JSR Publications

Available Online

J. Sci. Res. **18** (1), 81-89 (2026)

JOURNAL OF SCIENTIFIC RESEARCH

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# Numerical Solution of Boundary Value Problems by Wavelet-Based Galerkin Method

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Received 4 May 2025, accepted in final revised form 29 September 2025

**Abstract**

Differential equations are the formulation of scientific theory for many real-world physical problems. Boundary value problems (BVPs) occur frequently in the fields of engineering and science, such as gas dynamics, nuclear physics, atomic structures, and chemical reactions. In most cases, BVPs do not always find the exact solutions to these problems. Boubaker wavelets are wavelet functions derived from Boubaker polynomials. They serve as an effective numerical tool for tackling a range of scientific and engineering problems, including differential and variational equations. Their strength lies in generating accurate approximate solutions by transforming complicated equations into simpler linear systems. In this paper, a wavelet-based Galerkin method using Boubaker wavelets for the numerical solution of BVPs is proposed. Here, Boubaker wavelets are used as weight functions that are the assumed basis elements that allow us to obtain the numerical solution of the BVPs. The numerical results from the proposed method are compared with the exact solution to assess accuracy against existing schemes (Galerkin method using other wavelets, such as Laguerre and Fibonacci wavelets). Some BVPs are taken to demonstrate the validity and applicability of the proposed method.

*Keywords*: Boubaker wavelets; Function approximation; Boundary value problems; Galerkin method.



doi: https://dx.doi.org/10.3329/jsr.v18i1.81332 J. Sci. Res. **18** (1), 81-89 (2026)

## 1. Introduction

In recent years, studies of boundary value problems in second-order ordinary differential equations have attracted the attention of many mathematicians and physicists. Also, most of the differential equations arising from the modelling of physical phenomena do not always have known analytical solutions. Thus, the need for the development of numerical approaches to find approximate solutions becomes essential.

Recently, some of the numerical methods have been used for the numerical solutions of the second-order ordinary differential equations. For example, the Haar wavelet collocation

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method [1], the Legendre wavelet collocation method [2], the Taylor wavelet-based Galerkin method [3] etc.

Wavelet analysis became important in the 1980s after proving useful in signal and image processing. The approach uses repeated shifting and scaling of one function to build a smooth orthonormal basis, which was crucial for developing compression algorithms that keep signals and images within certain amplitude limits. Major developments include wavelet series in applied mathematics, sub-band coding for voice and image compression, and multiresolution signal processing for computer vision. Special interest has been dedicated to the construction of compactly supported smooth wavelet bases. Already we know that spectral methods have good spectral localization but poor spatial localization, while finite element methods have good spatial localization but poor spectral localization. Wavelet bases execute to combine the advantages of both spectral and finite element bases. An approach to studying differential equations is the use of wavelet function bases in place of other conventional piecewise polynomial trial functions in finite element type methods [4,5].

The Galerkin method is considered the most widely used in applied mathematics because of its implementation and simplicity. This transforms the differential equations into algebraic ones that can be solved numerically. The resulting linear system of algebraic equations for the unknown coefficients is then solved to obtain numerical solutions of the differential equations [6,7].

The advantage of the wavelet-Galerkin method over the finite difference or finite element method has led to tremendous applications in science and engineering. To a certain extent, the wavelet technique is a strong competitor to the finite element method. Although the wavelet method provided an efficient alternative technique for solving differential equations, especially boundary value problems, numerically.

In this paper, a wavelet-based Galerkin method using Boubaker wavelets was developed for the numerical solution for BVPs. This method is based on expanding the solution by Boubaker wavelets with unknown coefficients. The properties of Boubaker wavelets together with the Galerkin method are utilized to evaluate the unknown coefficients, and then a numerical solution of the BVPs is obtained.

The organization of the paper is as follows. Boubaker wavelets and function approximations are given in section 2. Section 3 deals with the wavelet-based Galerkin method for the solution of BVPs. Numerical implementation is given in section 4. Finally, conclusions of the proposed work are discussed in section 5.

