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Perimetric Contraction on n-gon and Related Fixed Point Results

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Abstract

This paper introduces a novel extension of the classical Banach Contraction Principle, focusing on "perimetric contractions" in n-gon. Unlike traditional contractions that deal with the distances between pairs of points, perimetric contractions are concerned with the contraction of the entire perimeter of an n-gon, considering the distances between consecutive points along the boundary. This new perspective enables the development of fixed-point results in higher-dimensional metric spaces. The core objective is to establish a fixed-point theorem for mappings that contract the perimeters of n-gon, providing a generalization of Banach's original theorem. The paper demonstrates that such mappings are continuous and presents conditions under which fixed points exist and are unique. Additionally, the relationships between perimetric contractions and conventional contraction mappings are examined, thus expanding the applicability of fixed-point theorems in more complex settings.

Keywords: Fixed point theorems; Metric spaces; Perimetric contraction on n-gon.

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1. Introduction

Banach developed the Contraction Mapping Principle in his 1920 dissertation, which was later published in 1922 [1]. Banach was the first to formulate this result in an accurate abstract form appropriate for a variety of applications, even though the concept of successive approximations in a number of concrete situations (solution of differential and integral equations, approximation theory) had previously been presented in the works of Chebyshev, Picard, Caccioppoli, and others [1]. One hundred years later, fixed-point theorems continue to pique the interest of mathematicians worldwide. This is supported by the large number of articles and monographs that have been written about fixed point theory and its applications in recent decades; for example, the monographs [2-4] for a survey on fixed point results.

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Over time, there have been numerous generalizations of the Banach contraction principle. The authors [5] pointed out that metric extensions are typically verified against three classical fixed point theorems in addition to Banach's fixed point theorem. These are Caristi's theorem [8], the extension of Banach's theorem to nonexpansive mappings [7], and Nadler's famous set-valued extension of Banach's theorem [6]. At the same time, there are at least two different kinds of generalizations of these theorems that may be distinguished: the first weakens the contractive character of the mapping, as seen in [9-18]; the second weakens the topology [19-32].

Assume that X is a metric space. It has been proved that the fixed point theorem for a new class of mappings, $T: X \rightarrow X$, in this work. These mappings can be generalized as mappings contracting perimeters of n gon. This work's main theorem proof is based on the concepts of the proof of Banach's classical theorem and generalized for perimetric contraction of triangles. However, the key distinction is that our mappings' definition is based on mapping n points of the space rather than three.

Additionally, it requires the necessary condition that $T(Tx) \neq x$ for any $x \in X$ such that $Tx \neq x$, which stops the mapping T from having points with least period two [33]. A significant subclass of these mappings are the ordinary contraction mappings, from which we can quickly derive the classical Banach's theorem as a straightforward consequence. For a space X with $|X| = \aleph_0$, where $|X|$ is the cardinality of the set X, an example of a mapping that contracts the perimeters of n-gon but is not a contraction mapping is built.

The core of this paper is to establish a fixed-point theorem for mappings that contract the perimeters of n-gon, thereby extending the established theory of Banach's contraction principle. This work addresses the properties of such mappings, investigates their uniqueness and existence of fixed points, and provides conditions under which these mappings exhibit interesting geometric behaviors. We will also explore the relationship between these perimetric contractions and ordinary contraction mappings, highlighting their connections and differences in the context of metric spaces. By introducing the perimetric contraction of n-gon, this paper contributes to the ongoing research in metric space theory, extending the applicability of fixed-point theorems to more complex and higher-dimensional structures.

2. Mapping Contracting Perimeters of n–gon

Definition 2.1. Let (X, d) be a metric space with $|X| \geq n$. We shall say that $T: X \rightarrow X$ is a *mapping contracting perimeters of n-gon* on X if there exists $\alpha \in [0,1)$ such that the inequality

$$d(Tx_1, Tx_2) + d(Tx_2, Tx_3) + \ldots + d(Tx_n, Tx_1) \leq \alpha\big(d(x_1, x_2) + d(x_2, x_3) + \ldots + d(x_n, x_1)\big). \quad (2.1)$$

Holds for all n pairwise distinct points $x_1, x_2, x_3, \ldots\ldots, x_n \in X$.

