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Thermodynamics of Apparent Horizons in a (2+1)-Dimensional Modified Friedmann-Robertson-Walker Universe

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Abstract

In this paper, thermodynamic effects in modified (2+1)-dimensional FRW cosmology is explored. The impacts of fluctuations in the Hubble parameter and its derivatives on the universe's evolution and corresponding thermodynamic behavior are examined. Using the modified FRW metric as a starting point, and the conditions $\nabla^{\mu}\tilde{r}\nabla_{\mu}\tilde{r} = 0$, we obtained the adjusted apparent horizon radius $\tilde{r}_A{}^{-2} = H^2 + k/a^2$ and matching surface gravity. The energy and generalized entropy at the horizon are obtained using the Misner–Sharp formalism. This is accomplished by using our modified (2+1)-dimensional FRW cosmology in conjunction with the unified first law of thermodynamics. The evolution equations were formulated for $\dot{H}$ and H^2 based on modified Friedmann and acceleration equations with dimensionless constants $\alpha_1, \alpha_2, \beta_1$ and β_2. The dynamics of (2+1)-dimensional cosmology is improved by these changes. In order to guarantee conformity with the generalized first law, we apply requirements to the modified gravity parameters by examining the consistency of the thermodynamics equation $Tds = d\mathrm{E} + Wdv$. These restrictions result in invariant relations between the corrections coefficients, namely $\frac{\alpha_2}{\alpha_1} = \frac{1-\beta_2}{1-\beta_1}$ and $\frac{\beta_2}{\beta_1} = \frac{1-\alpha_2}{1-\alpha_1}$, which are symmetric under the interchange $\alpha_i \leftrightarrow \beta_i$. This formulation establishes a connection between thermodynamics and gravitational dynamics in (2+1)-dimensions, thereby facilitating the systematic examination of the influence of modified gravity on cosmic evolution.

Keywords: Thermodynamics; Modified FRW cosmology; Viscous fluid; Cosmology; (2+1)-Dimensional gravity.

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1. Introduction

In nearly all other branches of physics, smaller dimensional models have been shown to be quite useful. These models are important because they stimulate new ideas and new ways of looking at their equivalents in higher dimensions. They also provide an environment that

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is simple enough to illustrate some basic physical processes without the mathematical complexity that is sometimes present in four dimensions. Interesting characteristics shared by four-dimensional gravity are present in (2+1)- dimensional gravity. The curvature tensor and the characteristics of the Einstein field equations reveal certain peculiarities of Einstein gravity in three space-time dimensions [1]. General Theory of Relativity (GTR or simply GR) in (2+1)-dimensional spacetime is regarded to possess a number of special features that simplifies the studying problem. As revealed from the works of several scientists, there are no gravitational waves, there are no black holes without a negative cosmological constant, and the Weyl curvature is the same in all cases [2-4]. In two spatial dimensions, the weak field limit of the theory does not match Newtonian gravity. Of course, here 'simplifying features' means that considerable progress can be made in search of the general cosmological solution to the three-dimensional Einstein equations. It has been observed that the cosmological solution is rather cumbersome and dominated by nonintegrability in (2 + 1)-dimension; on the other hand, the theory makes it possible to determine the general solutions. The state and features of the Einstein field equations in two spatial and one temporal dimension are what make (2 + 1)-dimension so intriguing [5-8]. In (2+1)-dimensional space, the Einstein and Riemann tensors are the same. This means that spacetime is flat outside of sources where there is no gravitational field and the Newtonian limit doesn't apply. It has been investigated that all hydrostatic structures in (2+1)-dimensional GR contain matter-filled spaces without any matching to the external vacuum solution and thus represent a specific static cosmology [9]. Several cosmological observations specify that there must be some kind of dark energy with a repulsive pressure in the late-time universe. Therefore, it has created interest in the study of cosmological scaling solutions of minimally coupled scalar fields in 3 dimensions. Fujiwara et al. [10] have explained the case of nucleation of the universe in a (2 + 1)-dimensional gravity model with a negative cosmological constant. It is understood that the formula $S = A / 4$ is largely employed to state a black hole's energy as a function of the event horizon's area. This inseparable correlation could be formed as signaling that a black hole's perimeter stores all of its degrees of freedom. The understanding that a physical system's comprehensive structures can be found on its border in quantum gravity under unique conditions has long been pursued in an endeavor to expand this interconnectedness to more solutions in gravity. This idea is known as the holographic principle [11]. A lot of interest has largely been generated in the research of gravitational theories other than the four. Although there are many other factors for this, quantum gravity, grand unified theory, and string theory facilitate them to sustain the main driving forces. For instance, the Weyl curvature is exactly zero, the weak field limit of the three-space-time general theory of relativity doesn't match Newtonian gravity in (2+1)-dimensions, there are no black holes or gravitational waves without a negative cosmological constant, and there are many other special features that make things easier. The structure of universal relativistic gravity in (2+1)-dimensional spacetime have been investigated [10-12]. (2+1)-gravity is an especially intriguing instance to explore because of the unique characteristics of Einstein's field equations in two space and one time dimension. As signified by quantum field theory, lower-dimensional systems are always researched in physical systems, and applying this method to gravity produces highly qualitative and insightful results. It is predicated that (2+1)-gravity will produce a unique outlook for understanding the physical importance of (3+1)-gravity [13,14]. Kaloper *et al*. achieved a particular extension of the holographic principle to cosmology.

