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ENHANCING INFERENCE FOR RAMA DISTRIBUTION: CONFIDENCE INTERVALS AND THEIR APPLICATIONS

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SUMMARY

This research introduces and investigates four approaches for constructing confidence intervals (CIs) associated with the parameter of the Rama distribution—a model often applied in lifetime data modeling. The methods under consideration comprise the likelihood-based, Wald-type, bootstrap-t, and bias-corrected and accelerated (BCa) bootstrap intervals. To assess their practical utility, both Monte Carlo simulations and real data applications were utilized, emphasizing key performance indicators such as empirical coverage probability (ECP) and average width (AW) under various experimental conditions. To improve computational efficiency, a closed-form expression for the Wald-type CI was formulated. Simulation findings indicated that, across most situations, the ECPs obtained from both the likelihood-based and Wald-type CIs remained closely aligned with the nominal 95% confidence level. However, when the sample size was small, both the bootstrap-t and BCa bootstrap CIs yielded ECPs that fell short of the nominal level. As the sample size increased, the ECPs associated with these methods progressively approached the targeted confidence level, though variations in parameter values continued to influence their performance. The practical utility of these CIs was further validated through their application to two real-world datasets: monthly tax revenue in Egypt and plasma concentrations of indomethacin. The results from these applications were consistent with the findings of the simulation study, confirming the robustness and applicability of the proposed methods.

Keywords and phrases: bootstrap confidence interval, interval estimation, likelihood function, lifetime distribution, Wald-type confidence interval

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1 Introduction

Lifetime data analysis involves estimating the duration until a specific event occurs, such as a failure or an incident (McAtamney, 2019). A defining feature of this analysis is the handling of censoring, where certain subjects do not experience the event by the end of the study, necessitating specialized statistical techniques. Lifetime data typically follow non-normal distributions, such as the Weibull, exponential, gamma, log-normal, or Lindley distributions. Among existing distributions, the exponential and Lindley distributions are frequently applied in lifetime data analysis. Nevertheless, both have notable limitations—for instance, the exponential distribution presumes a constant hazard rate and possesses the memoryless property, which often proves unrealistic in practical applications. Although the Lindley distribution provides an improvement over the exponential distribution (Ghitany et al., 2008), both remain insufficient for capturing data characterized by non-constant hazard rates. Therefore, the development of new and flexible probability distributions has become essential to improve the accuracy of distribution fitting in lifetime data analysis.

Researchers use diverse methodologies to develop new probability distributions or enhance the efficacy of classical ones. One common approach involves incorporating additional parameters to increase flexibility, as seen in the two-parameter Lindley distribution (Shanker et al., 2013), the three-parameter Lindley distribution (Shanker et al., 2017), the exponentiated Shanker distribution (Abdollahi Nanvapisheh et al., 2019), and the two-parameter Shanker distribution (Olufemi-Ojo et al., 2024). However, while such modifications improve flexibility, they also introduce complexity, making the models harder to interpret. This complexity can lead to overfitting, requiring larger datasets to achieve accurate results and complicating the estimation of parameter values. Additionally, the inclusion of extra parameters increases computational demands, which can hinder practical applications.

In contrast, several researchers have proposed new mixed probability distributions that do not rely on additional parameters. Examples include the Shanker distribution (Shanker, 2015b), Aradhana distribution (Rama, 2016), Gharaibeh distribution (Gharaibeh, 2021), Iwueze distribution (Elechi et al., 2022), Juchez distribution (Mbegbu and Echebiri, 2022), and Ola distribution (Al-Ta'ani and Gharaibeh, 2023). These mixed distributions, which extend the Lindley distribution, often outperform classical distributions in terms of flexibility and accuracy. They provide a robust framework for modeling diverse data types, making them highly versatile.

Among these, the Rama distribution, introduced by Shanker (2017a), stands out as a mixture of exponential and gamma distributions. It has demonstrated exceptional performance in modeling lifetime data, particularly in applications. This highlights its potential as a reliable tool for analyzing complex datasets in various fields.

The confidence interval (CI) is an essential tool for accurately estimating the distribution parameters in statistical inference. However, a review of the current literature found no researches to construct CIs for the parameter of the Rama distribution. Therefore, this research addressed this gap by suggesting four methods for estimating CIs for the parameter of the Rama distribution using the likelihood-based CI, Wald-type CI, bootstrap-t CI, and bias-corrected and accelerated (BCa) bootstrap CI. Monte Carlo simulation experiments were conducted and two real-world datasets were also analyzed to evaluate the performances of these four CI estimation approaches.

2 Methodology

This section provides an overview of the Rama distribution, including point parameter estimation and the construction of confidence intervals.

2.1 The Rama distribution and its parameter estimation

The Rama distribution is derived as a mixture of the exponential and a gamma distribution, with specific mixing probabilities applied to combine these two. In this formulation, the gamma distribution is defined with a fixed scale parameter, denoted as θ, and a shape parameter set to 4. Let X denote a random variable that follows the Rama distribution with parameter θ. The probability density function (pdf) of the Rama distribution is derived from two classical continuous distributions, in which each component is associated with a corresponding mixing weight, denoted by w_1 and w_2. The resulting pdf is given by:

$$f(x;\theta) = w_1 f_{\text{Exp}}(x;\theta) + w_2 f_{\text{Gamma}}(x;\theta,4),$$

where $w_1 = \frac{\theta^3}{\theta^3+6}$, $w_2 = \frac{6}{\theta^3+6}$, $f_{\text{Exp}}(x;\theta)$ is the pdf of the exponential distribution with rate parameter θ and $f_{\text{Gamma}}(x;\theta,4)$ is the pdf of the gamma distributions with rate parameter θ and shape parameter 4. Therefore, the pdf of the Rama distribution are expressed as follows

$$f(x;\theta) = \frac{\theta^3}{\theta^3+6}\left(\theta e^{-\theta x}\right) + \frac{6}{\theta^3+6}\left(\frac{1}{6}\theta^4 x^3 e^{-\theta x}\right) = \frac{\theta^4}{\theta^3+6}(1+x^3)e^{-\theta x},$$

where $x > 0$ and $\theta > 0$. The cumulative distribution function of the Rama distribution is given by

$$F(x;\theta) = 1 - \left(1 + \frac{\theta^3 x^3 + 3\theta^2 x^2 + 6\theta x}{\theta^3+6}\right)e^{-\theta x}.$$

Figure 1 illustrates the probability density function plot of the Rama distribution for various parameter values. The mean ($\mu_X = \mathbb{E}(X)$) and the r^{th} non-central moment (μ'_r) of X can be represented as

$$\mathbb{E}(X) = \frac{\theta^3+24}{\theta(\theta^3+6)}, \text{ and } \mu'_r = \frac{r![\theta^3 + (r+1) + (r+2)(r+3)]}{\theta^r(\theta^3+6)}, \quad r = 1,2,3,\ldots$$

2.2 Point parameter estimation

The point estimator for the parameter θ of the Rama distribution can be derived using the maximum likelihood (ML) method, as outlined in the following steps

