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ANALYTICAL DISTRIBUTIONS FOR FUNCTIONS OF INDEPENDENT RICIAN RANDOM VARIABLES: APPLICATION TO WIRELESS COMMUNICATIONS

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SUMMARY

This paper presents two analytical formulations for the distribution of the square of a sum of products involving independent Rician random variables. The expressions are derived through the application of the central limit theorem (CLT) and Laguerre series. To assess their accuracy, the derived expressions are compared with the empirical distribution acquired through a comprehensive simulation study. The comparison is made based on the Kolmogorov-Smirnov, Anderson-Darling, and Cramer-von Mises statistics, supplemented by information criteria such as AIC and BIC as indicators of goodness of fit. Finally, an application in wireless communications is presented to illustrate the importance of the proposed work. As a conclusion, it is shown that the expression obtained by Laguerre expansion method is more accurate and simpler than the one obtained by CLT.

Keywords and phrases: central limit theorem, Laguerre expansion, rician random variable

AMS Classification: 60E05; 60F05

1 Introduction and Motivation

With the tremendous increase in data rates and number of users, the need to design new/efficient wireless communications techniques to satisfy these requirements increases. Fifth generation (5G) and sixth generation (6G) wireless networks try to come up with new techniques, which could provide the needed services for system users. Although most of the degradation on transmitted signals happens due to channel impairments such as deep fading, shadowing, path loss, inter-symbol interference, etc., the thought over the whole previous time was that such impacts of channel or communication medium are uncontrollable. Therefore, most of the previous efforts to design and propose new techniques are usually done at the communicating nodes of wireless network themselves, which

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are the source and the destination. Such techniques include multiple antennas, security enhancement techniques, millimeter wave communications, multiple-access techniques, etc.

Recently and away from the previous techniques, the reconfigurable intelligent surface (RIS) emerges as a novel and auspicious approach for present and forthcoming wireless communications networks. Functioning as an artificial surface constructed from electromagnetic materials, an RIS exhibits the ability to tailor the propagation of radio waves it encounters (Basar et al., 2019; Di Renzo et al., 2020). This introduces a fresh, cost-effective, and simplified solution for achieving wireless communications with heightened spectrum and energy efficiencies. It is implemented between the source and destination and its main aim is to adjust its phase shift to compensate for the phase shifts of communication medium or channel. In ideal conditions, an RIS can fully compensate for the phase shifts introduced by wireless channels from source to RIS and from RIS to destination by simply choosing its phase shift to be equal to negative of summation of first hop and second hop channels' phase shifts. Currently, the RIS topic is being investigated from various aspects by many researchers. Among the important aspects is the channel modelling and performance analysis of this system. In this paper, we try to model the channel of RIS system and derive its statistics, including probability density function (PDF) and cumulative distribution function (CDF) using some mathematical approaches. Then, we utilize these derived results to derive a well-known performance measure of wireless systems, which is the *outage probability*.

1.1 Literature review

Many researchers, particularly those in the wireless communications field, focus on evaluating the effectiveness of diversity combining receivers within specified probability distributions. Hasna and Alouini (2003) presented a mathematical representation for the PDF of the end-to-end (e2e) signal-to-noise ratio (SNR), with the assumption that the intervening channels follow a random distribution according to the Rayleigh distribution. Sagias et al. (2005) investigated the performance of diversity combining receivers operating over independent Weibull fading channels, which might not have identical distributions. Additionally, a mathematical formulation for the PDF of the summation of squared Nakagami-m random variables (RVs) with non-identical fading parameters, all of which have integer-order fading parameters, was established by Karagiannidis et al. (2006).

A mathematical formulation for the PDF of the summation of independent Gamma-Gamma RVs, which might not have identical distributions, was derived by Chatzidiamantis and Karagiannidis (2011). The research was conducted while investigating the performance of radio frequency wireless systems dealing with composite fading and multiple-input multiple-output (MIMO) systems affected by atmospheric turbulence. In a related context, Peppas (2011) provided approximate mathematical representations for the PDF of the total of independent identically distributed (IID) generalized-K RVs. These representations were employed to characterize the channel conditions in satellite communications. Recently, Yang et al. (2020) developed precise approximations for the PDF of the summation of IID Rayleigh RVs, each possessing specific mean and variance. These approximations were developed to model the channel distributions in two distinct wireless system setups based on RIS and assuming Rayleigh fading channels.

Further, there has been extensive exploration in the literature on the distribution of the product of two RVs, notably with significant implications for wireless communications applications. Specifically, within a single-channel M-ary phase-shift-keying communication system, the output of a linear combiner can be effectively characterized through the product of two RVs. This principle extends its applicability to scenarios such as cascaded fading channels and keyhole channels within MIMO systems, where the received signal is modeled as the product of RVs. Bhargav et al. (2018) examined the PDF of the product of two κ-μ RVs, particularly in the context of body area networks and device-to-device channel measurements. Additionally, Li et al. (2020) examined the PDF of the product of two correlated Gaussian RVs. Hazra and Arti (2020) examined the PDF of the product of two squared correlated Shadowed-Rician RVs. Further insights into this topic can be found in references such as El Bouanani and da Costa (2018), Du et al. (2020), and Illi et al. (2020).

