JSR Publications

Available Online

J. Sci. Res. **17** (3), 849-859 (2025)

JOURNAL OF SCIENTIFIC RESEARCH

www.banglajol.info/index.php/JSR

Fractional Holographic Dark Energy in Self-Creation Theory and Lyra Geometry

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Received 26 January 2025, accepted in final revised form 26 May 2025

Abstract

In this paper, fractional holographic dark energy in self-creation theory and Lyra geometry have discussed the exact solution of Einstein's field equations for LRS Bianchi type- *I* cosmological model. In order to obtain a determinant solution, special law of variation for Hubble's parameter proposed by Berman (1983) has been considered. For each model, we evaluate key dynamical parameters, including the equation of state (EoS) parameter, the deceleration parameter and the total energy density parameter of dark energy. Our findings indicate that these models describe an accelerated expansion of the universe, with theoretical results showing reasonable agreement with observational data.

Keywords: LRS Bianchi type-*I*; Self creation theory; Lyra geometry.

doi: https://dx.doi.org/10.3329/jsr.v17i3.79171 J. Sci. Res. **17** (3), 849-859 (2025)

1. Introduction

The recent observational studies have given evidence for the accelerated expansion of the universe [1-5], suggesting the presence of a mysterious form of energy that drives this acceleration. This enigmatic force is commonly referred to as "dark energy." Observations indicate that over 70 % of the universe is composed of dark energy, which is responsible for the negative pressure that accelerates the cosmic expansion, while the remaining 30% consists of matter, most of which is dark matter-non-baryonic matter that does not emit or interact with electromagnetic radiation. There have been numerous other dark energy models proposed, including quintessence [6], phantom [7], quintom [8], tachyon [9], ghost [10], K-essence [11], phantom [12], chaplygin gas [13], polytropic gas [14] and holographic dark energy (HDE) [15] and many more to explain the accelerated expansion of the universe. The current understanding of the universe posits that it is predominantly made up of cosmic fluids consisting of dark matter and dark energy, evolving independently.

A variety of cosmological models have been studied to understand the behavior of dark energy and the dynamics of the universe within the framework of modified gravitational

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theories. Among these, the Self-Creation Theory (SCT) of gravitation, introduced by Barber [16], and Lyra Geometry, an extension of Riemannian geometry proposed by Lyra [17], have been extensively used in cosmological investigations. In SCT, a variable gravitational constant G is considered, which influences the evolution of the universe.

Lyra geometry modifies the standard Riemannian geometry by introducing a gauge function, and as a result, the gravitational dynamics and the structure of space time differ from those predicted by general relativity (GR). This framework opens up new possibilities for modeling the evolution of the universe and the role of dark energy within it.

Several studies have been dedicated to exploring cosmological models within these frameworks. For instance, Mahanta [18] studied locally rotationally symmetric Bianchi type I cosmological model. In the gravitational theory based on Lyra geometry and in the presence of a bulk viscous fluid, LRS Bianchi type I cosmological models were studied by Pradhan and Pandey [19] and Kandalkar and Samdurkar [20]. While Pradhan *et al.* [21] presented a new class of LRS Bianchi type-I cosmological model in the presence of bulk viscous fluid with variable deceleration parameter in the general relativity theory. Hegazy and Rahaman [22] studied Bianchi type VI_0 cosmological model in the second self-creation theory in general relativity and in Lyra geometry. Pawar and Solanke [23] studied magnetized anisotropic dark energy models in Barber's second self-creation theory. Interacting two-fluid viscous dark energy models in self-creation cosmology was given by Chirde and Shekh [24]. Kaluza-Klein cosmological model with bulk viscosity in Barber's second self-creation cosmology was given by Kumar and Reddy [25] Reddy and Naidu [26] studied Kaluza-Klein cosmological models in self-creation cosmology. Evolution of spatially homogeneous and isotropic FRW cosmological model with bulk-viscosity in self-creation theory of gravitation was analyzed by Katore *et al*. [27]. Pawar *et al.* [28] presented higher dimensional spherically symmetric string cosmological model with zero mass scalar field in Lyra geometry. Samanta and Mishra [29] studied anisotropic cosmological model in presence of holographic dark energy and quintessence. Kumawat *et al.* [30] investigated anisotropic Bianchi type VI_0 space-Time with barotropic fluid in Saez - Ballester theory of gravitation.