Step 1: *Find the likelihood function.* The likelihood function $L(\theta; x_i)$ represents the joint probability of observing the random sample $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ of size n drawn from the Rama distribution. It is expressed as follows

$$L(\theta;\mathbf{x}) = \left(\frac{\theta^4}{\theta^3+6}\right)^n \prod_{i=1}^{n}(1+x_i^3)e^{-\theta\sum_{i=1}^{n}x_i}.$$

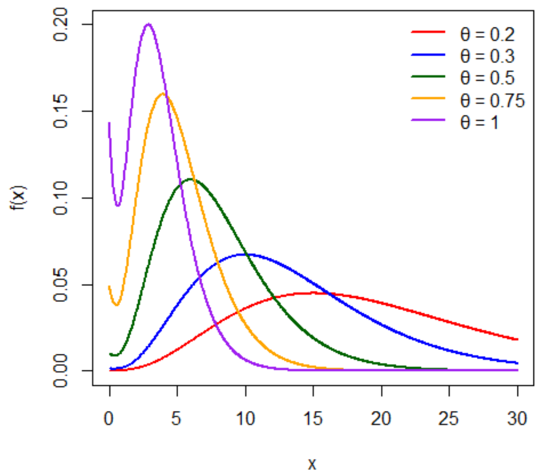

Figure 1: The probability density function plot of the Rama distribution for several parameter values.

Step 2: *Find the log-likelihood function*. Due to the complexity of differentiating the likelihood function, the log-likelihood function is derived to simplify the differentiation. The log-likelihood function is

$$\log L(\theta; \mathbf{x}) = n \log \left(\frac{\theta^4}{\theta^3 + 6} \right) + \sum_{i=1}^{n} \log(1 + x_i^3) - \theta \sum_{i=1}^{n} x_i.$$

Step 3: *Differentiate the log-likelihood function*. To determine the value of θ that maximizes the log-likelihood function, the derivative of $\log L(\theta; x_i)$ with respect to θ is computed. This yields the score function $S(\theta; x_i)$, which is expressed as follows

$$\begin{aligned} S(\theta; \mathbf{x}) &= \frac{\partial}{\partial \theta} \log L(\theta; x_i) \\ &= \frac{\partial}{\partial \theta} \left[4n \log(\theta) - n \log(\theta^3 + 6) + \sum_{i=1}^{n} \log(1 + x_i^3) - \theta \sum_{i=1}^{n} x_i \right] \\ &= \frac{4n}{\theta} - \frac{3n\theta^2}{\theta^3 + 6} - \sum_{i=1}^{n} x_i. \end{aligned}$$

Step 4: *Set the derivative equal to zero and solve for the ML estimator*. The value of θ is obtained by solving the equation $S(\theta; x_i) \overset{\text{set}}{=} 0$, which involves setting the score function to zero:

$$S(\theta; \mathbf{x}) = \frac{4n}{\theta} - \frac{3n\theta^2}{\theta^3 + 6} - \sum_{i=1}^{n} x_i \overset{\text{set}}{=} 0.$$

This process identifies the critical points that may correspond to local maxima of the likelihood function with respect to θ. As no explicit solution exists for the ML estimator of the parameter θ, this study applies numerical optimization methods to address the resulting non-linear equation. Specifically, the Newton-Raphson algorithm, available through the `maxLik` package (Henningsen and Toomet, 2011), was utilized in R programming language (R Core Team, 2023) to compute the ML estimates.

2.3 Confidence intervals

2.3.1 Likelihood-based confidence interval

The likelihood-based CI method estimates parameter by using the likelihood function and its parameter match the observed data. The main idea behind this method is to find the range of parameter value that make the likelihood function as large as possible, while still meeting the desired confidence level. This is done by adjusting the log-likelihood function based on the parameter being estimated.

The ML estimate of the parameter θ is found by solving the score equation $S(\theta; x_i) \overset{\text{set}}{=} 0$, which means the derivative of the log-likelihood function with respect to θ is zero. This value of θ gives the best match to the observed data based on the assumed probability model. Using this estimate, a likelihood-based CI can then be constructed. This approach employs the likelihood ratio, defined as

$$\lambda(\theta) = \frac{L(\theta; \mathbf{x})}{L(\widehat{\theta}; \mathbf{x})},$$

which quantifies how the likelihood at a specific value of θ compares to its maximum at $\widehat{\theta}$.

Under suitable regularity conditions, Wilks' theorem (Wilks, 1938) states that the statistic $-2\log\lambda(\theta)$ asymptotically follows a chi-square distribution with degrees of freedom equal to the number of parameters being estimated. As a result, the likelihood-based CI for θ at the $(1-\alpha)100\%$ confidence level can be expressed as

$$\begin{aligned}&\left\{\theta \;\middle|\; -2\log\frac{L(\theta; \mathbf{x})}{L(\widehat{\theta}; \mathbf{x})} \leq \chi^2_{1-\alpha,1}\right\}\\ &= \left\{\theta \;\middle|\; -2\log\left[\frac{\theta^{4n}(\widehat{\theta}^3+6)^n}{\widehat{\theta}^{4n}(\theta^3+6)^n}\exp\left(-\theta\sum_{i=1}^{n}x_i + \widehat{\theta}\sum_{i=1}^{n}x_i\right)\right] \leq \chi^2_{1-\alpha,1}\right\},\end{aligned}$$

where $\chi^2_{1-\alpha,1}$ denotes the $(1-\alpha)$ quantile of the chi-square distribution with one degree of freedom (Saleh, 2006; Severini, 2000).

For the Rama distribution, the likelihood ratio test (LRT) presents additional complexity due to the model's mixed nature. To overcome these computational difficulties, likelihood-based CI is typically obtained through numerical optimization techniques.

The ML estimator for the parameter θ of the Rama distribution was obtained by numerically solving the score equation using Brent's method (Brent, 1973), a hybrid root-finding algorithm. The equation to be solved is:

$$f(\theta) = S(\theta; \mathbf{x}) = \frac{4n}{\theta} - \frac{3n\theta^2}{\theta^3 + 6} - \sum_{i=1}^{n} x_i \overset{\text{set}}{=} 0,$$

where $S(\theta; \mathbf{x})$ is the score function. Brent's method finds a root of $f(\theta) = 0$ by combining the reliability of bracketing methods with the efficiency of interpolation techniques.

When $f(a) \cdot f(b) < 0$, the method begins with a bisection step to ensure that a root lies within the interval $[a, b]$. Depending on the function's behavior and the position of the interval endpoints, the algorithm alternates adaptively between the secant method and inverse quadratic interpolation. Specifically, the secant update is given by:

$$\theta_{\text{secant}} = \theta_n - f(\theta_n) \cdot \frac{\theta_n - \theta_{n-1}}{f(\theta_n) - f(\theta_{n-1})},$$

while the inverse quadratic interpolation step can be expressed as

$$\theta_{\text{quad}} = \frac{f(\theta_{n-1}) f(\theta_{n-2})}{(f(\theta_n) - f(\theta_{n-1}))(f(\theta_n) - f(\theta_{n-2}))} \theta_n + \dots$$

The likelihood-based CI is obtained by iteratively computing the statistic $-2\log(\lambda(\theta))$ and comparing it against the 0.95 quantile of the chi-square distribution with one degree of freedom. As shown in Figure 2, the plot displays the profile of $-2\log(\lambda(\theta))$ (blue curve), the critical chi-square threshold $\chi^2_{0.95,1}$ (red line), and the corresponding 95% likelihood-based CI (green segment), derived from a simulated sample of size 20 drawn from the Rama distribution with $\theta = 1$.