In many prior studies, the central limit theorem (CLT) has been employed to approximate the distribution of performance metrics such as the end-to-end SNR or outage probability, primarily due to its analytical simplicity and closed-form tractability. However, such approximations are asymptotic in nature and can result in substantial errors when applied to small or moderate values of n, which are common in practical RIS-assisted systems. This limitation underscores the need for more accurate analytical tools that preserve tractability while offering improved approximation accuracy under realistic conditions.

To this end, the present work proposes and systematically evaluates a novel closed-form approximation based on Laguerre expansion, specifically designed for estimating the outage probability, a fundamental performance metric in wireless communication systems. The proposed method is shown to offer enhanced accuracy across a broad range of system configurations and channel conditions.

1.2 Wireless communications dilemmas

In various wireless communications systems, it is crucial to explore the CDF of the following form of function of RVs

$$Z = \Big(\sum_{i=1}^{n} X_i Y_i \Big)^2. \tag{1.1}$$

This form of RV finds extensive applications in various contexts within the literature. As an illustration, Equations (1) and (11) in Yang et al. (2020) and Basar et al. (2019), respectively, where the RV Z signifies the SNR of a wireless signal transmitted from a source to a destination through RIS. To evaluate the performance of this system, a comprehensive understanding of the statistical properties of this RV is essential, encompassing its PDF and CDF. In the investigations conducted by Yang et al. (2020) and Basar et al. (2019), the distribution of Z is examined under the assumption that X_i and Y_i are IID Rayleigh RVs with a mean of $\sqrt{\pi}/2$ and a variance of $(4-\pi)/4$. Rayleigh fading is widely accepted as an effective model for the fading encountered in numerous wireless communications systems. In their study, Samuh and Salhab (2024) investigated the distribution of Z assuming X_i and Y_i are independent Nakagami RVs with different shape and scale parameters. It is known that the Nakagami distribution offers a significantly improved fit for fading channel distributions. This enhancement arises from the inclusion of an additional independent parameter (m), spanning a range from $1/2$ to ∞, thereby affording increased flexibility. Moreover, the Nakagami distribution often yields the most accurate fit for multipath propagation in various environments, such as land-mobile and indoor-mobile scenarios, along with scintillating ionospheric radio links (Simon and Alouini, 2005). However, various measurement campaigns have demonstrated that the Rician distribution offers a superior fit for fading channel distributions. This advantage stems from an additional free parameter (K ranging from 0 to ∞), allowing for enhanced flexibility. Additionally, this kind of fading is frequently encountered in the initial line-of-sight (LOS) routes within microcellular urban and suburban land-mobile environments, as well as in indoor picocellular environments. It also corresponds to the primary LOS trajectory for satellite and ship-to-ship radio communication links.

This paper investigates the distribution of Z under the assumption that $X_i\,(i = 1, 2, \ldots, n)$ follows IID Rician RVs, each characterized by a shape parameter K_1 and a scale parameter Ω_1, and $Y_i\,(i = 1, 2, \ldots, n)$ is assumed to be IID Rician RVs, each characterized by a shape parameter K_2 and a scale parameter Ω_2.

1.3 Theoretical background

Definition 1.1 (Rician Distribution (Rice, 1948))**.** The Rician RV X, characterized by the shape parameter K and scale parameter Ω, has PDF

$$f(x; K, \Omega) = \frac{2(K+1)x}{\Omega} \exp\left(-K - \frac{(K+1)x^2}{\Omega}\right) I_0\left(2\sqrt{\frac{K(K+1)}{\Omega}}x\right), \quad x \geq 0, \tag{1.2}$$

where $I_\nu(\cdot)$ represents the modified Bessel function of the first kind with order ν, as defined in (Gradshteyn and Ryzhik, 2000, Eq. 8.445), and $\Omega = E(X^2)$ denotes the mean power.

The expectation and variance of the Rician RV are given by

$$\mathbb{E}(X) = \frac{1}{2}\sqrt{\pi}\, e^{-K/2}\sqrt{\frac{\Omega}{K+1}}\left((K+1)I_0\left(\frac{K}{2}\right) + KI_1\left(\frac{K}{2}\right)\right), \tag{1.3}$$

$$\mathrm{Var}(X) = \Omega - \frac{\pi e^{-K}\Omega\left((K+1)I_0\left(\frac{K}{2}\right) + KI_1\left(\frac{K}{2}\right)\right)^2}{4(K+1)}. \tag{1.4}$$

This formulation provides a concise representation of the PDF, allowing for a comprehensive understanding of the statistical characteristics associated with the Rician distribution.

To derive the distribution of the product $V = XY$, where X and Y are independent continuous RVs, we use the following integral representation:

$$f_V(v) = \int_{-\infty}^{\infty} f_X(x) f_Y\left(\frac{v}{x}\right)\frac{1}{|x|}\,dx,$$

as found in Rohatgi (1976).