Remark 2.2. Note that the requirement for $x_1, x_2, x_3, \ldots\ldots, x_n \in X$ to be pairwise distinct is essential. One can see that otherwise this definition is equivalent to the definition of contraction mapping.

Proposition 2.3. *Mappings contracting perimeters of n-gon are continuous.*

Proof. Let (X,d) be a metric space with $|X| \geq n$, $T: X \rightarrow X$ be a mapping contracting perimeters of n–gon on X and let x_0 be an arbitrary. If x_0 is an isolated point of X, then it is clearly T is continuous at x_0. Therefore it remains to show that for any $\epsilon > 0$ there exist a $\delta > 0$ such that

$$d(Tx_0, Tx) < \epsilon \text{ for all } x \in X \text{ satisfying } d(x_0, x) < \delta.$$

Let $\epsilon > 0$ be an arbitrary. Choose $\delta > 0$ be such that $0 < \delta < \frac{\epsilon}{(2n-2)\alpha}$.

Since x_0 is a accumulation point of, there exist $x_1, x_2, x_3, \ldots\ldots, x_{n-2} \in X$ with $x_1 \neq x_2 \neq x_3 \neq \cdots \neq x_{n-2} \neq x_0$ such that

$$d(x_0, x_1) < \delta, d(x_0, x_2) < \delta, d(x_0, x_3) < \delta, \ldots\ldots, d(x_0, x_{n-2}) < \delta.$$

Now, for all $x \in X$ with $x \neq x_0$ satisfying $d(x_0, x) < \delta$. we have

$$d(Tx_0, Tx) \leq d(Tx_0, Tx) + d(Tx, Tx_1) + d(Tx_1, Tx_2) + \ldots\ldots\ldots + d(Tx_{n-2}, Tx_0).$$

By using (2.1) and triangle inequality we have,

$$\begin{aligned} d(Tx_0, Tx) &\leq \alpha\big(d(x_0, x) + d(x, x_1) + d(x_1, x_2) + \ldots\ldots\ldots + d(x_{n-2}, x_0)\big) \\ d(Tx_0, Tx) &\leq 2\alpha\big(d(x_0, x) + d(x_0, x_1) + d(x_0, x_2) + \ldots\ldots\ldots + d(x_0, x_{n-2})\big) \\ &\leq 2\alpha(\delta + \delta + \delta + \ldots\ldots\ldots + \delta) \\ &\leq 2\alpha(n-1)\delta \\ &\leq 2(n-1)\alpha\delta \\ &\leq (2n-2)\alpha\delta \\ &< \epsilon \quad \text{and hence the result follows.} \end{aligned}$$

Theorem 2.4. *Let (X,d), $|X| \geq n$, be a complete metric space and let the mapping $T: X \rightarrow X$ satisfy the following to conditions:*

(i) $T^m(x) \neq x$ for all $x \in X$ with $m = 2, 3, 4, \ldots\ldots, n-1$ such that $T(x) \neq x$.

(ii) T is a mapping contracting perimeters of n-gon on X.

Then T has a fixed point. The number of fixed points is at most $n-1$.

Proof. Let $x_0 \in X$, $Tx_0 = x_1$, $Tx_1 = x_2, \ldots\ldots, Tx_n = x_{n+1}, \ldots\ldots$. Suppose that x_i is not a fixed point of the mapping T for every $i = 0, 1, 2, \ldots\ldots$. Let us show that all x_i are different. Since x_i is not fixed , then $x_i \neq x_{i+1} = Tx_i$.

By condition (i) If $m = 2$, then $T^2(x_i) = T(Tx_i) = T(x_{i+1}) = x_{i+2} \neq x_i$.

If $m = 3$, then $T^3(x_i) = T(T^2 x_i) = T(x_{i+2}) = x_{i+3} \neq x_i$.