It is known that at the event horizon, the black hole emits thermal radiation with a temperature equal to the surface gravity. Additionally, there is an entropy associated with the horizon of the black hole [16,17]. The first law of thermodynamics says that the Hawking temperature and entropy are correct. Since, Hawking radiation is a quantum process, statistical physics and quantum gravitational theories are connected in black hole thermodynamics. As a result, researchers have been trying to connect Einstein's equation and thermodynamics in recent decades. Jacobson acquired Einstein equations from the entropy and temperature by using the Clausius relation $\delta Q = Tds$, where δQ and T are point energy flux and Unruh temperature [18,19]. The relationship between thermodynamics and gravity has also been extended in the brane world cosmology [25,26], and numerous authors have studied how the thermodynamic properties match the apparent horizon of the FRW universe with perfect fluid [20-23]. The Einstein field equations can be reformulated as a first law of thermodynamics in general static spherically symmetric spacetime, as demonstrated in literature [27-29]. The perfect fluid cosmological models have been studied in greater detail in the literature, and the viscous fluid was introduced much later in the study of the universe [30]. Additionally, the authors considered the viscous generalized Chaplygin gas as a dark energy model and discovered that, in the special case, it corresponds to modified Chaplygin gas [31]. The viscous cosmology and thermodynamics of the apparent horizon in the FRW background were investigated by Akbar *et al*. [32], where the differential form of the Friedman equations of the FRW universe can be recorded as a similar form of the first law, $T\,AdSA = dE + WdV$, of thermodynamics at the apparent horizon of the FRW universe filled with the viscous fluid. We have also considered a number of recent contributions on related topics in light of recent developments. Ditta *et al.* [33] examined thermal stability and quantum fluctuations in black hole spacetimes and examined emergent gravity phenomena in lower-dimensional models. Additional developments that are pertinent to our research have been published elsewhere [34,35]. These publications offer contrasting viewpoints on dark energy models, modified gravity, and thermodynamics. Our current analysis of generalized Friedmann dynamics in (2+1)-dimensional FRW cosmology can be placed in a wider context. In this paper, the generalized Friedman equations of the FRW universe are employed, and the first law of thermodynamics is achieved in some special conditions. The study of cosmology in lower-dimensional spacetimes, particularly in (2+1) dimensions, provides a simplified yet insightful framework for understanding fundamental aspects of gravitational dynamics and their thermodynamic interpretations. In a (2+1)-dimensional Friedmann–Robertson–Walker (FRW) universe, the geometry remains homogeneous and isotropic, characterized by a time-dependent scale factor and a constant spatial curvature. Despite the reduced dimensionality, the essential features of cosmic evolution, such as expansion, curvature effects, and the role of matter and energy, are preserved, making it a valuable setting for exploring theoretical extensions. In contemporary cosmology, the mysterious elements of dark energy and dark matter are the main drivers of the ongoing effort to comprehend the universe's late-time accelerated expansion. The dynamics of interacting dark energy and dark matter, controlled by a quadratic equation of state with time-dependent parameters, are examined in this paper using a five-dimensional Bianchi type-I cosmological model. In order to ascertain the evolutionary behavior of the equation of state parameter and the cosmological term, especially in the asymptotic limit as time approaches infinity, we solve the Einstein field equations within this framework. For certain variants of the Hubble parameter, the cosmological dynamics are further investigated in both linear and power-law regimes, and the physical behavior of important quantities is explored both at the beginning and as time increases to infinity. This study's key discovery highlights the crucial role that the interactive coupling between the dark energy and dark matter components plays in determining the dynamics of the universe by directly driving a phase of cosmic expansion that is noticeably faster than in non-interacting scenarios [36].