Theorem 1 (PDF, Expectation, and Variance of the Product of Two Independent Rician RVs)**.** *Let X and Y denote two continuous RVs, each following a Rician distribution. Specifically, X is characterized by the shape (K_1) and scale (Ω_1) parameters, while Y is characterized by the shape (K_2)*

and scale (Ω_2) parameters. Assuming independence between X and Y. The product $V = XY$ has PDF

$$f_V(v) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{v^{i+j+1}\left(4K_1^j K_2^i (\sqrt{\Omega_1\Omega_2})^{i+j+2}\right) K_{j-i}(2v\sqrt{\Omega_1\Omega_2})}{(j!)^2(i!)^2\exp(K_1+K_2)}, \tag{1.5}$$

where $K_\nu(\cdot)$ represents the modified Bessel function of the second kind with order ν, as defined in (Gradshteyn and Ryzhik, 2000, Eq. 8.432).

The expectation of V is given by

$$\begin{aligned}\mathbb{E}(V) = \frac{1}{4}\pi e^{-\frac{K_1}{2}-\frac{K_2}{2}} \sqrt{\frac{\Omega_1}{K_1+1}} \left((K_1+1)I_0\left(\frac{K_1}{2}\right) + K_1 I_1\left(\frac{K_1}{2}\right)\right) \\ \times \sqrt{\frac{\Omega_2}{K_2+1}} \left((K_2+1)I_0\left(\frac{K_2}{2}\right) + K_2 I_1\left(\frac{K_2}{2}\right)\right),\end{aligned} \tag{1.6}$$

and the variance of V is

$$\begin{aligned}\mathrm{Var}(V) = \Omega_1\Omega_2 - \frac{\pi^2\Omega_1\Omega_2 e^{-(K_1+K_2)}\left((K_1+1)I_0\left(\frac{K_1}{2}\right) + K_1 I_1\left(\frac{K_1}{2}\right)\right)^2}{16(K_1+1)(K_2+1)} \\ \times \left((K_2+1)I_0\left(\frac{K_2}{2}\right) + K_2 I_1\left(\frac{K_2}{2}\right)\right)^2.\end{aligned} \tag{1.7}$$

By the CLT, the sum $U = \sum_{i=1}^{n} V_i$ can be approximated by a normal distribution with mean $\mu_U = n\mathbb{E}(V)$ and variance $\sigma_U^2 = n\mathrm{Var}(V)$, provided that n is sufficiently large. For formal definitions, see Wackerly et al. (2008).

2 Analytical Approaches for Modeling and Analyzing the Distribution of Z

In this section, two distinct methodologies, namely the CLT and Laguerre expansion (LEx), are used to establish an analytical formulations for the distribution of the Z RV (Equation 1.1). Notably, Z is represented by the square of the sum of the products of two other RVs. For clarity and consistency throughout this paper, let's introduce the following notations: $V_i = X_iY_i$, $U = \sum_{i=1}^{n} V_i = \sum_{i=1}^{n} X_iY_i$, and $Z = U^2$ as defined in Equation 1.1. In both methodologies, the first step involves deriving the probability distribution of U. Subsequently, the probability distribution of Z is determined using the technique of transforming RVs.

2.1 Using CLT

The conventional methodology for approximating the distribution of the sum of independent RVs involves the utilization of the CLT. A more detailed exposition can be found in Theorem 2.

Theorem 2. *Consider a set of IID Rician RVs X_i ($i = 1, 2, \ldots, n$) characterized by shape (K_1) and scale (Ω_1) parameters. Similarly, let Y_i ($i = 1, 2, \ldots, n$) represent IID Rician RVs with shape (K_2)*

and scale (Ω_2) parameters. Assume mutual independence between the RVs X and Y. In accordance with the CLT, the summation $U = \sum_{i=1}^{n} V_i$ converges asymptotically to a normal distribution with mean $\mu_U = nE(V)$ and variance $\sigma_U^2 = n\mathit{Var}(V)$, where $E(V)$ and $\mathit{Var}(V)$ are defined in Equations 1.6 and 1.7, respectively.

Proof. When n becomes large, and in accordance with the CLT, U tends to a normally distributed RV. By virtue of Theorem 1, and due to the linearity of expectation, it follows that $\mu_U = E(U) = nE(V)$. Furthermore, owing to the independence property, the variance $\sigma_U^2 = \mathrm{Var}(U) = n\mathrm{Var}(V)$.