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If $m = n-1$, then

$$\begin{aligned} T^{n-1}(x_i) &= T(T^{n-2}x_i) \\ &= T(T(T^{n-3}x_i)) \\ &= T(T(T(T^{n-4}x_i))) \\ &\;\;\vdots \\ &= T\Big(T\Big(T\Big(T\Big(\ldots\ldots\ldots\ldots\ldots\big(T(x_i)\big)\Big)\Big)\Big)\Big) \quad \ldots(n-2 \text{ times of T}) \\ T^{n-1}(x_i) &= x_{i+n-1} \neq x_i. \end{aligned}$$

Since, x_i is not fixed point for all $\mathrm{m} = 2, 3, 4, \ldots\ldots, \mathrm{n}-1$ and by the supposition that x_{i+1} is not fixed point we have $x_{i+1} \neq x_{i+2} = Tx_{i+1}$.

By condition (i) If $\mathrm{m} = 2$, then $T^2(x_{i+1}) = T(Tx_{i+1}) = T(x_{i+2}) = x_{i+3} \neq x_{i+1}$.
If $\mathrm{m} = 3$, then $T^3(x_{i+1}) = T(T^2x_{i+1}) = T(x_{i+3}) = x_{i+4} \neq x_{i+1}$.

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If $\mathrm{m} = \mathrm{n}-1$, then $T^{n-1}(x_{i+1}) = x_{i+1+(n-1)} \neq x_{i+1}$.
Since x_{i+1} is not fixed point for all $\mathrm{m} = 2,3,4,\ldots\ldots,\mathrm{n}-1$. Therefore $x_i, x_{i+1}, x_{i+2}, \ldots\ldots, x_{i+n-1}$ are pairwise distinct. Further, set

$$p_0 = d(x_0,x_1) + d(x_1,x_2) + d(x_2, x_3) + \ldots\ldots\ldots + d(x_{n-1},x_0),$$
$$p_1 = d(x_1,x_2) + d(x_2, x_3) + d(x_3, x_4) + \ldots\ldots\ldots + d(x_n,x_1),$$
$$p_2 = d(x_2, x_3) + d(x_3, x_4) + d(x_4, x_5) + \ldots\ldots\ldots + d(x_{n+1},x_2),$$
$$p_3 = d(x_3, x_4) + d(x_4, x_5) + d(x_5, x_6) + \ldots\ldots\ldots + d(x_{n+2},x_3),$$

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$$p_n = d(x_n,x_{n+1}) + d(x_{n+1},x_{n+2}) + d(x_{n+2},x_{n+3}) + \ldots\ldots\ldots + d(x_{n+m-1},x_n).$$

Since $x_i, x_{i+1}, x_{i+2}, \ldots\ldots, x_{i+n-1}$ are pairwise distinct by (2.1) we have

$$p_1 \leq \alpha\, p_0,\ p_2 \leq \alpha p_1, \ldots\ldots\ldots, p_n \leq \alpha\, p_{n-1} \text{ and } p_0 > p_1 > p_2 > \ldots\ldots > p_n > \ldots\ldots \quad (2.2)$$

Suppose that $j \geq n$ is a minimal natural number such that $x_j = x_i$ for some i such that $0 \leq i < j-(n-1)$. Then $x_{j+1} = x_{i+1}, x_{j+2} = x_{i+2}, x_{j+3} = x_{i+3}, \ldots\ldots., x_{j+(n-1)} = x_{i+(n-1)}$. Hence $p_i = p_j$ which contradict to (2.2).
Further, let us show that $\{x_n\}$ is a Cauchy sequence. It is clear that

$$d(x_1,x_2) \leq p_0\,,$$
$$d(x_2,x_3) \leq p_1 \leq \alpha\, p_0\,,$$
$$d(x_3,x_4) \leq p_2 \leq \alpha\, p_1 \leq \alpha^2\, p_0\,,$$

……

$$d(x_n,x_{n+1}) \leq p_{n-1} \leq \alpha^{n-1}\, p_0\,,$$
$$d(x_{n+1},x_{n+2}) \leq p_n \leq \alpha^n\, p_0\,,$$