A key concept of this framework is the apparent horizon, which serves as a causal boundary for the observable universe. The apparent horizon also plays a crucial role in connecting gravity with thermodynamics. By associating temperature and entropy to this horizon, one can draw parallels between gravitational dynamics and the laws of thermodynamics. This thermodynamic perspective suggests that the evolution of the universe can be interpreted as a thermodynamic process governed by the flow of energy across the horizon.

To explore deviations from standard cosmology, generalized Friedmann equations are considered. These equations incorporate correction terms that may arise from quantum gravitational effects, higher-order curvature contributions, or modified gravity theories. Such corrections typically involve terms that depend on the expansion rate of the universe and its time derivative, modulated by dimensionless parameters. These additional contributions act as effective sources or driving terms, altering the conventional relationship between geometry and matter.

The compatibility of these generalized cosmological equations with the first law of thermodynamics is of particular interest. By examining the energy contained within the apparent horizon and the effects of cosmic expansion on this energy, we can develop a revised version of the first law that incorporates both matter and the alterations in geometry. Under specific conditions on the correction parameters, the generalized Friedmann equations and the thermodynamic law can be made fully consistent. These conditions reveal intrinsic symmetries and lead to invariant relations among the correction coefficients.

Overall, the investigation of generalized cosmological dynamics in (2+1) dimensions, through the lens of thermodynamics, provides a powerful approach to understanding the deeper connections between gravity, quantum theory, and the thermodynamic behavior of spacetime.

2. The Generalized Version of the Friedman Equation and the First Law of Thermodynamics

The spatially flat FRW universe is described by the following metric [38,39],

$$ds^2 = dt^2 - a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\theta^2\right), \tag{1}$$

where $a(t)$ is scale factor. the coordinate $(t, r, \theta\ )$ symbolize the co-moving coordinates, and the spatial curvature constant $k = 0, +1$ and -1 stands a flat, closed and open universe, respectively. The energy momentum tensor in the FRW metric in (2+1)-dimension turn into,

$$T_{\mu\vartheta} = (\rho + p)u_\mu u_\vartheta - p g_{\mu\vartheta}\ , \tag{2}$$

where ρ is the energy density, p is pressure, u^μ is the velocity three vector with $u^\mu u_\vartheta = -1$. The analysis involves matter characterized by the energy-momentum tensor $T_{\mu\vartheta} = (\rho, -p,\ -p)$. The Einstein field equation in (2+1)-dimension can be written as [40],

$$G_{\mu\vartheta} = R_{\mu\vartheta} - \frac{1}{2} g_{\mu\vartheta} R = -2\pi G T_{\mu\vartheta}\ . \tag{3}$$

The field equations (3) with the help of line element (1) and the energy-momentum conservation equation in (2+1)-dimensions are given by

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = 2\pi G\rho, \tag{4}$$

$$\frac{\ddot{a}}{a} = -2\pi G p, \tag{5}$$

$$\dot{\rho} = -2H(\rho + p). \tag{6}$$

The above metric (1) can be rewritten in spherical form

$$ds^2 = h_{ab} dx^a dx^b + \tilde{r}^2, \tag{7}$$

where, $\tilde{r} = a(t)r$ and $x^0 = t, x^1 = r$ and two-dimensional metric $h_{ab} = diag\left(1, \frac{a^2(t)}{1-kr^2}\right)$. The dynamical apparent horizon is resolved by the relation $h^{ab}\partial_a\tilde{r}\partial_b\tilde{r} = 0$, which suggest that the vector $\nabla\tilde{r}$ is null on the apparent horizon for the FRW metric present the apparent horizon radius [41]

$$\tilde{r}_A{}^{-2} = H^2 + \frac{k}{a^2}\ . \tag{8}$$

The related temperature $T = \frac{\kappa}{2\pi}$ at the apparent horizon is resolved through the surface gravity [41]

$$\kappa = -\frac{1}{2\sqrt{-h}}\partial_a\left(\sqrt{-h}h^{ab}\partial_b\tilde{r}\right)\ . \tag{9}$$