Theorem 3. *Let $U = \sum_{i=1}^{n} V_i = \sum_{i=1}^{n} X_i Y_i$, where $\{X_i\}$ and $\{Y_i\}$ are IID Rician random variables. Then, as $n \to \infty$, the random variable $Z = U^2$ converges in distribution to a scaled non-central chi-square distribution with one degree of freedom. That is, $Z \xrightarrow{d} \sigma_U^2 W^2$, where $W \sim \mathcal{N}(\mu_U/\sigma_U, 1)$, and $\lambda = (\mu_U/\sigma_U)^2$ is the non-centrality parameter.*

Proof. From Theorem 2, the sum $U = \sum_{i=1}^{n} V_i$ converges in distribution to a normal random variable:

$$U \xrightarrow{d} \mathcal{N}(\mu_U, \sigma_U^2),$$

where $\mu_U = n\mathbb{E}[V]$ and $\sigma_U^2 = n\,\mathrm{Var}(V)$. Define the standardised variable

$$W = \frac{U}{\sigma_U} \sim \mathcal{N}\left(\frac{\mu_U}{\sigma_U}, 1\right),$$

so that $W^2 \sim \chi_1^2(\lambda)$ is a non-central chi-square distribution with 1 degree of freedom and non-centrality parameter $\lambda = (\mu_U/\sigma_U)^2$. Then

$$Z = U^2 = \sigma_U^2 W^2.$$

Hence, the PDF of Z is given by the transformation of a non-central chi-square distribution:

$$f_Z(z) = \frac{1}{2\sigma_U^2} \exp\left\{-\frac{1}{2}\left(\frac{z}{\sigma_U^2} + \lambda\right)\right\} \left(\frac{z}{\sigma_U^2 \lambda}\right)^{-\frac{1}{4}} I_{-1/2}\left(\sqrt{\frac{\lambda z}{\sigma_U^2}}\right), \tag{2.1}$$

where $I_{-1/2}(\cdot)$ is the modified Bessel function of the first kind of order $-\frac{1}{2}$, as defined in (Gradshteyn and Ryzhik, 2000, Eq. 8.445).

To obtain the CDF of Z, we use the known result that the CDF of a non-central chi-square distribution with one degree of freedom and non-centrality parameter λ can be written in terms of the Marcum Q-function of order $\frac{1}{2}$ as:

$$P(W^2 \le w) = 1 - Q_{1/2}(\sqrt{\lambda}, \sqrt{w}),$$

see Simon (2002). Substituting $w = z/\sigma_U^2$, we obtain:

$$F_Z(z) = P(Z \le z) = 1 - Q_{1/2}\left(\sqrt{\lambda}, \sqrt{\frac{z}{\sigma_U^2}}\right), \tag{2.2}$$

which completes the proof. □

2.2 Using Laguerre expansion

To extend the estimation of the distribution of U beyond the CLT, an alternative closed-form expression is derived herein through the application of Laguerre series expansion (Section 2.2.2 Primak et al., 2004). This analytical expression remains valid for any arbitrary value of n. The choice of Laguerre series expansion is motivated by its appropriateness in capturing the non-negative characteristics, exponential decay, and orthogonality inherent in the Rician-distributed RVs X_i and Y_i. Utilizing the initial term of the Laguerre series expansion, we approximate the PDF of $U = \sum_{i=1}^{n} X_i Y_i$, resulting in an expression resembling a gamma distribution. Specifically, the PDF of U can be expressed as

$$f_U(u) \approx \frac{1}{\beta^{\alpha+1}\Gamma(\alpha+1)} u^{\alpha} \exp(-u/\beta), \tag{2.3}$$

where the shape parameter $\alpha = (\mu_U^2/\sigma_U^2) - 1$ and the scale parameter $\beta = \sigma_U^2/\mu_U$.

Utilizing the method of transforming RVs, the PDF of Z is approximated as follows

$$f_Z(z) \approx \frac{1}{2\beta^{\alpha+1}\Gamma(\alpha+1)} z^{\frac{1}{2}(\alpha-1)} \exp(-\sqrt{z}/\beta). \tag{2.4}$$

Subsequently, the CDF $F_Z(z)$ is obtained by integrating the derived PDF $f_Z(t)$ with respect to t. The integration is performed over the interval $(0, z)$, resulting in the following CDF

$$F_Z(z) \approx \frac{\gamma(\alpha+1, \frac{\sqrt{z}}{\beta})}{\Gamma(\alpha+1)}, \tag{2.5}$$

where the function $\gamma(\cdot,\cdot)$ denotes the incomplete gamma as defined in (Gradshteyn and Ryzhik, 2000, Eq. 8.350.1).

3 Statistical Goodness-of-fit Analysis and Model Selection Metrics

Goodness-of-fit metrics play a crucial role in measuring the disparity between a theoretical distribution and an empirical one. Considering the order statistics $x_{(1)}, x_{(2)}, \ldots, x_{(n)}$ corresponding to a dataset X defined as $X = \{x_1, x_2, \ldots, x_n\}$, the empirical distribution is expressed as

$$F_n(x) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{I}(x_{(i)} \leq x). \tag{3.1}$$

Here, the symbol $\mathbb{I}(\cdot)$ represents the indicator function.

Various quantitative metrics are commonly utilized to assess the agreement between theoretical and empirical distributions. Three notable metrics are discussed below.

1. Kolmogorov-Smirnov (KS) metric (Kolmogorov, 1933; Smirnov, 1948), defined as

$$D_{KS} = \sup \{|F_n(x) - F(x)|\}. \tag{3.2}$$

   Here, the function $\sup$ denotes the supremum.