By the triangle inequality,

$$d(x_n,x_{n+p}) \leq d(x_n,x_{n+1}) + d(x_{n+1},x_{n+2}) + d(x_{n+2},x_{n+3}) + \ldots\ldots + d(x_{n+p-1},x_{n+p})$$
$$\leq \alpha^{n-1} p_0 + \alpha^n p_0 + \alpha^{n+1} p_0 + \ldots\ldots\ldots + \alpha^{n+p-2} p_0 = \alpha^{n-1}(1+\alpha+\alpha^2+ \ldots + \alpha^{p-1})p_0 = \alpha^{n-1}\left(\frac{1-\alpha^p}{1-\alpha}\right)p_0.$$

Since, by the supposition $0 \leq \alpha < 1$, then $d(x_n,x_{n+p}) < \alpha^{n-1}\left(\frac{1}{1-\alpha}\right)p_0$. Hence $d(x_n,x_{n+p}) \to 0\ as\ n \to \infty$ for every > 0. Thus $\{x_n\}$ is a Cauchy sequence. By completeness of (X,d), this sequence has a limit $x^* \in X$.
Let us prove that $Tx^* = x^*$. By the triangle inequality and by inequality (2.1) we have

$$d(x^*,Tx^*) \leq d(x^*,x_n) + d(x_n,Tx^*) = d(x^*,x_n) + d(Tx_{n-1},Tx^*)$$
$$\leq d(x^*,x_n) + d(x_n,x_{n+1}) + d(x_{n+1},x_{n+2}) + \cdots + d(x_{n+i-2},x^*)$$

where $\mathrm{i} = 1,2,3,\ldots,\mathrm{n}$.

$$\leq d(x^*,x_n) + \alpha\big(d(x_{n+1},x^*) + d(x_{n+2},x^*) + \cdots + d(x_{n+i-2},x^*)\big).$$

Since all the terms in the previous sum tend to zero as $n \to \infty$, we obtain $d(x^*,Tx^*) = 0$.
Suppose that there exist at least n pairwise distinct fixed points $x_1, x_2, x_3, \ldots\ldots, x_n$. Then $Tx_1 = x_1,\ Tx_2 = x_2, Tx_3 = x_3, \ldots, Tx_n = x_n$, which contradicts to (2.1).

Remark 2.5. Suppose that under the supposition of the theorem the mapping T has a fixed point f_1 which is a limit of some iteration sequence $x_0,\ x_1 = Tx_0,\ x_2 = Tx_1, \ldots$ such that

$x_n \neq f_1$ for all $n = 1,2,3,....$ Then f_1 is a unique fixed point. Indeed suppose that T has another fixed point $f_2, f_3, f_4, \dots, f_{n-1}$ such that $f_2 \neq f_1$, $f_3 \neq f_2$, $f_4 \neq f_3, \dots, f_{n-1} \neq f_1$. It is clear that $x_n \neq f_2$, $x_n \neq f_3, \dots, x_n \neq f_{n-1}$ for all $n = 1, 2, 3, ...$ Hence we have that the points $f_1, f_2, f_3, f_4, \dots, f_{n-1}$ and x_n are n pairwise distinct for all $n = 1, 2, 3, ...$ Consider the ratio

$$R_n = \frac{d(Tf_1,Tf_2) + d(Tf_2,Tf_3) + \cdots + d(Tf_{n-1},Tx_n) + d(Tx_n,Tf_1)}{d(f_1,f_2) + d(f_2,f_3) + \cdots + d(f_{n-1},x_n) + d(x_n,f_1)}$$
$$= \frac{d(f_1,f_2) + d(f_2,f_3) + \cdots + d(f_{n-1},x_{n+1}) + d(x_{n+1},f_1)}{d(f_1,f_2) + d(f_2,f_3) + \cdots + d(f_{n-1},x_n) + d(x_n,f_1)}$$

Taking into consideration that $d(f_2, f_3) \to 0, d(f_3, f_4) \to 0, \dots, d(f_{n-1}, x_{n+1}) \to 0$, $d(f_{n-1}, x_n) \to 0$ and $d(x_{n+1}, f_1) \to d(f_1, f_2), d(x_n, f_1) \to d(f_1, f_2)$, we obtain $R_n \to 1$ as $n \to \infty$ which contradicts to condition (2.1).