2.3.2 Wald-type confidence interval

The Wald-type CI is a classic approach to interval estimation that exploits the asymptotic normality of the ML estimator. By combining the ML estimator with its estimated standard error and the appropriate quantile of the normal distribution, one obtains an approximate CI for the parameter. Since the first-order derivative of the log-likelihood function equals zero at the ML estimate, the Wald statistic approximates the likelihood ratio by using the second-order term of the expansion:

$$\log L(\theta; \mathbf{x}) \approx \log L(\widehat{\theta}; \mathbf{x}) + (\theta - \widehat{\theta}) \left. \frac{\partial}{\partial \theta} \log L(\theta; \mathbf{x}) \right|_{\theta = \widehat{\theta}} + \frac{1}{2}(\theta - \widehat{\theta})^2 \left. \frac{\partial^2}{\partial \theta^2} \log L(\theta; \mathbf{x}) \right|_{\theta = \widehat{\theta}},$$

$$\log \frac{L(\theta; \mathbf{x})}{L(\widehat{\theta}; \mathbf{x})} \approx \frac{1}{2}(\theta - \widehat{\theta})^2 \left. \frac{\partial^2}{\partial \theta^2} \log L(\theta | x) \right|_{\theta = \widehat{\theta}}, -2\log \frac{L(\theta; \mathbf{x})}{L(\widehat{\theta}; \mathbf{x})} \approx (\theta - \widehat{\theta})^2 I(\widehat{\theta}),$$

where $I(\widehat{\theta})$ is the estimated Fisher information. When the sample size is sufficiently large, asymptotic theory allows the Wald statistic to serve as a valid approximation to the LRT statistic. Under such conditions, the log-likelihood ratio can be approximated by a quadratic form (Pawitan, 2001). We obtain the first and second derivatives of the Rama distribution's log-likelihood function as fol-

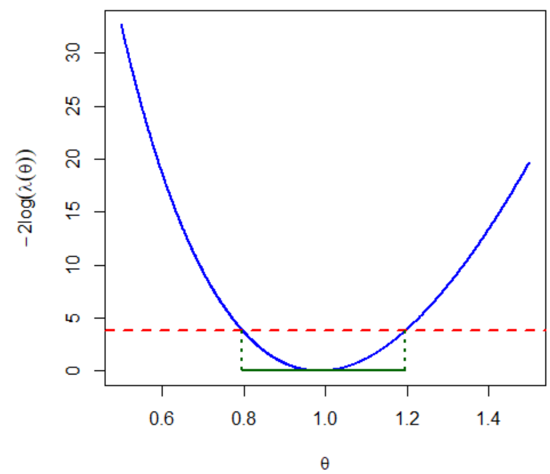

Figure 2: The plot of $-2\log(\lambda(\theta))$ versus θ.

lows

$$\frac{\partial}{\partial\theta}\log L(\theta;\mathbf{x}) = \frac{4n}{\theta} - \frac{3n\theta^2}{\theta^3+6} - \sum_{i=1}^{n} x_i,$$
$$\frac{\partial^2}{\partial\theta^2}\log L(\theta;\mathbf{x}) = -\frac{4n}{\theta^2} - \frac{6n\theta}{(\theta^3+6)} + \frac{9n\theta^4}{(\theta^3+6)^2} = -\frac{4n}{\theta^2} - \left(3n\theta\left(\frac{12-\theta^3}{(\theta^3+6)^2}\right)\right).$$

The estimated Fisher information is therefore given by the following expression

$$I(\widehat{\theta}) = \mathbb{E}\left[-\frac{\partial^2}{\partial\theta^2}\log L(\theta;\mathbf{x})|_{\theta=\widehat{\theta}}\right] = \frac{4n}{\widehat{\theta}^2} + 3n\widehat{\theta}\frac{12-\widehat{\theta}^3}{(\widehat{\theta}^3+6)^2}. \tag{2.1}$$

The Wald-type confidence interval for θ with a confidence level of $(1-\alpha)100\%$ takes the form

$$\left(\widehat{\theta} - z_{1-\frac{\alpha}{2}}\sqrt{I^{-1}(\widehat{\theta})},\ \widehat{\theta} + z_{1-\frac{\alpha}{2}}\sqrt{I^{-1}(\widehat{\theta})}\right),$$

where $z_{1-(\alpha/2)}$ denotes the $(1-\alpha/2)^{\text{th}}$ quantile of the standard normal distribution, and $I^{-1}(\widehat{\theta})$ is the inverse function of $I(\widehat{\theta})$ given in Equation (2.1).

2.3.3 Bootstrap-t confidence interval

The bootstrap-t CI constructs parameter interval estimates by combining the bootstrap resampling method with the properties of the t-distribution. Unlike the percentile bootstrap method, this approach explicitly accounts for the estimated standard error of the parameter, thereby enhancing the interval's accuracy and robustness. This technique is especially beneficial when working with small samples or when the distribution of the estimator is skewed or non-normal (Davison and Hinkley, 1997). The steps for constructing a bootstrap-t CI are outlined as follows

1. The procedure starts with drawing a random sample $\big(X_1, \ldots, X_n\big)$ from the original population. Based on this sample, the parameter θ is estimated using the ML estimator $\widehat{\theta}$.

2. Generate a total of $B = 1{,}000$ bootstrap resamples by repeating the following procedure: for each iteration $b = 1, \ldots, B$, construct a bootstrap sample $X^{*(b)} = \big(X_1^{*(b)}, \ldots, X_n^{*(b)}\big)$ of size n by drawing observations with replacement from the original dataset $\big(X_1, \ldots, X_n\big)$.

3. For each bootstrap sample $X^{*(b)}$ $(b = 1, \ldots, B)$, compute the bootstrap replicate of the ML estimator $\widehat{\theta}^{*(b)}$. Collecting these values yields the set $\big\{\widehat{\theta}^{*(1)}, \ldots, \widehat{\theta}^{*(B)}\big\}$, which serves as an empirical approximation to the sampling distribution of $\widehat{\theta}$.

4. For each bootstrap replicate $b = 1, \ldots, B$, compute the bootstrap-t statistic

$$t^{*(b)} \;=\; \frac{\widehat{\theta}^{*(b)} - \widehat{\theta}}{\sqrt{I^{-1}\big(\widehat{\theta}^{*(b)}\big)}},$$

   where $I^{-1}\big(\widehat{\theta}^{*(b)}\big)$ denotes the inverse of $I(\theta)$ evaluated at the bootstrap estimate $\widehat{\theta}^{*(b)}$; see Equation 2.1. Collecting these values yields the set $\{\, t^{*(1)}, \ldots, t^{*(B)}\}$, which approximates the sampling distribution of the Studentised statistic.