2. Anderson-Darling (AD) metric (Anderson and Darling, 2015), defined as

$$D_{AD} = n \int_{-\infty}^{\infty} \frac{(F_n(x) - F(x))^2}{F(x)(1 - F(x))} dx. \tag{3.3}$$

3. Cramer-von Mises (CvM) metric (Cramer, 1928; von Mises, 1972), defined as

$$D_{CvM} = n \int_{-\infty}^{\infty} (F_n(x) - F(x))^2 \, dx. \tag{3.4}$$

These metrics quantify the deviation between theoretical and empirical distributions, with the best-fitting distribution determined by the method that minimizes D_{KS}, D_{AD}, and/or D_{CvM}.

Furthermore, the subsequent information criteria are commonly employed for the purpose of selecting the most appropriate distribution.

1. Akaike's Information Criterion (AIC) (Akaike, 1973) is defined by

$$AIC = -2 \ln \left(L(x_1, x_2, \ldots, x_n; \hat{\Theta}_{MLE}) \right) + 2g, \tag{3.5}$$

where $L\left(x_1, x_2, \ldots, x_n; \hat{\Theta}_{MLE}\right)$ denotes the maximized likelihood function, and g represents the number of parameters.

2. Bayesian Information Criterion (BIC) (Schwarz, 1978) is defined by

$$BIC = -2 \ln \left(L(x_1, x_2, \ldots, x_n; \hat{\Theta}_{MLE}) \right) + g \ln(n). \tag{3.6}$$

Models with the lowest AIC and/or BIC values are preferred.

It is important to note that while AIC and BIC were originally designed for comparing nested models under a shared likelihood function, they have been widely adopted in the statistical literature for evaluating and ranking non-nested models; particularly in contexts involving empirical goodness-of-fit assessments across distinct distribution families (Burnham and Anderson, 2004; Shmueli, 2010; Dziak et al., 2020). In our case, although the CLT and LEx yield approximations derived from different theoretical foundations, we utilize AIC and BIC solely as practical tools for comparing how well each method fits the empirical data.

Therefore, the information criteria in this study should be interpreted not as indicators of model selection in a formal inferential sense, but rather as relative measures of approximation quality and empirical fit to the simulated distribution of Z. This usage aligns with prior work on distributional model selection and approximation comparisons in applied statistics and engineering.

4 Findings and Interpretation

In Section 2, we establish analytical formulations for the PDF and CDF of the RV Z (Equation 1.1) using two distinct methodologies: the CLT and LEx. To assess the performance of these methods, we conduct a comprehensive simulation study and evaluate their goodness-of-fit using the KS, AD, and CvM statistics. Additionally, we report AIC and BIC values for model comparison.

To ensure robust performance evaluation and address concerns about statistical variability, each configuration is simulated with $R = 10000$ independent Monte Carlo replications. For each replication, we generate Z values as follows:

1. Generate two independent random samples of size n from Rician distributions with parameters (K_1, Ω_1) and (K_2, Ω_2), respectively.
2. Compute $Z = \left(\sum_{i=1}^{n} x_i y_i\right)^2$, where $x_i \sim$ Rician(K_1, Ω_1) and $y_i \sim$ Rician(K_2, Ω_2).
3. Repeat Steps 1 and 2 a total of R times to obtain an empirical sample $\{z_1, z_2, \ldots, z_R\}$.
4. Fit the theoretical distributions derived from both the CLT and LEx approaches to the empirical distribution, and compute the corresponding performance metrics.

To investigate the validity of each method across different regimes, we examine a wide range of configurations, including both small and large values of n. In particular, we include:

- Low n values (e.g., $n = 2, 7$) where CLT is expected to perform poorly due to skewness and non-normality in the distribution of Z,
- Moderate $n = 20$ to reflect practical scenarios,
- Large $n = 50, 100$ to explore the asymptotic behavior where the CLT becomes more applicable.

The results are presented in Table 1. For each configuration, the average values of AIC, BIC, KS, AD, and CvM over the 10000 simulations are reported. As shown, the LEx method consistently yields better or comparable performance to the CLT method across all configurations. As expected, the gap between the two methods narrows with increasing n, indicating that the CLT becomes more effective in the asymptotic regime. However, even for $n = 50$ and $n = 100$, the LEx approximation either outperforms or closely matches the CLT-based estimates.

Figures 1 and 2 provide visual comparisons between the empirical distribution and the two theoretical formulations under two representative configurations. Each panel contains:

- A histogram and overlaid PDFs for the empirical, CLT, and LEx distributions (top-left),
- Empirical and theoretical CDFs (top-right),
- Q–Q plot (bottom-left) comparing empirical quantiles against theoretical quantiles,
- P–P plot (bottom-right) comparing empirical CDF values with theoretical values.

The results further demonstrate that the LEx method provides a better fit to the empirical data, especially when n is small or moderate. For larger n, the CLT-based approximation becomes more competitive, as seen in the narrowing gap between performance metrics.

Table 1: Comparison of AIC and BIC criteria, as well as KS, CvM, and AD statistics, across different configurations of the parameter vector $(n, K_1, K_2, \Omega_1, \Omega_2)$. Results are reported for the Central Limit Theorem (CLT) and Laguerre Expansion (LEx) methods.