Example 2.6. Let us construct an example of t he mapping T contracting perimeters of n-gon which has exactly n-1 fixed points. Let $X = \{x_1, x_2, x_3, \dots\dots, x_n\}, d(x_1, x_2) = d(x_2, x_3) = \cdots = d(x_n, x_1) = 1$ and let $T: X \to X$ be such that $Tx_1 = x_1$, $Tx_2 = x_2$, $Tx_3 = x_3, \dots, Tx_n = x_n$. One can easily see that conditions (i) and (ii) of Theorem 2.4 are fulfilled.

Example 2.7. Let us show that condition (i) of Theorem 2.4 is necessary. Let $X = \{x_1, x_2, x_3, \dots\dots, x_n\}, d(x_1, x_2) = d(x_2, x_3) = \cdots = d(x_n, x_1) = 1$ and let $T: X \to X$ be such that $Tx_1 = x_2, Tx_2 = x_1, Tx_3 = x_4, Tx_4 = x_3, \dots, Tx_{n-1} = x_n, Tx_n = x_{n-1}$. One can easily see that condition (ii) of Theorem 2.4 is fulfilled but T does not have any fixed point.

Let (X, d) be a metric space. Then a mapping $T: X \to X$ is called a contraction mapping on X if there exist $\alpha \in [0,1)$ such that

$$d(Tx, Ty) \leq \alpha\left(d(x, y)\right), \text{ for all} x, y \in X. \quad (2.3)$$

Corollary 2.8. ***(Banach fixed point theorem)*** *Let* (X, d) be a nonempty complete metric space with a contraction mapping $: X \to X$. Then T admits a unique fixed point.

Proof. For $|X| = 1, 2$ the proof is trivial. Let $|X| \geq n$. Suppose that there exist $x \in X$ such that $T^m x = x$ for $m = n$. Consequently

$d(x, T^{n-1}x) = d(T^{n-1}x, x) = d(T^{n-1}x, T(T^{n-1}x)$ which contradicts to (2.3). Thus condition (i) of Theorem 2.4 holds. Let $x_1, x_2, x_3, \dots\dots, x_n \in X$ be pairwise distinct. By (2.3) we obtain $d(Tx_1, Tx_2) \leq \alpha\, d(x_1, x_2)$,

$$d(Tx_2, Tx_3) \leq \alpha\, d(x_2, x_3),$$
$$\dots.$$
$$d(Tx_n, Tx_1) \leq \alpha\, d(x_n, x_1)$$ which immediately implies

condition (ii) of Theorem (2.4). This completes the proof of existence of fixed point. The uniqueness can be shown in a standard way.

Let (X, d) be a metric space and $x_1, x_2, x_3, \dots\dots, x_n \in X$. We shall say that the points $x_2, x_3, \dots\dots, x_{n-1}$ lies between x_1 and x_n in the metric space (X, d) if the extremal version of the triangle inequality

$d(x_1, x_n) = d(x_1, x_2) + d(x_2, x_n) + d(x_1, x_3) + d(x_3, x_n) + \cdots + d(x_1, x_{n-1}) + d(x_{n-1}, x_n)$ (2.4) holds.

Example 2.9. Let us construct an example of a mapping $T: X \to X$ contracting perimeters of n- gon that is not contraction mapping for a metric space X with $|X| = \aleph_0$. Let $X = \{x^*, x_1, x_2, x_3, \ldots..\}$ and let a be a positive real number. Define a metric d on X as follows:

$$d(x,y) = \begin{cases} \frac{a}{2^{\left\lfloor \frac{i}{n-1} \right\rfloor}}, & if\ x = x_i,\ y = x_{i+1}, i = 0,1,2,\ldots.. \\ \sum_{c=i}^{j-1} \frac{a}{2^{\left\lfloor \frac{c}{n-1} \right\rfloor}}, & if\ x = x_i\ ,\ y = x_j\ ;\ i+1<j \\ 2(n-1)a - \sum_{c=0}^{i-1} \frac{a}{2^{\left\lfloor \frac{c}{n-1} \right\rfloor}}, & if\ x = x_i,\ y = x^* \\ 0, & if\ x = y \end{cases}$$

where $\lfloor . \rfloor$ is the floor function. Then (X, d) is a complete metric space with a single limit point x^*.