5. Repeat Steps 1-3 for all $b = 1, \ldots, B$ to obtain the collection $\big\{\widehat{\theta}^{*(1)}, \widehat{\theta}^{*(2)}, \ldots, \widehat{\theta}^{*(B)}\big\}$. This set constitutes an empirical distribution of the estimator, which serves as a plug-in approximation to the sampling distribution of the pivotal quantity.

6. Using the corresponding bootstrap-t statistics $\big\{t^{*(1)}, t^{*(2)}, \ldots, t^{*(B)}\big\}$, construct their empirical distribution $F_{t^*}(t)$. This distribution provides a non-parametric estimate of the true distribution of the studentised statistic and will be used to form the CI.

7. Determine the lower and upper critical values $t^*_{(\alpha/2)}$ and $t^*_{(1-\alpha/2)}$ as the $(\alpha/2)^{\text{th}}$ and $\big(1 - \alpha/2\big)^{\text{th}}$ empirical quantiles of the bootstrap–t distribution:

$$\frac{\#\big\{\, t^* \le t^*_{(\alpha/2)} \,\big\}}{B} \;=\; \alpha/2, \qquad \frac{\#\big\{\, t^* \le t^*_{(1-\alpha/2)} \,\big\}}{B} \;=\; 1 - \alpha/2,$$

   where $\#(\,\cdot\,)$ denotes the counting operator.

8. The resulting $(1 - \alpha)$ bootstrap–t confidence interval for θ is

$$\left(\widehat{\theta} + t^*_{(\alpha/2)} \sqrt{I^{-1}(\widehat{\theta})},\; \widehat{\theta} + t^*_{(1-\alpha/2)} \sqrt{I^{-1}(\widehat{\theta})}\right).$$

2.3.4 Bias-corrected and accelerated bootstrap confidence interval

The bias-corrected and accelerated (BCa) bootstrap CI improves the simple percentile method by introducing two corrections: the bias correction factor z_0, and the acceleration parameter a. These tweaks shift the empirical quantiles to give more accurate intervals for small or skewed samples, all without parametric assumptions (Bittmann, 2021). The procedure is outlined as follows

1. The procedure begins by drawing a random sample $(X_1, \ldots, X_n)$ from the population, after which the parameter θ is estimated by its point estimator $\widehat{\theta}$.

2. Next, generate $B = 1{,}000$ bootstrap resamples, $\{X_1^{*(b)}, \ldots, X_n^{*(b)}\}_{b=1}^{B}$, by sampling n observations with replacement from the original dataset for each b. Thus, every bootstrap sample remains the same size n as the original sample.

3. For each bootstrap sample $b = 1, \ldots, B$, compute the corresponding estimate
$$\widehat{\theta}^{*(b)} \ = \ g\big(X_1^{*(b)}, \ldots, X_n^{*(b)}\big),$$
where $g(\cdot)$ denotes the ML estimator. Collecting these values yields the set $\{\widehat{\theta}^{*(1)}, \ldots, \widehat{\theta}^{*(B)}\}$, which approximates the sampling distribution of $\widehat{\theta}$.

4. Determine the proportion p of the bootstrap estimates $\widehat{\theta}_b^*$ that are less than the original estimate $\widehat{\theta}$, that is,
$$p = \frac{1}{B} \sum_{b=1}^{B} \mathbb{I}\left(\widehat{\theta}_b^* < \widehat{\theta}\right),$$
where $\mathbb{I}(\cdot)$ is the indicator function. The bias correction factor z_0 is then defined as the standard normal quantile corresponding to the probability p, i.e.,
$$z_0 = \Phi^{-1}(p),$$
where $\Phi^{-1}(\cdot)$ denotes the inverse cumulative distribution function (quantile function) of the standard normal distribution.

5. To account for the asymmetry in the sampling distribution of the estimator, the acceleration value a is computed. This adjustment reflects the rate of change in the standard error with respect to the true parameter value. The acceleration factor is commonly estimated via the jackknife method that assess the asymmetry of the estimator's distribution.

6. To refine the percentile bounds for constructing the confidence interval, the bias-corrected normal deviates are transformed using the acceleration factor. The adjusted lower and upper percentiles are computed as
$$p_L^* = \Phi\left(z_0 + \frac{z_0 + z_{\alpha/2}}{1 - a(z_0 + z_{\alpha/2})}\right) \quad \text{and} \quad p_U^* = \Phi\left(z_0 + \frac{z_0 + z_{1-\alpha/2}}{1 - a(z_0 + z_{1-\alpha/2})}\right),$$
where Φ denotes the cumulative distribution function of the standard normal distribution. The values $z_{\alpha/2}$ and $z_{1-\alpha/2}$ represent the lower and upper quantiles of the standard normal distribution corresponding to the desired confidence level.

7. The BCa bootstrap CI is obtained by selecting the empirical quantiles that correspond to the adjusted lower and upper percentiles p_L^* and p_U^*. Formally, the BCa bootstrap confidence interval is given by:

$$\left(\widehat{\theta}^*_{(p_L^*)},\ \widehat{\theta}^*_{(p_U^*)}\right),$$

where $\widehat{\theta}^*_{(p_L^*)}$ and $\widehat{\theta}^*_{(p_U^*)}$ represent the p_L^* and p_U^* quantiles, respectively, of the ordered bootstrap estimates $\widehat{\theta}^*_b$.

3 Simulation Studies and Results

In this study, we propose four methods for constructing confidence intervals (CIs) for the parameter of the Rama distribution. Their performance is assessed through an extensive Monte Carlo simulation study, implemented in R, across a wide range of scenarios.

To evaluate the efficiency of each method, we examined seven sample sizes–n = 10, 20, 30, 50, 100, 200, and 500–and ten parameter values, θ = 0.2, 0.3, 0.5, 0.75, 1, 1.5, 2, 3, 4, and 5. For every (n, θ) combination, 1,000 Monte Carlo replications were performed. Performance was assessed with two metrics: (i) the empirical coverage probability (ECP), the proportion of intervals that contain the true parameter value; and (ii) the average width (AW), the mean length of the intervals, which reflects their precision.

