i\delta_j|Y_iY_j] = \overline{G}(T_i)\overline{G}(T_j),$$

we get under (H4) and (H5), for $i \neq j$,

$$\begin{aligned}
Cov(Z_i^k, Z_j^k) &= Cov\left(\frac{1}{h_K^k}\delta_i\overline{G}^{-1}(T_i)K_i(x)H_i'(z_y)\beta_i^k(x), \frac{1}{h_K^k}\delta_j\overline{G}^{-1}(T_j)K_j(x)H_j'(z_y)\beta_j^k(x)\right) \\
&= \mathbb{E}\left[\frac{1}{h_K^{2k}}\delta_i\delta_j\overline{G}^{-1}(T_i)\overline{G}^{-1}(T_j)K_i(x)K_j(x)H_i'(z_y)H_j(z_y)\beta_i^k(x)\beta_j^k(x)\right] \\
&\quad -\mathbb{E}\left[\frac{1}{h_K^k}\delta_i\overline{G}^{-1}(T_i)K_i(x)H_i'(z_y)\beta_i^k(x)\right]\mathbb{E}\left[\frac{1}{h_K^k}\delta_j\overline{G}^{-1}(T_j)K_j(x)H_j'(z_y)\beta_j^k(x)\right] \\
&\leq C[\mathbb{E}[\mathbf{1}_{B(x,h_K)\times B(x,h_K)}(X_i, X_j)\mathbb{E}[H_i'(z_y)H_j'(z_y)|X_iX_j]]] \\
&\quad +C[\mathbb{E}[\mathbf{1}_{B(x,h_K)}(X_i)\mathbb{E}[H_i'(z_y)|X_i]]\mathbb{E}[\mathbf{1}_{B(x,h_K)}(X_j)\mathbb{E}[H_i'(z_y)|X_j]]] \\
&\leq C(h_H^2\varphi_x(h_K) + \phi_x^2(h_K)) \\
&\leq Ch_H^2\psi_x(h_K). \qquad\qquad (A.13)
\end{aligned}$$

Following the approach of **?**, we define

$$\begin{aligned}
E_1 &= \{(i,j) \text{ such that } 1 \leq i-j \leq m_n\}, \\
E_2 &= \{(i,j) \text{ such that } m_n + 1 \leq i-j \leq n-1\},
\end{aligned}$$

where $m_n \longrightarrow \infty$, as $n \longrightarrow \infty$.

Thus

$$S_n^{2*} \leq \sum_{E_1} Cov(Z_i^k, Z_j^k) + \sum_{E_2} Cov(Z_i^k, Z_j^k) := \varsigma_{1,n} + \varsigma_{2,n}.$$

Equation (A.13) implies that

$$\varsigma_{1,n} \leq Cnm_nh_H^2\psi_x(h_K).$$

To bound $\varsigma_{2,n}$, we apply the Davydov-Rio's inequality for bounded mixing processes

$$\forall i \neq j \quad |Cov(Z_i^k, Z_j^k)| \leq C\alpha(|i-j|).$$

Therefore, using $\sum_{j\geq x+1} j^{-a} \leq \int_{u\geq x} u^{-a} = \left[(a-1)x^{a-1}\right]^{-1}$, and the first part of (H4), we get

$$\varsigma_{2,n} \leq Cnm_n^{-a+1}.$$

Thus

$$S_n^{2*} \leq C(nm_nh_H^2\psi_x(h_K) + nm_n^{1-a}).$$

By choosing $m_n = (h_H^2\psi_x(h_K))^{-1/a}$, we obtain

$$S_n^{2*} = O(nh_H^2\psi_x(h_K)^{(a-1)/a}).$$

Finally, as $a > 2$, then

$$S_n^2 = O(nh_H\psi_x^{1/2}(h_K)). \qquad (A.14)$$

Taking $\varepsilon = \eta \frac{\sqrt{S_n^2 \log n}}{nh_H \phi_x(h_K)}$,Combining Proposition A.1 with equation (A.14), we can deduce the following

$$\begin{aligned}
\mathbb{P}\left\{|D_{2(k+1)} - \mathbb{E}[D_{2(k+1)}]| > \varepsilon\right\} &= \mathbb{P}\left\{\left|\sum_{i=1}^{i=n} Z_i^k\right| > \varepsilon n h_H \phi_x(h_K)\right\} \\
&\leq C\left(1 + \frac{\varepsilon^2 n^2 h_H^2 (\phi_x(h_K))^2}{S_n^2 r}\right)^{-r/2} + Cnr^{-1}\left(\frac{2r}{\varepsilon n h_H \phi_x(h_K)}\right)^{a+1} \\
&\leq C \exp\left(\frac{-r}{2\log\left(1 + \frac{\eta^2 \log n}{r}\right)}\right) \\
&+ Cnr^{-1}\left(\frac{r}{n}\right)^{a+1} (nh_H \psi_x^{1/2}(h_K) \log n)^{-(a+1)/2} \\
&\leq C(\theta_1 + \theta_2). \qquad \text{(A.15)}
\end{aligned}$$