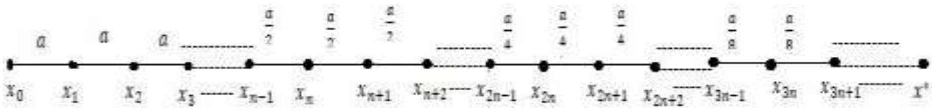

Fig. 1. The points of the space (*X, d*) with consecutive distances between them.

Define a mapping $T: X \to X$ as $Tx_{i_m} = x_{i_m+1}$, for all i $= 0, 1, 2, \ldots \ldots ..$ and m $= 1, 2, \ldots.,$ n $- 1, Tx^* = x^*$.
Since $d\left(Tx_{(n-1)i_m}, Tx_{(n-1)i_m+1}\right) = d\left(x_{(n-1)i_m}, x_{(n-1)i_m+1}\right)$ for all $i_m = 0, 1, 2, \ldots$ and m $= 1, 2, \ldots., $ n $- 1$, using (2.3) we see that T is not contraction mapping.
Let us show that inequality (2.1) holds for every n pairwise distinct points from the space X. Consider the points $x_{i_1}, x_{i_2}, x_{i_3}, \ldots, x_{i_{n-1}}, x^* \in X$ with $0 \le i_1 < i_2 < i_3 < \cdots < i_{n-1}$. Then we have

$$d(x_{i_1}, x_{i_2}) + d(x_{i_2}, x_{i_3}) + \ \ldots, + d(x_{i_{n-1}}, x^*) + d(x^*, x_{i_1}) = 2d((x_{i_1}, x^*)$$
$$= 4(n-1)a - 2\sum_{c=0}^{i_1-1} \frac{a}{2^{\left\lfloor \frac{c}{n-1} \right\rfloor}}$$

and

$$d(Tx_{i_1}, Tx_{i_2}) + d(Tx_{i_2}, Tx_{i_3}) + \ \ldots, + d(Tx_{i_{n-1}}, Tx^*) + d(Tx^*, Tx_{i_1}) = 2d((Tx_{i_1}, Tx^*)$$
$$= 2d(x_{i_1+1}, x^*) = 4(n-1)a - 2\sum_{c=0}^{i_1} \frac{a}{2^{\left\lfloor \frac{c}{n-1} \right\rfloor}}.$$

Now we have,

$$d(x_0, x_{i_m}) = \begin{cases} 2(n-1)a\left(1 - \left(\frac{1}{2}\right)^p\right), & if\ i_m = (n-1)p, \\ 2(n-1)a\left(1 - \left(\frac{1}{2}\right)^p\right) - \frac{a}{2^{p-1}}, & if\ i_m = (n-1)p - 1, \\ 2(n-1)a\left(1 - \left(\frac{1}{2}\right)^p\right) - \frac{a}{2^{p-2}}, & if\ i_m = (n-1)p - 2, \\ \vdots & \vdots \\ 2(n-1)a\left(1 - \left(\frac{1}{2}\right)^p\right) - \frac{a}{2^{p-(n-2)}}, & if\ i_m = (n-1)p - (n-2). \end{cases}$$

where $p = 1,2,3, \ldots.$ (2.5)

Consider the ratio,

$$R_{i_1} = \frac{d(Tx_{i_1}, Tx_{i_2}) + d(Tx_{i_2}, Tx_{i_3}) + \ ... + d(Tx_{i_{n-1}}, Tx^*) + \ d(Tx^*, Tx_{i_1})}{d(x_{i_1}, x_{i_2}) + d(x_{i_2}, x_{i_3}) + \ ... + d(x_{i_{n-1}}, x^*) + \ d(x^*, x_{i_1})}$$

$$= \frac{4(n-1)a - 2\ \sum_{c=0}^{i_1} \frac{a}{2^{\lfloor \frac{c}{n-1} \rfloor}}}{4(n-1)a - 2\ \sum_{c=0}^{i_1 - 1} \frac{a}{2^{\lfloor \frac{c}{n-1} \rfloor}}}$$