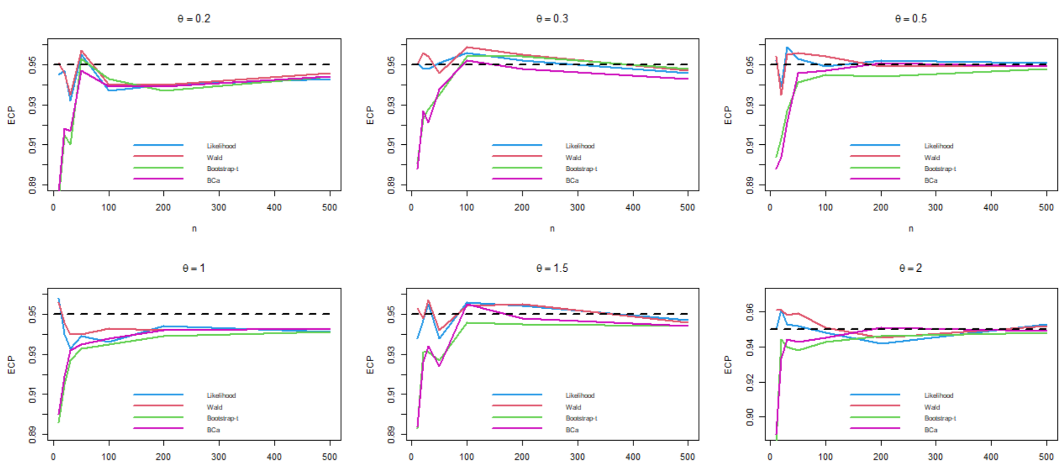

Figure 3: Plots of the ECPs of the CIs for the parameter of the Rama distribution.

Tables 1-3 and Figures 3-4 compare the ECP and AW of four CIs for the Rama-distribution parameter. Overall, the likelihood-based and Wald-type intervals achieve the highest ECPs, remaining close to the nominal level of 0.95 in nearly every situation and stabilising as n increases. By contrast, for small samples ($n = 10, 20$, or 30) the bootstrap-t and BCa intervals frequently undercover,

Table 1: Empirical Coverage Probability and Average Width of 95% CIs for θ in the Rama Distribution where θ = 0.2, 0.3, 0.5 and 0.75

θ	n	Empirical Coverage Probability				Average Width			
		Likelihood	Wald	Boot-t	BCa	Likelihood	Wald	Boot-t	BCa
0.2	10	0.945	0.951	0.882	0.886	0.1267	0.1264	0.1121	0.1178
	20	0.947	0.946	0.915	0.918	0.0887	0.0886	0.0841	0.0861
	30	0.932	0.935	0.910	0.917	0.0721	0.0721	0.0695	0.0706
	50	0.955	0.957	0.953	0.947	0.0557	0.0557	0.0544	0.0550
	100	0.937	0.940	0.943	0.939	0.0393	0.0393	0.0387	0.0390
	200	0.940	0.940	0.937	0.939	0.0277	0.0277	0.0275	0.0277
	500	0.943	0.946	0.944	0.944	0.0175	0.0175	0.0175	0.0176
0.3	10	0.951	0.950	0.901	0.898	0.1909	0.1905	0.1708	0.1796
	20	0.948	0.956	0.923	0.927	0.1334	0.1333	0.1251	0.1275
	30	0.948	0.954	0.928	0.921	0.1083	0.1082	0.1036	0.1052
	50	0.951	0.946	0.935	0.938	0.0832	0.0831	0.0809	0.0816
	100	0.956	0.959	0.954	0.952	0.0585	0.0585	0.0578	0.0583
	200	0.952	0.955	0.954	0.948	0.0416	0.0416	0.0412	0.0416
	500	0.946	0.947	0.948	0.943	0.0262	0.0262	0.0259	0.0262
0.5	10	0.952	0.954	0.904	0.898	0.3125	0.3122	0.2800	0.2921
	20	0.939	0.935	0.913	0.904	0.2186	0.2185	0.2060	0.2097
	30	0.959	0.955	0.927	0.921	0.1772	0.1772	0.1705	0.1722
	50	0.953	0.956	0.941	0.946	0.1373	0.1373	0.1334	0.1347
	100	0.949	0.954	0.945	0.947	0.0970	0.0970	0.0955	0.0964
	200	0.952	0.949	0.944	0.951	0.0684	0.0684	0.0680	0.0685
	500	0.951	0.949	0.948	0.949	0.0432	0.0432	0.0429	0.0433
0.75	10	0.957	0.960	0.911	0.907	0.4537	0.4535	0.4063	0.4221
	20	0.950	0.958	0.923	0.935	0.3164	0.3164	0.2975	0.3016
	30	0.943	0.942	0.922	0.921	0.2580	0.2579	0.2479	0.2502
	50	0.956	0.959	0.946	0.942	0.1990	0.1990	0.1958	0.1976
	100	0.960	0.959	0.952	0.950	0.1408	0.1408	0.1383	0.1394
	200	0.953	0.956	0.949	0.948	0.0997	0.0997	0.0983	0.0991
	500	0.947	0.944	0.945	0.953	0.0632	0.0632	0.0629	0.0633

Table 2: Empirical Coverage Probability and Average Width of 95% CIs for θ in the Rama Distribution where θ = 1, 1.5, 2 and 3

θ	n	Empirical Coverage Probability				Average Width			
		Likelihood	Wald	Boot-t	BCa	Likelihood	Wald	Boot-t	BCa
1	10	0.958	0.956	0.896	0.900	0.5828	0.5818	0.5260	0.5521
	20	0.940	0.946	0.915	0.919	0.4097	0.4094	0.3915	0.3993
	30	0.933	0.940	0.927	0.932	0.3326	0.3325	0.3198	0.3239
	50	0.939	0.940	0.933	0.935	0.2573	0.2573	0.2506	0.2531
	100	0.936	0.943	0.935	0.938	0.1821	0.1821	0.1802	0.1818
	200	0.944	0.942	0.939	0.942	0.1282	0.1282	0.1271	0.1279
	500	0.941	0.943	0.941	0.943	0.0813	0.0813	0.0808	0.0816
1.5	10	0.938	0.953	0.893	0.894	0.7219	0.7182	0.6504	0.6978
	20	0.947	0.948	0.931	0.926	0.4991	0.4980	0.4760	0.4885
	30	0.955	0.957	0.931	0.934	0.4078	0.4072	0.3914	0.3985
	50	0.938	0.942	0.927	0.924	0.3144	0.3141	0.3071	0.3110
	100	0.956	0.954	0.946	0.955	0.2208	0.2207	0.2181	0.2201
	200	0.954	0.955	0.945	0.948	0.1566	0.1566	0.1542	0.1560
	500	0.947	0.946	0.944	0.944	0.0985	0.0985	0.0980	0.0987
2	10	0.950	0.961	0.885	0.890	0.8770	0.8688	0.7777	0.8531
	20	0.961	0.961	0.944	0.933	0.6015	0.5989	0.5701	0.5915
	30	0.953	0.958	0.940	0.944	0.4863	0.4849	0.4666	0.4772
	50	0.952	0.959	0.938	0.943	0.3751	0.3745	0.3629	0.3697
	100	0.948	0.951	0.943	0.945	0.2639	0.2637	0.2606	0.2632
	200	0.942	0.945	0.946	0.951	0.1860	0.1860	0.1844	0.1861
	500	0.953	0.952	0.948	0.949	0.1177	0.1177	0.1162	0.1170
3	10	0.948	0.963	0.893	0.882	2.5554	2.4993	2.1393	2.6827
	20	0.932	0.946	0.906	0.895	1.6883	1.6682	1.5368	1.6746
	30	0.950	0.960	0.926	0.915	1.3325	1.3216	1.2560	1.3180
	50	0.942	0.952	0.926	0.930	1.0117	1.0067	0.9725	1.0048
	100	0.960	0.958	0.947	0.952	0.6980	0.6962	0.6811	0.6937
	200	0.952	0.950	0.942	0.944	0.4925	0.4919	0.4863	0.4928
	500	0.951	0.948	0.952	0.950	0.3111	0.3109	0.3092	0.3126