Next, using equation (A.15) with $r = C \log n (\log_2 n)^{1/a}$ and Taylor series expansion of $\log(1 + x)$, we obtain

$$\theta_1 \leq C \exp(-C\eta^2 \log n) = Cn^{-C\eta^2}, \qquad \text{(A.16)}$$

and

$$\theta_2 \leq Cn^{(1-a)/2} \eta^{-(a+1)} (h_H \psi_x^{1/2}(h_K))^{-(a+1)/2} (\log n)^{(a-1)/2} \log_2 n. \qquad \text{(A.17)}$$

We can use equation (A.15) and (A.17), together with assumption (H10)(ii), to show that $s_n \theta_2 = O(n^{-1} \log^{-1} n \log_2^{-2} n)$ is the general term of a convergent Bertrand series. Similarly, we can choose η such that $s_n \theta_1$ is the general term of a convergent series. Therefore, we can deduce that

$$s_n \mathbb{P}\left(|D_{2(k+1)} - \mathbb{E}[D_{2(k+1)}]| > \eta \sqrt{\frac{\psi_x^{(1/2)}(h_K) \log n}{nh_H \phi_x^2(h_K)}}\right) < \infty.$$

- Concerning (A.8): **?** have shown that

$$D_1 = O(1) \text{ and } \mathbb{E}[D_i] = O(1) \text{ for } i = 3, 5,$$

In the following, we will focus on the case where $i = 2, 4$.

Given that the pairs $(X_i, Y_i), \ i = 1, \ldots, n$ are identically distributed, we obtain

$$\mathbb{E}[D_2] = \frac{\mathbb{E}[\delta_1 \overline{G}^{-1}(T_1) K_1(x) H_1'(z_y)]}{h_H \phi_x(h_K)} \ \ and \ \ \mathbb{E}[D_4] = \frac{\mathbb{E}[\delta_1 \overline{G}^{-1}(T_1) K_1(x) H_1'(z_y) \beta_1(x)]}{h_H h_K \phi_x(h_K)}.$$

We need to assess the value of

$$\mathbb{E}[\delta_1 \overline{G}^{-1}(T_1) K_i(x) H_i'(z_y) \beta_i^l(x)], \text{ for } l = 0, 1.$$

By applying the properties of conditional expectation and equation (A.1), we obtain the following result for all $l = 0, 1$:

$$\mathbb{E}[\delta_1 \overline{G}^{-1}(T_1) K_i(x) H_i'(z_y) \beta_i^l(x)] = O(h_H \mathbb{E}[K_i(x)\beta_i^l(x)]).$$

By applying Lemma 3 from **?**, we get

$$\mathbb{E}[\delta_1 \overline{G}^{-1}(T_1) K_i(x) H_i'(z_y) \beta_i^l(x)] = O(h_H h_K^l \phi_x(h_K)) \tag{A.18}$$

Therefore, equation (A.18) gives us

$$\mathbb{E}[D_i] = O(1), \text{ for } i = 2, 4.$$

- Concerning (A.9) and (A.10): following similar steps as in the proof of (A.14), we obtain

$$Cov(D_2, D_3) = O\left(\frac{\psi_x^{(1/2)}(h_K)}{n\phi_x^2(h_K)}\right) = o\left(\sqrt{\frac{\psi_x^{(1/2)}(h_K)\log n}{nh_H\phi_x^2(h_K)}}\right)$$

and

$$Cov(D_4, D_5) = O\left(\frac{\psi_x^{(1/2)}(h_K)}{n\phi_x^2(h_K)}\right) = o\left(\sqrt{\frac{\psi_x^{(1/2)}(h_K)\log n}{nh_H\phi_x^2(h_K)}}\right)$$

Hence

$$\sup_{y\in\Omega}\left|\widetilde{f}_N(z_y) - \mathbb{E}[\widetilde{f}_N(z_y)]\right| = o_{a.co.}\left(\sqrt{\frac{\psi_x^{(1/2)}(h_K)\log n}{nh_H\phi_x^2(h_K)}}\right) \tag{A.19}$$

The proof of Lemma 3.4 can be completed by considering equations (A.2), (A.3), and (A.19). ■

Proof of Lemma 3.6 We can see that

$$\begin{aligned} |\widehat{f}_N^x(y) - \widetilde{f}_N^x(y)| &\leq \frac{1}{n(n-1)h_H\mathbb{E}[W_{12}]}\sum_{i\neq j}|\delta_i W_{ij}(x)H_i'(y)(\overline{G}_n^{-1}(T_i) - \overline{G}^{-1}(T_i))| \\ &\leq \frac{\sup_{y\in\Omega}|\overline{G}_n(t) - \overline{G}(t)|}{\overline{G}_n(\tau)}\frac{1}{n(n-1)h_H\mathbb{E}[W_{12}]}\sum_{i\neq j}\delta_i\overline{G}^{-1}(T_i)W_{ij}(x)H_i'(y) \\ &= \frac{\sup_{y\in\Omega}|\overline{G}_n(t) - \overline{G}(t)|}{\overline{G}_n(\tau)}\widetilde{F}_N^x(y). \end{aligned}$$

Based on the fact that $\overline{G}(\tau) > 0$, the strong law of large numbers, and the law of the iterated logarithm on the censoring law (see formula (4.28) in **?**), we can immediately conclude the result from Lemmas 3.2 and 3.4. ■

Remark 1. The proofs of Lemmas 3.3, 3.5, and 3.7 adhere to the methodologies employed in the proofs of Lemmas 3.2, 3.4, and 3.6.