$$R_{i_1} = \begin{cases} 1 - \dfrac{\frac{a}{2^p}}{2(n-1)a - 2(n-1)a\left(1 - \left(\frac{1}{2}\right)^p\right)}, & if\ i_1 = (n-1)p, \\ 1 - \dfrac{\frac{a}{2^{p-1}}}{2(n-1)a - 2(n-1)a\left(1 - \left(\frac{1}{2}\right)^p\right) + \frac{a}{2^{p-1}}}, & if\ i_1 = (n-1)p - 1 \\ 1 - \dfrac{\frac{a}{2^{p-1}}}{2(n-1)a - 2(n-1)a\left(1 - \left(\frac{1}{2}\right)^p\right) + \frac{a}{2^{p-2}}}, & if\ i_1 = (n-1)p - 2 \\ \vdots \\ 1 - \dfrac{\frac{a}{2^{p-1}}}{2(n-1)a - 2(n-1)a\left(1 - \left(\frac{1}{2}\right)^p\right) + \frac{a}{2^{p-(n-2)}}}, & if\ i_1 = (n-1)p - (n-2) \end{cases}$$

$$R_{i_1} = \begin{cases} \dfrac{2n-3}{2n-2}, & if\ i_1 = (n-1)p, \\ \dfrac{n-1}{n}, & if\ i_1 = (n-1)p - 1 \\ \dfrac{n}{n+1}, & if\ i_1 = (n-1)p - 2 \\ \vdots \\ \dfrac{(n-2) + 2^{n-3}}{(n-1) + 2^{n-3}}, & if\ i_1 = (n-1)p - (n-2). \end{cases}$$

Now, let us consider the points $x_{i_1}, x_{i_2}, x_{i_3}, \dots, x_{i_n} \in X$ with $0 \le i_1 < i_2 < i_3 < \cdots < i_n$. Then, we have,

$$d(x_{i_1}, x_{i_2}) + d(x_{i_2}, x_{i_3}) + \ \dots, + d(x_{i_n}, x_{i_1}) = 2d((x_{i_1}, x_{i_n}) = 2 \sum_{c = i_1}^{i_n - 1} \frac{a}{2^{\lfloor \frac{c}{n-1} \rfloor}}$$

and $d(Tx_{i_1}, Tx_{i_2}) + d(Tx_{i_2}, Tx_{i_3}) + \ \dots, + d(Tx_{i_n}, Tx_{i_1}) = 2d((Tx_{i_1}, Tx_{i_n})$

$$= 2d(x_{i_1+1}, x_{i_n+1})$$

$$= 2d(x_{i_1}, x_{i_n}) - 2[d(x_{i_1}, x_{i_1+1}) - d(x_{i_n}, x_{i_n+1})]$$

$$= 2 \sum_{c = i_1}^{i_n - 1} \frac{a}{2^{\lfloor \frac{c}{n-1} \rfloor}} - 2\left(\frac{a}{2^{\lfloor \frac{i_1}{n-1} \rfloor}} - \frac{a}{2^{\lfloor \frac{i_n}{n-1} \rfloor}}\right)$$

Consider the ratio,

$$R_{i_1, i_n} = \frac{d(Tx_{i_1}, Tx_{i_2}) + d(Tx_{i_2}, Tx_{i_3}) + \ ... + d(Tx_{i_n}, Tx_{i_1})}{d(x_{i_1}, x_{i_2}) + d(x_{i_2}, x_{i_3}) + \ ... + d(x_{i_n}, x_{i_1})}$$

$$= \frac{2\ \sum_{c = i_1}^{i_n - 1} \frac{a}{2^{\lfloor \frac{c}{n-1} \rfloor}} - 2\left(\frac{a}{2^{\lfloor \frac{i_1}{n-1} \rfloor}} - \frac{a}{2^{\lfloor \frac{i_n}{n-1} \rfloor}}\right)}{2\ \sum_{c = i_1}^{i_n - 1} \frac{a}{2^{\lfloor \frac{c}{n-1} \rfloor}}}$$

$$= 1 - \frac{\left(\frac{a}{2^{\left\lfloor\frac{i_1}{n-1}\right\rfloor}} - \frac{a}{2^{\left\lfloor\frac{i_n}{n-1}\right\rfloor}}\right)}{\sum_{c=i_1}^{i_n-1} \frac{a}{2^{\left\lfloor\frac{c}{n-1}\right\rfloor}}}.$$