Table 3: Empirical Coverage Probability and Average Width of 95% CIs for θ in the Rama Distribution where θ = 4 and 5

θ	n	Empirical Coverage Probability				Average Width			
		Likelihood	Wald	Boot-t	BCa	Likelihood	Wald	Boot-t	BCa
	10	0.949	0.957	0.879	0.836	4.2201	4.1481	3.4849	4.3082
	20	0.948	0.966	0.930	0.916	2.7506	2.7247	2.4975	2.7074
	30	0.948	0.951	0.928	0.927	2.1702	2.1557	2.0378	2.1370
4	50	0.950	0.955	0.935	0.928	1.6353	1.6283	1.5694	1.6126
	100	0.945	0.948	0.936	0.938	1.1344	1.1320	1.1114	1.1270
	200	0.965	0.965	0.952	0.958	0.7973	0.7965	0.7821	0.7911
	500	0.950	0.953	0.946	0.943	0.5018	0.5016	0.4970	0.5012
	10	0.947	0.960	0.893	0.868	5.8087	5.7333	4.8028	5.8397
	20	0.949	0.950	0.911	0.909	3.7856	3.7610	3.4046	3.6384
	30	0.945	0.946	0.923	0.924	3.0451	3.0316	2.8718	2.9621
5	50	0.947	0.953	0.927	0.924	2.3021	2.2955	2.2193	2.2636
	100	0.959	0.958	0.943	0.945	1.6098	1.6076	1.5643	1.5866
	200	0.957	0.959	0.944	0.946	1.1265	1.1257	1.1028	1.1156
	500	0.953	0.952	0.942	0.947	0.7077	0.7075	0.6992	0.7060

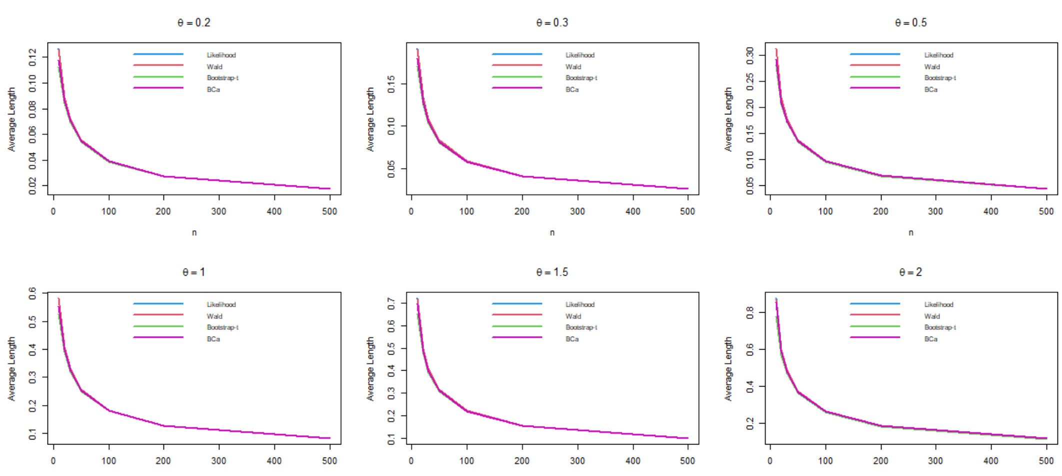

Figure 4: Plots of the AWs of the CIs for the parameter of the Rama distribution.

with ECPs below 0.95. This gap narrows at larger sample sizes ($n = 100, 200$, and 500), where the ECPs of all four methods converge on the nominal target, indicating that bootstrap-based intervals are more sensitive to limited data.

Parameter values also affected performance. For small θ (0.20-0.50), the likelihood-based and Wald-type CIs maintained ECPs near the nominal 0.95. As θ increased (1–5), the bootstrap-t and BCa intervals began to under-cover, most notably at n = 10 and θ = 2, where their ECPs dropped to 0.885 and 0.890, respectively, compared with 0.950 and 0.961 for the likelihood-based and Wald-type intervals. Hence, bootstrap methods are more sensitive to large parameter values when data are sparse. All four CIs widened with increasing θ; however, the bootstrap-based intervals remained the narrowest, trading precision for reduced coverage in the most challenging situations.

4 Applications to Real Data

To benchmark the practical utility of the Rama distribution, we compared its fit with several one–parameter lifetime distributions whose pdfs are defined on $(0, \infty)$ and are each characterised by a single positive scale parameter $\theta > 0$:

1. The Ola distribution (Al-Ta'ani and Gharaibeh, 2023) $f(x;\theta) = \frac{\theta^8}{\theta^7+6\theta^4+5040}\left(x^7+x^3+1\right)e^{-\theta x}$.
2. The Pratibha distribution (Shanker, 2023b) $f(x;\theta) = \frac{\theta^3}{\theta^3+\theta+2}(\theta+x+x^2)\,e^{-\theta x}$.
3. The Komal distribution (Shanker, 2023a) $f(x;\theta) = \frac{\theta^2}{\theta^2+\theta+1}(1+\theta+x)\,e^{-\theta x}$.
4. The Chris-Jerry distribution (Onyekwere and Obulezi, 2022) $f(x;\theta) = \frac{\theta^2}{\theta+2}(1+\theta x^2)\,e^{-\theta x}$.
5. The Juchez distribution (Mbegbu and Echebiri, 2022) $f(x;\theta) = \frac{\theta^4}{\theta^3+\theta^2+6}(1+x+x^3)\,e^{-\theta x}$.
6. The Adya distribution (Shanker et al., 2021) $f(x;\theta) = \frac{\theta^3}{\theta^4+2\theta^2+2}(\theta+x)^2\,e^{-\theta x}$.
7. The Ishita distribution (Shanker and Shukla, 2017) $f(x;\theta) = \frac{\theta^3}{\theta^3+2}(\theta+x^2)\,e^{-\theta x}$.
8. The Rani distribution (Shanker, 2017b) $f(x;\theta) = \frac{\theta^5}{\theta^5+24}(\theta+x^4)e^{-\theta x}$.
9. The Sujatha distribution (Shanker, 2016b) $f(x;\theta) = \frac{\theta^3}{\theta^2+\theta+2}\left(1+x+x^2\right)e^{-\theta x}$.
10. The Garima distribution (Shanker, 2016a) $f(x;\theta) = \frac{\theta}{\theta+2}(1+\theta+\theta x)\,e^{-\theta x}$.
11. The Shanker distribution (Shanker, 2015b) $f(x;\theta) = \frac{\theta^2}{\theta^2+1}(\theta+x)e^{-\theta x}$.
12. The Akash distribution (Shanker, 2015a) $f(x;\theta) = \frac{\theta^3}{\theta^2+2}(1+x^2)e^{-\theta x}$.
13. The Lindley distribution (Ghitany et al., 2008) $f(x;\theta) = \frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}$.
14. The exponential distribution $f(x;\theta) = \theta\,e^{-\theta x}$.