## 2. Boubaker Wavelets and Function Approximation

*Boubaker wavelets*

Wavelets constitute a family of functions constructed from dilation and translation of a single function called the mother wavelet. When the dilation parameter $a$ and the translation parameter $b$ vary continuously and have the following family of continuous wavelets [8,9]:

$$\psi_{a,b}(x) = |a|^{\frac{-1}{2}}\psi\left(\frac{x-b}{a}\right), \forall\, a,b \in R \,\&\, a \neq 0$$

Restrict the parameters $a$ & $b$ to discrete values as

$$a = a_0^{-n}, b = mb_0a_0^{-n}\ ;\ a_0 > 1, b_0 > 0$$

and the following family of discrete wavelets

$$\psi_{n,m}(x) = |a_0|^{\frac{1}{2}}\psi(a_0^n x - mb_0), n, m \in Z$$

Where $\psi_{n,m}$ form a wavelet basis for $a$, $b$. In particular, when $a_0 = 2$ & $b_0 = 1$, then $\psi_{n,m}(x)$ forms an orthonormal basis.

Boubaker wavelets are defined as follows:

$$\psi_{n,m}(x) = \begin{cases} \sqrt{2m+1}\frac{(2m!)}{(m!)^2}2^{\frac{k+1}{2}}B_m(2^{k+1}x - 2n + 1), \frac{n-1}{2^{k-1}} \leq x < \frac{n}{2^{k-1}} \\ 0, Otherwise \end{cases} \quad (2.1)$$

where, $k$ is any positive integer, $n = 1,2,3,\ldots\ldots\ldots, 2^{k-1}$ is an argument and $m = 0,1,2,3\ldots\ldots\ldots M-1$ is the order of Boubaker functions

$$B_0(x) = 1\ ,$$

$B_1(x) = \frac{1}{2}(2x-1)$ . $B_2(x) = \frac{1}{6}(6x^2 - 6x + 1)$ and so on.

For instance, for $k = 1$ and $M = 3$, the Boubaker wavelet bases as follows:

$\psi_{1,0}(x) = 2, \psi_{1,1}(x) = 2\sqrt{3}(8x-3), \psi_{1,2}(x) = 2\sqrt{5}(96x^2 - 72x + 13)$ and so on.

*Function approximation*

Suppose $y(x) \in L^2[0\ ,\ 1)$ is expanded in terms of Boubaker wavelets as:

$$y(x) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\,\psi_{n,m}(x) \quad (2.2)$$

Truncating the above infinite series, we get

$$y(x) = \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{n,m}\,\psi_{n,m}(x) \quad (2.3)$$

*Convergence of Boubaker wavelets*

**Theorem:** If a continuous function $y(x) \in L^2(R)$ defined on [0,1) be bounded, i.e. $y(x) \leq K$, then the Boubaker wavelets expansion of $y(x)$ converges uniformly to it [10].

## 3. Method of Solution

Consider the boundary value of the problem is of the form,

$$y'' + P(x)y' + Q(x)y = f(x) \quad (3.1)$$

With boundary conditions $y(0) = a, y(1) = b$ (3.2)

Where $P(x)$ & $Q(x)$ are constants or functions and $f(x)$ be a continuous function of $x$.

Write the Eq. (3.1) as $R(x) = y'' + P(x)y' + Q(x)y - f(x)$ (3.3)

where $R(x)$ the residual of Eq. (3.1) equals zero, the exact solution is identified, and the boundary conditions are satisfied.

Consider the trial series solution of Eq. (3.1), $y(x)$ defined over [0, 1) can be expanded as a modified Boubaker wavelet, satisfying the given boundary conditions, which involves unknown coefficients as follows:

$$y(x) = \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{n,m}\,\psi_{n,m}(x) \quad (3.4)$$

Where $c_{n,m}'s$ are unknown coefficients to be determined.

Accuracy in the solution is increased by choosing higher-degree Boubaker wavelet polynomials.

Differentiate Eq. (3.4) twice with respect to $x$ and substitute the values of in $y, y', y''$ Eq. (3.3). To find $c_{n,m}'s$, choose the function as assumed bases elements and integrate on boundary values together with the residual to zero [11].

i.e. $\int_0^1 \psi_{1,m}(x)\, R(x)dx = 0, m = 0,1,2, \ldots\ldots$

then obtained a system of linear algebraic equations, on solving this system, to get unknown coefficients. Substitute these unknowns in the trail solution i.e. Eq. (3.4), obtained the numerical solution of Eq. (3.1).