The precise expansion of the surface gravity at apparent horizon of Friedman-Robertson-Walker universe is given by [42];

$$\kappa = -\frac{1}{\tilde{r}_A}\left(1 - \frac{\dot{\tilde{r}}_A}{2H\tilde{r}_A}\right), \tag{10}$$

where, over dot represent the time derivative and Hubble parameter $H = \frac{\dot{a}}{a}$. The analysis shows that $T_h = \frac{1}{2\pi\tilde{r}_A}$ be close to surface gravity $|\kappa| = \frac{1}{\tilde{r}_A}$. Focus is now placed on defining the energy of the universe enclosed by the apparent horizon. The quantities are added to the whole matter energy $E = V\rho$ inside a sphere of radius $\tilde{r}$ that is also the Mizner-Sharp energy [41]

$$E = \frac{\tilde{r}}{2}(1 - h^{ab}\partial_a\tilde{r}\partial_b\tilde{r})\ , \tag{11}$$

within the apparent horizon. At the time that the apparent horizon ($h^{ab}\partial_a\tilde{r}\partial_b\tilde{r} = 0$), the Mizner-Sharp energy which is determined by, is actually the total matter energy within the sphere of radius $\tilde{r}_A$,

$$E = V\rho. \tag{12}$$

The total entropy inside the horizon $S_h = \frac{A}{4}$, where horizon area $A = 2\pi\tilde{r}_A$. It should be noted that natural units are employed, with $\hbar = c = G = \kappa_B = 1$. The modified Friedman and acceleration described in [24] will be examined here.

$$H^2 + \frac{k}{a^2} = 2\pi\rho + \alpha_1 H^2 + \alpha_2\dot{H} \tag{13}$$

and

$$\dot{H} - \frac{k}{a^2} = -2\pi(\rho + p) + \beta_1 H^2 + \beta_2\dot{H} \tag{14}$$

The coefficients α_1, α_2, β_1 and β_2 are dimensionless constant with some conditions are given in literature [37]. The terms H^2 and $\dot{H}$ contains the dimensionless constants relates to the extra driving terms. So as to investigate the first law of thermodynamics from (13) and (14), it is necessary to acquire H^2 and $\dot{H}$ individually, which are given by

$$H^2 = \frac{2\pi a^2\alpha_2(\rho+p) + 2\pi(\beta_2 - 1)a^2\rho + k(1 - \alpha_2 - \beta_2)}{a^2 A}, \tag{15}$$

$$\dot{H} = \frac{2\pi a^2(1-\alpha_1)(\rho+p) - 2\pi a^2\beta_1\rho + k(\alpha_1 + \beta_1 - 1)}{a^2 A}, \tag{16}$$

where,

$$A = \alpha_1 + \beta_2 + \alpha_2\beta_1 - \beta_2\alpha_1 - 1.$$

From equation (7) and taking the derivative, the result can be obtained

$$-2\frac{d\tilde{r}_A}{\tilde{r}_A{}^3 dt} = \left(2H\dot{H} - 2k\frac{H}{a^2}\right), \tag{17}$$

After putting value of $\dot{H}$ and simplifying gives

$$-2\frac{d\tilde{r}_A}{\tilde{r}_A{}^3} = \left[B\dot{\rho} + D\dot{p} - 2C\frac{k\dot{a}}{a^3}\right]dt, \tag{18}$$

where,

$$B = \frac{2\pi\alpha_2 + 2\pi(\beta_2 - 1)}{A}, C = \frac{1 - \alpha_2 - \beta_2 + A}{A}, D = \frac{2\pi\alpha_2}{A}. \tag{19}$$

From equation (17) and (18) one can obtain

$$2H\dot{H} - 2k\frac{H}{a^2} = B\dot{\rho} + D\dot{p} - 2C\frac{kH}{a^2} \tag{20}$$

By putting the value of $\dot{H}$ from equation (16) in above equation, the following equation is derived:

$$-\frac{d\tilde{r}_A}{\tilde{r}_A{}^3} = H\left[(\gamma\rho + \theta p) + (\eta - 1)\frac{k}{a^2}\right]dt, \tag{21}$$

where,

$$\gamma = \frac{2\pi(1-\alpha_1) - 2\pi\beta_1}{A},$$

$$\theta = \frac{2\pi(1-\alpha_1)}{A},$$

$$\eta = \frac{\alpha_1 + \beta_1 - 1}{A}.$$

Where the temperature and curvature as $T = \kappa/2\pi$ and $\kappa = -\frac{1}{\tilde{r}_A}\left(1 - \frac{\dot{\tilde{r}}_A}{2H\tilde{r}_A}\right)$. As is known, the definition of entropy $S = A/4$ and $A = 2\pi\tilde{r}_A$, the following equation can be derived:

$$Tds = -\frac{1}{4\tilde{r}_A}\left(1 - \frac{\dot{\tilde{r}}_A}{2H\tilde{r}_A}\right)d\tilde{r}_A. \tag{22}$$