4.1 Monthly tax revenue in Egypt

Monthly tax revenue data for Egypt from January 2006 to November 2010 (Nassar and Nada, 2011) exhibit pronounced right skewness. The series, measured in units of $1{,}000$ million Egyptian pounds, contains the following observations: 5.9, 20.4, 14.9, 16.2, 17.2, 7.8, 6.1, 9.2, 10.2, 9.6, 13.3, 8.5, 21.6, 18.5, 5.1, 6.7, 17.0, 8.6, 9.7, 39.2, 35.7, 15.7, 9.7, 10.0, 4.1, 36.0, 8.5, 8.0, 9.2, 26.2, 21.9, 16.7, 21.3, 35.4, 14.3, 8.5, 10.6, 19.1, 20.5, 7.1, 7.7, 18.1, 16.5, 11.9, 7.0, 8.6, 12.5, 10.3, 11.2, 6.1, 8.4, 11.0, 11.6, 11.9, 5.2, 6.8, 8.9, 7.1, and 10.8. Descriptive statistics are summarised in Table 4, while Figure 5 provides graphical presentation-including a histogram, Box-and-Whisker plot, kernel-density estimate, and violin plot.

Table 4: Descriptive statistics for Egypt's monthly tax revenue.

Sample Sizes	Minimum	Q1	Median	Mean	Q3	Maximum	St.Dev
59	4.10	8.45	10.60	13.49	16.85	39.20	8.0515

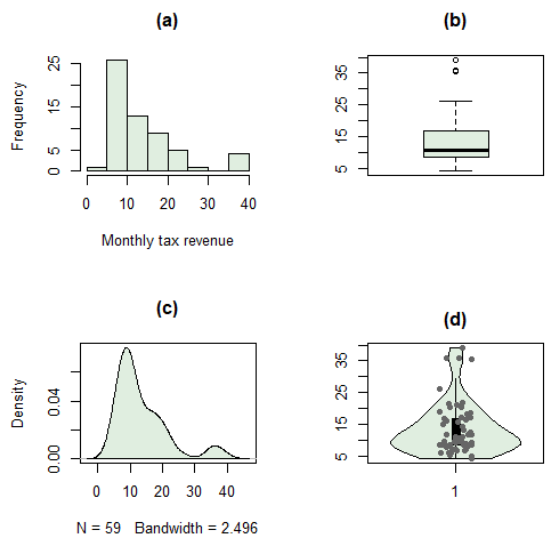

Figure 5: Visual summaries of Egypt's monthly tax revenue (Jan 2006–Nov 2010)

ML estimation for the parameter was used to fit each candidate distribution, and the resulting models were compared with three goodness-of-fit criteria: the negative log-likelihood $\left(-\log(\widehat{L})\right)$, the Akaike information criterion (AIC) (Akaike, 1974), and the Bayesian (Schwarz) information criterion (BIC) (Schwarz, 1978). These information criteria are defined by

$$\mathrm{AIC} = 2k - 2\log(\widehat{L}), \qquad \mathrm{BIC} = 2k\log(n) - 2\log(\widehat{L}),$$

where k is the number of estimated parameters and $\widehat{L}$ is the maximised likelihood.[1] In Table 5, Estimate (SE) reports the ML estimate of θ together with its standard error in parentheses, while the column labelled $-\log(\widehat{L})$ gives the negative log-likelihood evaluated at that estimate.

Table 5: Model-fit analysis of candidate distributions for monthly tax revenue data for Egypt

Distributions	Estimate (SE)	$-\log(\widehat{L})$	AIC	BIC
Rama	0.2956 (0.01918)	193.3599	**388.7198**	**390.7974**
Ola	0.5931 (0.02720)	205.6559	413.3118	415.3893
Pratibha	0.2144 (0.01608)	194.5842	391.1684	393.2459
Komal	0.1379 (0.01266)	200.9136	403.8273	405.9048
Chris-Jerry	0.2117 (0.01619)	196.8246	395.6493	397.7268
Juchez	0.2935 (0.01897)	193.5395	389.0790	391.1565
Adya	0.2184 (0.01616)	194.0959	390.1918	392.2694
Ishita	0.2214 (0.01655)	193.8824	389.7648	391.8423
Rani	0.3706 (0.02168)	194.7634	391.5267	393.6043
Sujatha	0.2125 (0.01581)	195.0663	392.1326	394.2101
Garima	0.1137 (0.01272)	208.3080	418.6160	420.6935
Shanker	0.1462 (0.01333)	198.5302	399.0605	401.1380
Akash	0.2189 (0.01634)	194.4629	390.9257	393.0032
Lindley	0.1462 (0.01333)	200.7719	403.5439	405.6214
Exponential	0.0741 (0.00965)	212.5068	427.0136	429.0912

Note: Boldfaced values indicate the distribution that yields the minimum AIC and BIC.

The values of the negative log-likelihood, AIC, and BIC reported in Table 5 indicate that the Rama distribution offers the best overall fit among the candidate distributions. The ML estimate for the dataset was 0.1623. Table 6 summarises the 95% confidence intervals for the Rama distribution parameter. The likelihood-based CI extends from 0.2596 to 0.3348 (width $= 0.0752$), while the Wald-type CI, 0.2580–0.3332, is practically identical in both location and width. By contrast, the bootstrap-t and BCa bootstrap CIs are substantially wider, signalling lower estimation precision. These differences underscore the importance of method choice and highlight how precision can vary markedly across competing CI procedures.

[1] Lower AIC, BIC, and $-\log(\widehat{L})$ values indicate a superior balance between fit and parsimony.

Table 6: The 95% two-sided CIs and widths in Egypt's monthly tax revenue.

Methods	Confidence intervals	Width
Likelihood-based	(0.2596, 0.3348)	0.0752
Wald-type	(0.2580, 0.3332)	0.0752
Bootstrap-t	(0.2536, 0.3431)	0.0895
BCa bootstrap	(0.2497, 0.3402)	0.0905

4.2 Plasma concentrations of Indomethacin

The plasma concentrations of indomethacin (measured in mcg/ml) are documented by Kwan et al. (1976). The dataset comprises 66 observations, recorded as follows: 1.50, 0.94, 0.78, 0.48, 0.37, 0.19, 0.12, 0.11, 0.08, 0.07, 0.05, 2.03, 1.63, 0.71, 0.70, 0.64, 0.36, 0.32, 0.20, 0.25, 0.12, 0.08, 2.72, 1.49, 1.16, 0.80, 0.80, 0.39, 0.22, 0.12, 0.11, 0.08, 0.08, 1.85, 1.39, 1.02, 0.89, 0.59, 0.40, 0.16, 0.11, 0.10, 0.07, 0.07, 2.05, 1.04, 0.81, 0.39, 0.30, 0.23, 0.13, 0.11, 0.08, 0.10, 0.06, 2.31, 1.44, 1.03, 0.84, 0.64, 0.42, 0.24, 0.17, 0.13, 0.10, and 0.09. Descriptive statistics for this dataset are summarized in Table 7. Figure 6 provides visual representations illustrating the dataset's asymmetry and confirming its non-normal distribution.