Parameters	**Method**	**AIC**	**BIC**	**KS**	**CvM**	**AD**
(2, 1.5, 3.0, 1.0, 1.0)	CLT	900	917	0.0773	0.3161	1.9632
(2, 1.5, 3.0, 1.0, 1.0)	LEx	**886**	**902**	**0.0621**	**0.1775**	**1.0984**
(7, 1.5, 3.0, 1.0, 1.0)	CLT	1680	1697	0.0652	0.2032	1.2464
(7, 1.5, 3.0, 1.0, 1.0)	LEx	**1676**	**1693**	**0.0597**	**0.1620**	**0.9935**
(20, 1.5, 3.0, 1.0, 1.0)	CLT	2315	2331	0.0602	0.1678	1.0301
(20, 1.5, 3.0, 1.0, 1.0)	LEx	**2313**	**2330**	**0.0588**	**0.1559**	**0.9515**
(50, 1.5, 3.0, 1.0, 1.0)	CLT	2867	2884	0.0610	0.1684	1.0136
(50, 1.5, 3.0, 1.0, 1.0)	LEx	**2866**	**2883**	**0.0602**	**0.1619**	**0.9723**
(100, 1.5, 3.0, 1.0, 1.0)	CLT	3283	3300	0.0607	0.1650	0.9924
(100, 1.5, 3.0, 1.0, 1.0)	LEx	3283	**3299**	**0.0605**	**0.1648**	**0.9857**
(2, 3.0, 2.0, 1.0, 2.0)	CLT	1172	1189	0.0755	0.2995	1.8491
(2, 3.0, 2.0, 1.0, 2.0)	LEx	**1161**	**1177**	**0.0630**	**0.1834**	**1.1438**
(7, 3.0, 2.0, 1.0, 2.0)	CLT	1949	1965	0.0642	0.1946	1.1988
(7, 3.0, 2.0, 1.0, 2.0)	LEx	**1945**	**1962**	**0.0602**	**0.1593**	**0.9873**
(20, 3.0, 2.0, 1.0, 2.0)	CLT	2583	2599	0.0627	0.1850	1.1198
(20, 3.0, 2.0, 1.0, 2.0)	LEx	**2582**	**2598**	**0.0614**	**0.1739**	**1.0490**
(50, 3.0, 2.0, 1.0, 2.0)	CLT	3135	3151	0.0613	0.1729	1.0381
(50, 3.0, 2.0, 1.0, 2.0)	LEx	**3134**	3151	**0.0605**	**0.1670**	**1.0040**
(100, 3.0, 2.0, 1.0, 2.0)	CLT	3553	3569	0.0602	0.1656	1.0142
(100, 3.0, 2.0, 1.0, 2.0)	LEx	**3552**	3569	**0.0601**	**0.1633**	**1.0012**

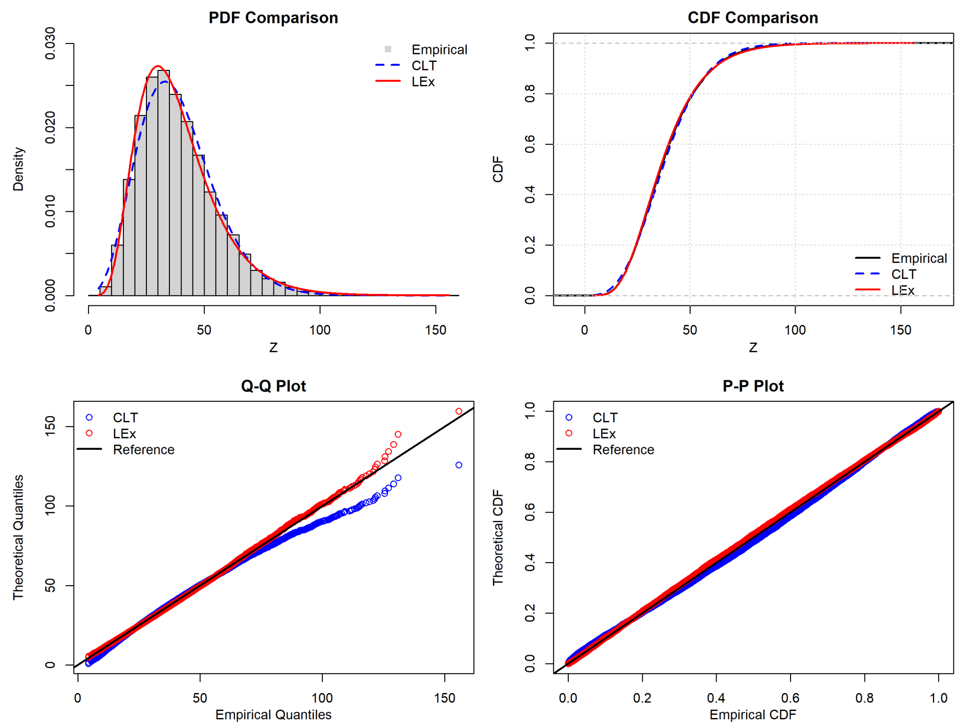

Figure 1: Comparison of simulated data with the two analytical formulations (CLT, LEx) for the parameter configuration $(n, K_1, K_2, \Omega_1, \Omega_2) = (7, 1.5, 3, 1, 1)$

In summary, the LEx method exhibits superior approximation quality across small, moderate, and large n, particularly when the distribution of Z is skewed or highly non-normal. While the CLT approach gains accuracy as n increases, it still underperforms compared to LEx for the parameter configurations considered. Hence, the LEx method is recommended as a more robust and accurate approximation for the distribution of Z.