In order to know the accuracy of BWGM on the test problems, use the maximum absolute error as a measure of error. The formulas for the calculations are listed as follows:

(i) Maximum absolute error = $E_{\max} = \max \left| y(x)_e - y(x)_n \right|$, where $y(x)_e$ and $y(x)_n$ are exact and numerical solution

(ii) $L_2$ - Norm= $\left\| (\sum_{m=1}^{n} E_m^2)^{1/2} \right\|$ (iii) $L_\infty$ - Norm= $\| Max(E_m) \|, m = 1,2, \ldots\ldots, 9$

## 4. Numerical Implementation

**Problem 4.1** First, consider the boundary value problem(In Eq. (3.1) $P(x) = -1$, $Q(x) = 0$ & $f(x) = -(e^{x-1} + 1)$) i.e.

$$y'' - y' = -(e^{x-1} + 1), 0 \le x \le 1 \quad (4.1)$$

With boundary conditions: $y(0) = 0, y(1) = 0$ (4.2)

The implementation of the Eq. (4.1) as per the method explained in section 3 is as follows: and its residual can be written as:

$$R(x) = y'' - y' + (e^{x-1} + 1) \quad (4.3)$$

Now, choosing the function $w(x) = x(1 - x)$ for Boubaker wavelet bases to satisfy the given boundary conditions Eq. (4.2), i.e. $\psi(x) = w(x) \times \psi(x)$

$$\psi_{1,0}(x) = \psi_{1,0}(x) \times x(1 - x) = 2x(1 - x)$$

$$\psi_{1,1}(x) = \psi_{1,1}(x) \times x(1 - x) = 2\sqrt{3}(8x - 3)x(1 - x)$$

$$\psi_{1,2}(x) = \psi_{1,2}(x) \times x(1 - x) = 2\sqrt{5}(96x^2 - 72x + 13)x(1 - x)$$

Assuming the trail solution of Eq. (4.1) for $k = 1$ and $M = 3$ is given by

$$y(x) = c_{1,0}\psi_{1,0}(x) + c_{1,1}\psi_{1,1}(x) + c_{1,2}\psi_{1,2}(x) \quad (4.4)$$

Then the Eq. (4.4) becomes

$$y(x) = c_{1,0}\{2x(1 - x)\} + c_{1,1}\{2\sqrt{3}(8x - 3)x(1 - x)\} + c_{1,2}\{2\sqrt{5}(96x^2 - 72x + 13)x(1 - x)\} \quad (4.5)$$

Differentiate Eq. (4.5) w.r.t. $x$ twice and substitute the values of $y', y''$ in Eq. (4.3), to get the residual of Eq. (4.1). The “weight functions” are the same as the basis functions. Then by the weighted Galerkin method, consider the following:

$$\int_0^1 \psi_{1,j}(x)\, R(x)dx = 0, j = 0,1,2 \quad (4.6)$$

For $j = 0,1,2$ in Eq. (4.6),

$$\text{i.e.} \left.\begin{aligned} \int_0^1 \psi_{1,0}(x)\, R(x) dx = 0 \\ \int_0^1 \psi_{1,1}(x)\, R(x) dx = 0 \\ \int_0^1 \psi_{1,2}(x)\, R(x) dx = 0 \end{aligned}\right\} \tag{4.7}$$

From Eq. (4.7), obtained a system of algebraic equations with unknown coefficients i.e. $c_{1,0}$, $c_{1,1}$and $c_{1,2}$. Solving this by the Gauss elimination method, finding the values of $c_{1,0} = 0.37193$, $c_{1,1} = 0.01208$ and $c_{1,2} = 0.00027$. On substituting these values in Eq. (4.5) then obtained the numerical solution of Eq. (4.1). Table 1 shows the comparison between BWGM and absolute errors, while Table 2 compares error norms against exact solutions for verification. Fig. 1 illustrates the numerical solution in relation to the exact solution of Eq. (4.1) is $y(x) = x(1 - e^{x-1})$[13].

Table 1. Comparison of numerical solutionand absolute error with exact solution of the problem 4.1.

| x | Numerical solution | | | Exact solution | Absolute error | | |
|---|---|---|---|---|---|---|---|
| | Ref [12] | Ref [13] | BWGM | | Ref [12] | Ref [13] | BWGM |
| 0.1 | 0.3079992 | 0.059383 | 0.059397 | 0.059343 | 1.02e-03 | 4.00e-05 | 5.40e-05 |
| 0.2 | 0.5880739 | 0.110234 | 0.110115 | 0.110134 | 7.00e-04 | 1.00e-04 | 1.90e-05 |
| 0.3 | 0.8094184 | 0.151200 | 0.150948 | 0.151024 | 4.00e-04 | 1.76e-04 | 7.60e-05 |
| 0.4 | 0.9515192 | 0.180617 | 0.180408 | 0.180475 | 4.60e-04 | 1.42e-04 | 6.70e-05 |
| 0.5 | 1.0001543 | 0.196983 | 0.196729 | 0.196735 | 1.50e-04 | 2.48e-04 | 6.00e-06 |
| 0.6 | 0.9513935 | 0.198083 | 0.197868 | 0.197808 | 3.40e-04 | 2.75e-04 | 6.00e-05 |
| 0.7 | 0.8092985 | 0.181655 | 0.181503 | 0.181427 | 2.80e-04 | 2.28e-04 | 7.60e-05 |
| 0.8 | 0.5878225 | 0.145200 | 0.145035 | 0.145015 | 3.80e-05 | 1.85e-04 | 2.00e-05 |
| 0.9 | 0.3084107 | 0.085710 | 0.085587 | 0.085646 | 6.10e-04 | 6.40e-05 | 5.90e-05 |