Table 7: Descriptive statistics for the plasma concentrations of indomethacin.

Sample Sizes	Minimum	Q1	Median	Mean	Q3	Maximum	St.Dev
66	0.050	0.110	0.340	0.592	0.833	2.720	0.6325

Table 8 shows that the Rama distribution delivers the best fit, posting the lowest AIC and BIC among all candidate distributions. A comparison of 95% CIs for the indomethacin plasma–θ parameter (Table 9) reveals that the likelihood-based $(2.4378, 3.1953)$ and Wald-type $(2.4111, 3.1658)$ intervals are virtually identical, with widths of 0.7575 and 0.7547, respectively. The bootstrap-t $(2.4312, 3.1850)$ and BCa bootstrap $(2.4177, 3.1636)$ intervals are slightly narrower–0.7538 and 0.7459–indicating a modest gain in precision.

The CIs derived from the two empirical datasets exhibited consistency, with average widths aligning closely with those obtained through Monte Carlo simulations. This alignment suggests that the empirical results are consistent with the theoretical expectations established via repeated sampling. The findings demonstrate that the proposed CI estimation methods perform effectively and reliably when applied to real-world data.

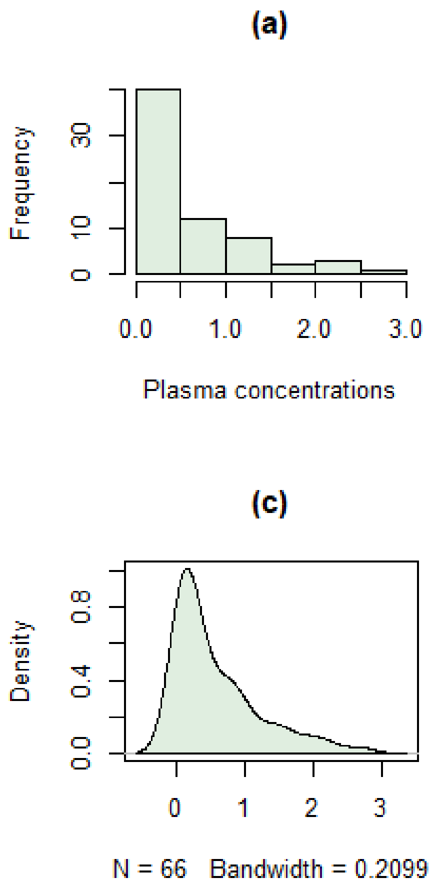

Figure 6: Visual summaries of plasma concentrations of indomethacin.

Table 8: Model-fit analysis of candidate distributions for the plasma concentrations of indomethacin.

Distributions	Estimate (SE)	$-\log(\widehat{L})$	AIC	BIC
Rama	2.7884 (0.19245)	29.6692	**61.3384**	**63.5281**
Ola	4.1473 (0.16346)	32.2058	66.4116	68.6012
Pratibha	2.3287 (0.18273)	31.1726	64.3451	66.5348
Komal	1.9447 (0.19883)	31.5141	65.0282	67.2179
Chris-Jerry	2.9792 (0.26688)	31.7391	65.4781	67.6678
Juchez	2.8642 (0.20854)	30.6502	63.3004	65.4900
Adya	2.3095 (0.18257)	31.8291	65.6581	67.8478
Ishita	2.2110 (0.17549)	30.6644	63.3288	65.5185
Rani	2.6773 (0.15212)	30.5441	63.0881	65.2778
Sujatha	2.6517 (0.22222)	31.9383	65.8765	68.0662
Garima	2.0965 (0.22581)	31.8037	65.6073	67.7970
Shanker	2.0295 (0.19159)	31.6345	65.2690	67.4587
Akash	2.5060 (0.20727)	30.5833	63.1666	65.3563
Lindley	2.0295 (0.19159)	32.5142	67.0283	69.2180
Exponential	1.6897 (0.20794)	31.3793	64.7586	66.9483

Table 9: The 95% CIs and widths in the plasma concentrations of indomethacin.

Methods	Confidence interval	Width
Likelihood-based	(2.4378, 3.1953)	0.7575
Wald-type	(2.4111, 3.1658)	0.7547
Bootstrap-t	(2.4312, 3.1850)	0.7538
BCa bootstrap	(2.4177, 3.1636)	0.7459

5 Conclusion and Recommendations

This paper developed and assessed four interval estimations for parameter of the Rama distribution including likelihood-based, Wald-type, bootstrap-t, and bias-corrected and accelerated (BCa) bootstrap. The Wald-type CI was formulated and presented as an explicit equation. All the CIs were assessed through Monte Carlo simulation studies based on empirical coverage probability (ECP) and the average width (AW) of the CIs. The simulation results indicated that likelihood-based and Wald-type CIs were superior for attaining ECPs at 95% confidence level, providing consistent performance across various sample sizes and parameter values. The bootstrap-t and BCa bootstrap CIs, while providing shorter AWs and thus more precise estimates, exhibited lower ECPs, especially when the sample size was small or the parameter value was high. As the sample size increased, the performance of the bootstrap-t and BCa bootstrap CIs improved, with ECPs approaching the nominal confidence level of 0.95 and AWs becoming comparable to the likelihood-based and Wald-type CIs. Therefore, for applications requiring high ECP, the likelihood-based and Wald-type CIs are recommended, particularly when dealing with small sample sizes or high parameter values. Conversely, the bootstrap-t and BCa bootstrap CIs, with their shorter AWs, may be more suitable when a more precise interval estimate is required that can accept a slight deviation in ECP. Selection of the approach depends on the specific requirements of the investigation, including the preferred balance between ECP and interval precision.

Although the ECPs of all methods are generally close to the nominal level of 0.95, noticeable differences remain, particularly in small samples and extreme values of θ. Likelihood-based and Wald-type CIs tend to maintain nominal coverage more reliably, whereas bootstrap methods may underperform in small samples but improve as the sample size increases. Differences in AWs, though small, reflect a balance between accuracy and stability. Moreover, while ECPs are theoretically expected to increase with larger sample sizes, slight deviations—especially for likelihood-based and Wald-type CIs–were observed, likely due to finite-sample variability and the sensitivity of asymptotic approximations in Monte Carlo simulations. These details highlight the importance of choosing a suitable method based on the sample conditions.

The computational requirements of bootstrap techniques, particularly the bootstrap-t and BCa bootstrap CIs, can pose a challenge to limited computational resources. To facilitate the computation of bootstrap CIs in R, several packages are accessible, with the 'boot' package (Canty and Ripley, 2017) and the 'bootstrap' package (Kostyshak, 2024) being notable examples. Future research could study hypothesis testing for the parameter of the Rama distribution. This topic represents valuable opportunities for future studies.

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