5 Wireless Communications Application

Consider a wireless communications setup incorporating a single-source transmitter equipped with a solitary antenna, a RIS composed of n reflective elements, and a single-antenna receiver situated at the destination (See Figure 3).

The mathematical expression representing the received signal at the destination, y_d, is

$$y_d = x \left(\sum_{i=1}^{n} h_i e^{j\phi_i} g_i \right) + \epsilon, \tag{5.1}$$

where x denotes the transmitted signal and $\epsilon \sim \mathcal{N}(0, N_0)$ represents an additive white Gaussian

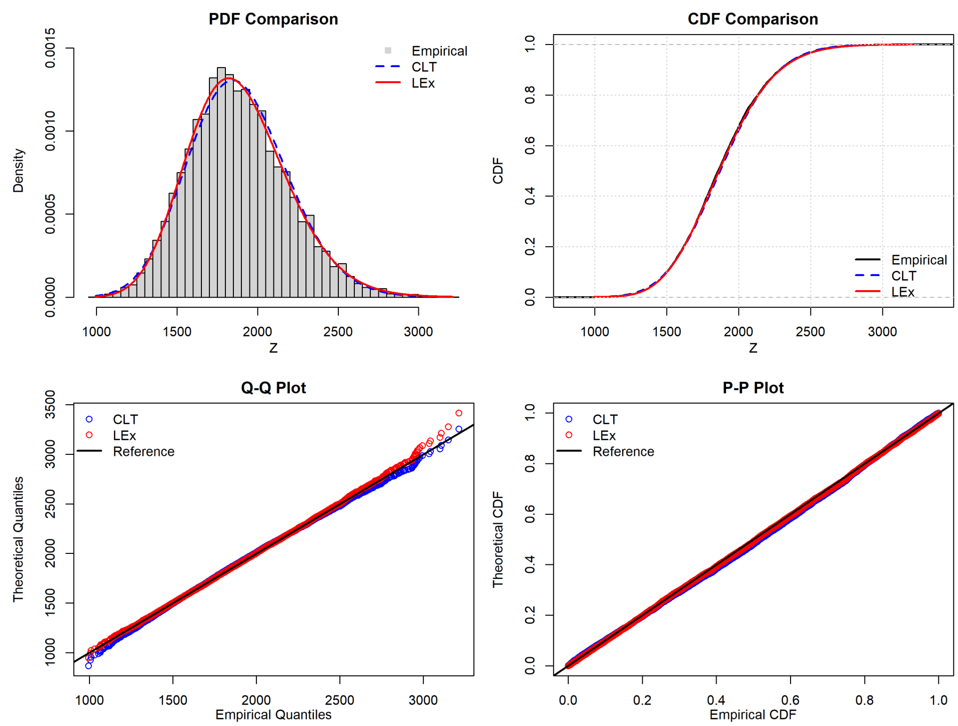

Figure 2: Comparison of simulated data with the two analytical formulations (CLT, LEx) for the parameter configuration $(n, K_1, K_2, \Omega_1, \Omega_2) = (50, 1.5, 3, 1, 1)$

noise (AWGN) term with zero mean and variance N_0. The signal transmission involves n reflecting elements, each characterized by specific parameters. The channel coefficient h_i signifies the link between the source and the i^{th} reflecting element, the channel coefficient g_i signifies the link between the i^{th} reflecting element and the destination, and ϕ_i denotes the adjustable phase induced by the i^{th} reflecting element. The channel coefficients h_i and g_i are given by the expressions $h_i = \alpha_i e^{-j\theta_i}$ and $g_i = \beta_i e^{-j\theta_i^{'}}$, where α_i and β_i represent the amplitudes, and θ_i and $\theta_i^{'}$ denote the phase shifts for the source-to-i^{th} reflecting element and i^{th} reflecting element-to-destination channels, respectively. It is assumed that $\alpha_1, \alpha_2, \ldots, \alpha_n$ and $\beta_1, \beta_2, \ldots, \beta_n$ are IID Rician RVs. Specifically, α_i follows a Rician distribution with shape K_1 and scale Ω_1 parameters, and β_i follows a Rician distribution with shape K_2 and scale Ω_2 parameters. It is crucial to note that the independence of α and β variables is maintained in this model.