Now multiplying equation (19) with $\tilde{r}_A^{\,3}\left(1-\frac{\dot{\tilde{r}}_A}{2H\tilde{r}_A}\right)$, after solving, the result is given as [43]

$$Tds=\frac{H}{4}\tilde{r}_A^{\,2}\left(1-\frac{\dot{\tilde{r}}_A}{2H\tilde{r}_A}\right)\left[\gamma\rho+\theta p+(\eta-1)\frac{k}{a^2}\right]dt, \tag{23}$$

The total matter energy is considered as $E=V\rho$. Differentiation then yields:

$$dE=\rho dV+V\frac{d\rho}{dt}dt$$

$$dE=2\pi\rho\tilde{r}_A d\tilde{r}_A+\pi\tilde{r}_A^{\,2}\dot{\rho}dt. \tag{24}$$

In above equation, first $\dot{\rho}$ is determined from equation (20)

$$\dot{\rho}=\frac{2\gamma}{B}H\rho+\frac{2\theta}{B}Hp-\frac{D}{B}\dot{p}+\frac{2}{B}[\eta-1+C]\frac{Hk}{a^2}. \tag{25}$$

By inserting the value of $\dot{\rho}$ into equation (25), dE is determined.

$$dE=2\pi\tilde{r}_A\rho d\tilde{r}_A+\frac{2\pi}{B}H\tilde{r}_A^{\,2}\left[\gamma\rho+\theta P+(\eta-1+C)\frac{k}{a^2}\right]dt-\frac{\pi D}{B}\tilde{r}_A^{\,2}dp$$

$$dE=2\pi\tilde{r}_A\rho d\tilde{r}_A+\frac{2\pi}{B}H\tilde{r}_A^{\,2}\left[\gamma\rho+\theta P+(\eta-1)\frac{k}{a^2}\right]dt+\frac{2\pi}{B}H\tilde{r}_A^{\,2}\frac{Ck}{a^2}dt$$
$$-\frac{\pi D}{B}\tilde{r}_A^{\,2}dp. \tag{26}$$

Using (24) and (26), one can ultimately get the following equation:

$$Tds=\frac{B}{8\pi}dE-\frac{1}{16\pi}\left[(\gamma+2B)\rho+\theta p+(\eta-1)\frac{k}{a^2}\right]dV+\frac{D}{8\pi}Vdp-$$
$$CH\frac{k}{a^2}\tilde{r}_A^{\,2}dt \tag{27}$$

This equation guides us to apply some requirement as $\eta=1$ and $C=0$. The first generalized law of thermodynamics applies in those situations, after putting the value of η and C and generalized law of thermodynamics, a relation connecting α_1, α_2, $\beta_1$and β_2 in modified Friedman-Robertson-Walker's equations is derived:

$$\frac{\alpha_2}{\alpha_1}=\frac{1-\beta_2}{1-\beta_1}\text{ and }\frac{\beta_2}{\beta_1}=\frac{1-\alpha_2}{1-\alpha_1}. \tag{28}$$

It is evident that these relations are fixed under transformation $\alpha_i\leftrightarrow\beta_i$. The result is a generalized first law of thermodynamics where $\frac{D}{8\pi}Vdp$ arises from the depression of the environment.

3. Conclusion

In this study, an investigation was carried out on the thermodynamic properties of a modified Friedmann-Robertson-Walker (FRW) universe in (2+1)-dimensions, incorporating additional correction terms in the field equations. By employing the unified first law of thermodynamics and examining the apparent horizon dynamics, the required constraints on the dimensionless parameters α_1, α_2, $\beta_1$and β_2 were found. Analysis reveals that the generalized first law of thermodynamics holds only when the parameters satisfy specific relations, remaining unchanged under the transformation $\alpha_i\leftrightarrow\beta_i$. These constraints ensure consistency between the modified cosmological equations and the thermodynamic laws at the apparent horizon. Furthermore, it was demonstrated that an additional term $\frac{D}{8\pi}Vdp$ arises in the thermodynamic relation, representing the work done due to pressure variations in the system. This term highlights the interplay between geometric corrections and thermodynamic behavior in modified gravity scenarios. The results suggest that the modified FRW framework maintains thermodynamic consistency under specific parameter constraints, reinforcing the deep connection between gravity, thermodynamics, and horizon dynamics. Future work could explore observational implications of these constraints and their role in alternative theories of gravity. While the present analysis has concentrated on entropy, Misner–Sharp energy, and the unified first law at the apparent horizon, other thermodynamic quantities such as heat capacity and free energies, which are important for stability analysis, will be considered in future work.

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