Table 2. Comparison for error norms $L_2$ & $L_\infty$ to compare with exact solutions for problem 4.1.

| Method | $L_2$ norm | $L_\infty$norm |
|---|---|---|
| Ref [12] | 1.60e-03 | 1.00e-03 |
| Ref [13] | 5.60e-04 | 2.75e-04 |
| BWGM | 1.64e-04 | 7.60e-05 |

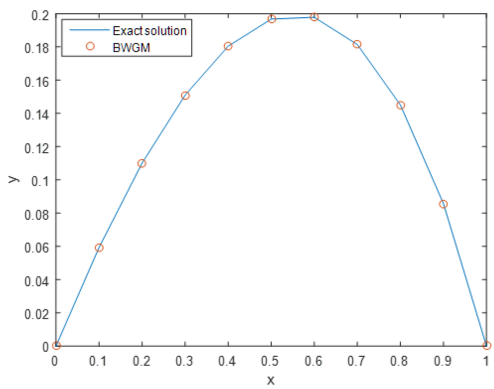


Fig. 1. Comparison of numerical solution with exact solution of the problem 4.1.

**Problem 4.2** Next, consider another the boundary value problem (In Eq. (3.1) $P(x) = 0, Q(x) = 1 \& f(x) = x^2$) i.e.

$$y'' + y = x^2, 0 \leq x \leq 1 \quad (4.8)$$

With boundary conditions: $$y(0) = 0, y(1) = 0 \quad (4.9)$$

As explained in section 3 and in the previous problem, obtained the values of $c_{1,0} = -0.07158$, $c_{1,1} = -0.00552$ and $c_{1,2} = -0.00016$. Substituting these values in Eq. (4.5), to find the numerical solution. The comparison of the numerical solution and the absolute errors are presented in Tables 3 and 4, while Table 5 compares error norms against exact solutions for verification, and numerical solution with the exact solution of Eq. (4.8) is $y(x) = \frac{sin(x)+2\,sin(1-x)}{sin(1)} + x^2 - 2$ [14] in Fig. 2.

Table 3. Comparison of numerical solutionand absolute error with exact solution of the problem 4.2.

| x | Numerical solution | | Exact solution | Absolute error | |
|---|---|---|---|---|---|
| | Ref [14] | BWGM | | Ref [14]] | BWGM |
| 0.125 | -0.0121 | -0.0119057 | -0.0119 | 2.00e-04 | 5.70e-06 |
| 0.375 | -0.0340 | -0.0334693 | -0.0334 | 6.00e-04 | 6.93e-05 |
| 0.625 | -0.0440 | -0.0434389 | -0.0435 | 5.00e-04 | 6.11e-05 |
| 0.875 | -0.0261 | -0.0258631 | -0.0259 | 2.00e-04 | 3.69e-05 |

Table 4. Comparison of numerical solutionand absolute error with exact solution of the problem 4.2.

| x | Numerical solution | | Exact solution | Absolute error | |
|---|---|---|---|---|---|
| | FDM | BWGM | | FDM | BWGM |
| 0.1 | -0.009628 | -0.009534 | -0.009555 | 7.30e-05 | 2.10e-05 |
| 0.2 | -0.019027 | -0.018901 | -0.018897 | 1.30e-04 | 4.00e-06 |
| 0.3 | -0.027804 | -0.027661 | -0.027635 | 1.69e-04 | 2.60e-05 |
| 0.4 | -0.035371 | -0.035209 | -0.035180 | 1.91e-04 | 2.90e-05 |
| 0.5 | -0.040954 | -0.040772 | -0.040759 | 1.95e-04 | 1.30e-05 |
| 0.6 | -0.043600 | -0.043407 | -0.043416 | 1.84e-04 | 9.00e-06 |
| 0.7 | -0.042180 | -0.042007 | -0.042025 | 1.55e-04 | 1.80e-05 |
| 0.8 | -0.035418 | -0.035296 | -0.035302 | 1.16e-04 | 6.00e-05 |
| 0.9 | -0.021878 | -0.021831 | -0.021815 | 6.30e-05 | 1.60e-05 |