Basar et al. (2019) demonstrated the calculation of the maximized e2e SNR in RIS-assisted networks using the following expression

$$\gamma = \frac{\left(\sum_{i=1}^{n} \alpha_i \beta_i\right)^2 E_s}{N_0} = \bar{\gamma} Z, \tag{5.2}$$

where E_s denotes the mean power of the transmitted signal, $\bar{\gamma}$ is the mean SNR, and N_0 denotes

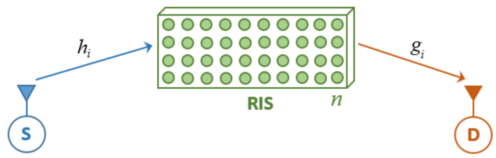

Figure 3: Wireless communications system with a RIS

the noise power. To assess the system's performance, the outage probability metric is utilized. This metric measures the likelihood that the e2e SNR falls below a preassigned outage threshold γ_{out}, stated as follows

$$P_{\text{out}} = P\left(\gamma \leq \gamma_{\text{out}}\right) = F_{\gamma}\left(\gamma_{\text{out}}\right). \tag{5.3}$$

A lower outage probability indicates superior system performance.

To derive the CDF of γ, we use the fact that γ is a bijective function of Z. Thus, based on the transformation of RVs technique, the resulting CDF is

$$F_{\gamma}\left(\omega\right) = F_{Z}(\omega/\bar{\gamma}), \tag{5.4}$$

where $F_Z(\cdot)$ represents the CDF of Z, as explained in Section 2. Consequently, there are two analytical formulations available for the CDF of γ. Subsequently, the outage probability, P_{out}, is analyzed using these two formulations.

Table 2 presents the outage probability P_{out} for a specific configuration defined by $(n, K_1, K_2, \Omega_1, \Omega_2) = (2, 1, 1, 2, 2)$, evaluated across various SNR levels. The results correspond to three approaches: Monte Carlo simulation (based on 10000 independent replications), the CLT approximation, and the LEx method, as shown in the 2nd, 3rd, and 4th columns, respectively.

The outage threshold is fixed at $\gamma_{\text{out}} = 20$ dB, which corresponds to a signal level of $10\log_{10}(100)$, i.e., one hundred times the noise power. As observed from Table 2, the LEx method provides outage probability estimates that are in close agreement with the simulation results, outperforming the CLT method in terms of accuracy. Notably, the CLT-based estimates exhibit considerable deviation from the simulated values, particularly at high SNR levels. Therefore, the CLT approach is not recommended for modeling channel distributions in RIS-assisted wireless systems with small n. However, it remains acceptable for approximating performance in low-SNR regimes, where its estimates more closely match those obtained from simulations.

6 Conclusion

This study introduced an alternative analytical method for approximating the PDF and CDF of the squared sum of products of independent Rician random variables using the Laguerre series expansion. Unlike the traditional CLT approach, which approximates the distribution by a non-central chi-squared form, the Laguerre expansion offers a flexible and accurate series representation. Closed-form expressions were derived for both the PDF and CDF, forming the foundation for precise analytical characterization.

Table 2: Outage probability estimates for $(n, K_1, K_2, \Omega_1, \Omega_2) = (2, 1, 1, 2, 2)$ across different average SNR values, based on 10,000 simulations.

SNR (dB)	Simulation	CLT	LEx
0	0.9985	0.99998	0.99829
5	0.9115	0.92640	0.91290
10	0.5304	0.46923	0.53455
15	0.1680	0.17389	0.16736
20	0.0377	0.07412	0.03213
25	0.0061	0.03705	0.00452
30	0.0015	0.01999	0.00053
35	0.0000	0.01109	0.00006
40	0.0000	0.00621	0.00001

Note: The values in the CLT and LEx columns are computed directly from the respective closed-form CDFs derived in Section 2, using the known model parameters. The zero values in the Simulation column for high SNRs (e.g., 35 and 40 dB) indicate that no outage was observed in 10000 replications; these should be interpreted as probabilities smaller than 0.0001 rather than exact zeros.

A comprehensive simulation study was conducted across a wide range of configurations, including non-identically distributed Rician variables and different values of n, to assess the performance of the two methods. Evaluation metrics such as the Kolmogorov–Smirnov, Cramér–von Mises, and Anderson–Darling statistics, as well as Akaike Information Criterion and Bayesian Information Criterion, consistently demonstrated the superiority of the Laguerre expansion. The Laguerre expansion method produced significantly lower values for all metrics, highlighting its improved approximation accuracy, especially in non-IID. and small-sample scenarios where the CLT-based method is less reliable.

Furthermore, the practical relevance of the proposed approach was illustrated through a wireless communications application involving RIS-assisted networks. By modeling the end-to-end SNR and estimating outage probabilities, the Laguerre expansion provided results closely aligned with simulation outcomes, in contrast to the notable deviation observed in the CLT-based estimates. This demonstrates the robustness and real-world utility of the Laguerre method in characterizing signal behavior under complex fading conditions.

It is worth noting that the simulation study encompasses various realistic configurations by adjusting the number of reflecting elements (n), the Rician parameters, and the SNR range. These scenarios reflect the diversity encountered in practical wireless communications settings, including non-IID environments. Consequently, the Laguerre expansion offers a promising and accurate tool for performance evaluation and design optimization in modern wireless systems, particularly where analytical tractability and accuracy are crucial.

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