It is to be noted that $i_n \geq i_1 + (n-1)$. Therefore

$$\left\lfloor\frac{i_n}{n-1}\right\rfloor \geq \left\lfloor\frac{i_1}{n-1}\right\rfloor + 1$$

$$\Rightarrow \; 2^{\left\lfloor\frac{i_n}{n-1}\right\rfloor} \geq 2.\,2^{\left\lfloor\frac{i_1}{n-1}\right\rfloor}$$

$$\Rightarrow \; \frac{1}{2^{\left\lfloor\frac{i_n}{n-1}\right\rfloor}} \leq \frac{1}{2.\,2^{\left\lfloor\frac{i_1}{n-1}\right\rfloor}}$$

$$\Rightarrow \; \frac{a}{2^{\left\lfloor\frac{i_n}{n-1}\right\rfloor}} \leq \frac{a}{2.2^{\left\lfloor\frac{i_1}{n-1}\right\rfloor}}. \tag{2.6}$$

Now from (2.5), we can write,

$$d(x_{i_m}, x^*) = \begin{cases} \frac{2(n-1)a}{2^p}, & if\ i_m = (n-1)p, \\ \frac{2(n-1)a}{2^p} + \frac{a}{2^{p-1}}, & if\ i_m = (n-1)p-1, \\ \frac{2(n-1)a}{2^p} + \frac{a}{2^{p-2}}, & if\ i_m = (n-1)p-2, \\ \vdots & \\ \frac{2(n-1)a}{2^p} + \frac{a}{2^{p-(n-2)}}, & if\ i_m = (n-1)p-(n-2). \end{cases} \tag{2.7}$$

from above equation (2.7) we get,

$$d(x_{i_m}, x^*) \leq 2(n-1)d(x_{i_m}, x_{i_m+1})$$

$$\Rightarrow d(x_{i_1}, x^*) \leq 2(n-1)d(x_{i_1}, x_{i_1+1})$$

$$\Rightarrow \; d(x_{i_1}, x_{i_n}) \leq d(x_{i_1}, x^*) \leq 2(n-1)d(x_{i_1}, x_{i_1+1})$$

$$\Rightarrow \; \sum_{c=i_1}^{i_n-1} \frac{a}{2^{\left\lfloor\frac{c}{n-1}\right\rfloor}} \leq 2(n-1)\frac{c}{2^{\left\lfloor\frac{i_1}{n-1}\right\rfloor}}. \tag{2.8}$$

Consequently, from (2.6) and (2.8) we have

$$R_{i_1, i_n} = 1 - \frac{\frac{a}{2^{\left\lfloor\frac{i_1}{n-1}\right\rfloor}} - \frac{1}{2}\frac{a}{2^{\left\lfloor\frac{i_1}{n-1}\right\rfloor}}}{2(n-1)\frac{a}{2^{\left\lfloor\frac{i_1}{n-1}\right\rfloor}}} = \frac{4n-5}{4n-4}.$$

Thus the inequality (2.1) holds for any n pairwise distinct points from X with

$$\alpha = \frac{4n-5}{4n-4} = \max = \left\{\frac{2n-3}{2n-2}, \frac{n-1}{n}, \frac{n}{n+1}, \dots, \frac{(n-2)+2^{n-3}}{(n-1)+2^{n-3}}, \frac{4n-5}{4n-4}\right\}.$$

Therefore, T is a perimetric contraction on n- gon in X.

3. Conclusion

The study introduced perimetric contraction mappings on n-gons and established a fixed-point theorem that generalizes Banach's classical contraction principle. Celebrated Banach Fixed Point Theorem can be proved has simple corollary of the theorem established in the paper. Key outcomes:

a) Mappings contracting perimeters of n-gons are continuous.

b) Under suitable conditions, such mappings possess at most (n – 1) fixed points.

Remark 2.5: If the limit of the iterative sequence $x_0, x_1 = Tx_0, x_2 = Tx_1, ...$ does not belong to this sequence, then T has unique fixed point.

c) It is demonstrated that ordinary contractions are subclass of perimetric contractions.

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