Table 5. Comparison for error norms $L_2$ & $L_\infty$ to compare with exact solutions for problem 4.2.

| Method | $L_2$ norm | $L_\infty$norm |
|---|---|---|
| Ref [14] | 8.31e-04 | 6.00e-04 |
| FDM | 4.78e-04 | 1.95e-04 |
| BWGM | 8.00e-05 | 6.00e-05 |

**Problem 4.3** Now consider the singular boundary value problem (In Eq. (3.1) $P(x) = \frac{1}{x}$, $Q(x) = 1 \& f(x) = x^2 - x^3 - 9x + 4$) i.e.

$$y'' + \frac{1}{x} y' + y = x^2 - x^3 - 9x + 4, 0 \leq x \leq 1 \quad (4.10)$$

With boundary conditions: $$y(0) = 0, y(1) = 0 \quad (4.11)$$

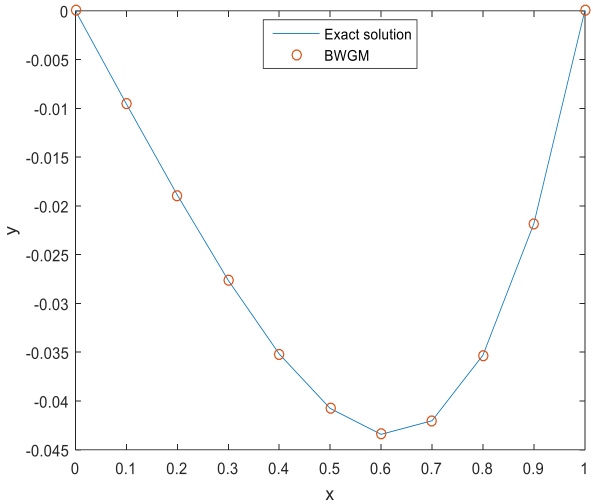


Fig. 2. Comparison of numerical solution with exact solution of the problem 4.2.

As explained in section 3 and in the previous problem, obtained the values of $c_{1,0} = 0.18746$, $c_{1,1} = 0.03609$ and $c_{1,2} = 0.00001$. Substituting these values in Eq. (4.5), to find the numerical solution. The comparison of the numerical solution and the absolute errors are presented in Table 6, while Table 7 compares error norms against exact solutions for confirmation, and the numerical solution with the exact solution of Eq. (4.10) is $y(x) = x^2 - x^3$ [15] in Fig. 3.

Table 6. Comparison of numerical solutionand absolute error with exact solution of the problem 4.3.

| x | Numerical solution | | | Exact solution | Absolute error | | |
|---|---|---|---|---|---|---|---|
| | Ref [15] | Ref [16] | BWGM | | Ref [15] | Ref [16] | BWGM |
| 0.1 | 0.010673 | 0.009677 | 0.008989 | 0.009000 | 1.67e-03 | 6.77e-04 | 1.10e-05 |
| 0.2 | 0.033159 | 0.032675 | 0.031988 | 0.032000 | 1.16e-03 | 6.75e-04 | 1.20e-05 |
| 0.3 | 0.063290 | 0.063354 | 0.062993 | 0.063000 | 2.90e-04 | 3.54e-04 | 7.00e-06 |
| 0.4 | 0.095881 | 0.095981 | 0.096003 | 0.096000 | 1.19e-04 | 1.90e-05 | 3.00e-06 |
| 0.5 | 0.125034 | 0.124731 | 0.125014 | 0.125000 | 3.40e-05 | 2.69e-04 | 1.40e-05 |
| 0.6 | 0.144429 | 0.143688 | 0.144023 | 0.144000 | 4.29e-04 | 3.12e-04 | 2.30e-05 |
| 0.7 | 0.147623 | 0.146841 | 0.147030 | 0.147000 | 6.23e-04 | 1.59e-04 | 3.00e-05 |
| 0.8 | 0.128350 | 0.128089 | 0.128029 | 0.128000 | 3.50e-04 | 8.90e-05 | 2.90e-05 |
| 0.9 | 0.080816 | 0.080862 | 0.081020 | 0.081000 | 1.84e-04 | 1.38e-04 | 2.00e-05 |

Table 7. Comparison for error norms $L_2$ & $L_\infty$ to compare with exact solutions for problem 4.3.

| Method | $L_2$ norm | $L_\infty$ norm |
|---|---|---|
| Ref [15] | 2.20e-03 | 1.70e-03 |
| Ref [16] | 1.10e-03 | 6.77e-04 |
| BWGM | 5.65e-05 | 3.00e-05 |

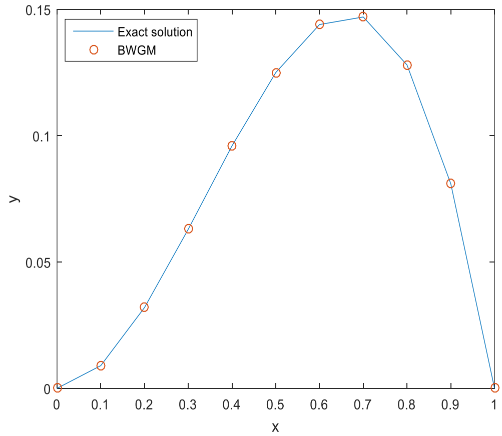


Fig. 3. Comparison of numerical solution with exact solution of the problem 4.3.

## 5. Conclusions

This paper presents a wavelet-based Galerkin method that employs Boubaker wavelets (BWGM) for the computation of numerical solutions to differential equations. The results indicate that the proposed method surpasses current techniques, such as the Finite difference method (FDM) and the Galerkin Method using Laguerre and Fibonacci wavelets, yielding solutions that align more closely with the exact results. Furthermore, the absolute error, $L_2$ & $L_\infty$ norms associated with this method is significantly lower when compared to the existing methods, namely the Finite difference method (FDM) and the Galerkin Method utilizing Laguerre and Fibonacci wavelets. This advancement represents a notable contribution to recent research in numerical analysis, offering substantial advantages to novice researchers. Consequently, the Boubaker wavelet-based Galerkin method proves to be highly effective for addressing boundary value problems.

## References

1. S. C. Shiralashetti and S. Kumbinarasaiah, Compu. Meth. Diff. Eqs. **7**, 177 (2019).
2. S. C. Shiralashetti and A. B. Deshi, Int. J. Compu. Mat. Sci. Eng. **6**, ID 1750014 (2017). https://doi.org/10.1142/S2047684117500142
3. L. M. Angadi, J. Math. Sci. Comp. Math. **6**, 77 (2025).
4. L. M. Angadi, J. Sci. Res. **16**, 31 (2024). http://dx.doi.org/10.3329/jsr.v16i1.63085
5. L. M. Angadi, Elect. J. Math. Anal. Appl. **13**, 1 (2025). http://dx.doi.org/10.21608/ejmaa.2024.305417.1238
6. K. Amaratunga and J. R. William, Inter. J. Num. Meth. Eng. **37**, 2703 (1994). https://doi.org/10.1002/nme.1620371602
7. J. W. Mosevic, Math. Comp. **31**, 139 (1977). https://doi.org/10.1090/S0025-5718-1977-0426447-0
8. M. A. Sarhan, S. Shihab, and M. Rasheed, J. South. Jiaot. Univ. **55** (2020). https://doi.org/10.35741/issn.0258-2724.55.2.3
9. S. C. Shiralashetti and L. Lamani, Math. Forum **28**, 114 (2020).

10. S. C. Shiralashetti, E. Harishkumar, and S. Hanaji, TWMS J. Appl. Eng. Math. **1**, 175 (2023).
11. J. E. Cicelia, Ind. J. Sci. Tech. **7**, 52 (2014). https://doi.org/10.17485/ijst/2014/v7sp3.3
12. L. M. Angadi, J. Stat. Math. Eng. **10**, 31 (2024). https://doi.org/10.1080/10724117.2024.2312037
13. T. Lot and K. Mahdiani, Math. Sci. **1**, 07 (2007).
14. S. Arora, Y. S. Brar, and S. Kumar, Int. J. Compu. Appl. **9**7, 33 (2014). https://doi.org/10.5120/17108-7759
15. L. M. Angadi, Int. J. Moder. Math. Sci. **19**, 34 (2021).
16. L. M. Angadi, J. Sci. Res. **17**, 227 (2025). https://dx.doi.org/10.3329/jsr.v17i1.75341