The concept of fractional holographic dark energy (FHDE) was proposed by Trivedi *et al.* [31] and explores a modification of the standard holographic dark energy (HDE) model by using fractional calculus. This modification leads to a power-law relationship between entropy and area, which is an important feature of the theory. The entropy is expressed as:

$$S_h = CA^{\frac{2+\gamma}{2\gamma}}, \tag{1}$$

where γ is a parameter that modifies the entropy-area relation, resembling the Barrow and Tsallis entropies. By applying the holographic inequality, a relationship between the cosmological parameters is derived:

$$\Lambda^3 L^3 \le \left(CA^{\frac{2+\gamma}{2\gamma}} \right)^{\frac{3}{4}}. \tag{2}$$

and the energy density of FHDE is given by:

$$\rho_f = \gamma L^{\frac{2-3\gamma}{\gamma}}, \tag{3}$$

where γ is a constant. In the limit where $\gamma = 2$, this gives us the standard HDE energy density. The proposed FHDE energy density is then written as:

$$\rho_f = 3c^2 L^{\frac{2-3\gamma}{\gamma}}, \tag{4}$$

where c is a free dimensionless O(1) parameter say arbitrary parameter, for the remaining part of the paper we will consider $c^2 = 1$.
With this in mind, the definition (5) with the Hubble Horizon cutoff $L = H^{-1}$ gives

$$\rho_f = 3c^2 H^{\frac{3\gamma-2}{\gamma}}, \tag{5}$$

Here we analyze the cosmological evolution at late times in the framework of the FHDE model using the Granda–Oliveros (G–O) cutoff.

$$L = \left(\ell_1 H^2 + \ell_2 \dot{H}\right)^{-1/2}, \tag{6}$$

where ℓ_1 and ℓ_2 are arbitrary dimensionless parameters. Here, by considering Hubble horizon described in Equation (2) as a candidate for Granda-Oliveros IR cutoff, the FHDE density from Equation(6) comes out to be

$$\rho_f = 3\left(\ell_1 H^2 + \ell_2 \dot{H}\right)^{\frac{3\gamma-2}{2\gamma}}. \tag{7}$$

The above discussion and investigations, we consider in this paper the fractional holographic dark energy model in LRS Bianchi type- I space-time in SCT and in gravity theory based on Lyra geometry. This work is organized as follows: In section 2, the Einstein's field equations in SCT and in Lyra geometry are derived with the help of a LRS Bianchi type- I space time metric in the presence of two minimally A interacting fields: dark matter and fractional holographic dark energy. Section 3, is devoted to the solution of Einstein's field equations in SCT and in Lyra geometry equations with the help of a special law of variation for Hubble's parameter proposed by Berman [32] and using physically relevant conditions. In section 4, kinematical parameters of the model are computed and discussed. The last section contains some concluding remarks.

2. The Metric and Field Equations

The spatially homogeneous LRS Bianchi type- I space-time as

$$ds^2 = -dt^2 + A^2 dx^2 + B^2[dy^2 + dz^2], \tag{8}$$

where A, B are functions of cosmic time t only.
The Einstein's field equations in SCT and in Lyra geometry are given as:

$$R_{ij} - \frac{1}{2} g_{ij} R + \frac{3}{2}\sigma_i \sigma_j - \frac{3}{4} g_{ij}\sigma_k \sigma^k = -8\pi\phi^{-1}\left(T_{ij} + \overline{T}_{ij}\right), \tag{9}$$

and

$$\Box\,\phi = \frac{8\pi}{3}\eta\left(T + \overline{T}\right). \tag{10}$$

Here $\Box\,\phi = \phi_{;k}^{;k}$ is invariant d'Alembertian and T is the trace of the energy momentum tensor describing all non-gravitational and non-scalar field matter and energy. Here η is a coupling constant to be determined from experiments. The Barber's second theory approaches the standard general relativity in every respect. The scalar field ϕ will be considered as a function of t only.

In equation (9), σ_i is the timelike displacement vector and takes the form:

$$\sigma_i = \left(\beta(t),0,0,0\right). \tag{11}$$

T_{ij} and $\overline{T}_{ij}$ are energy momentum tensors for matter and holographic dark energy, respectively. Which are defined as

$$T_{ij} = \rho_m u_i u_j,$$
$$\overline{T}_{ij} = (\rho_f + p_f)u_i u_j + g_{ij}p_f. \tag{12}$$

Here ρ_m and ρ_f are the energy densities of matter and barrow holographic dark energy and p_f is the pressure of holographic dark energy.

Also, the energy conservation equation is

$$(T_{ij} + \overline{T}_{ij})_{;j} = 0. \tag{13}$$

The field equation (9), for the metric equation (8) with the help of equations (10), (12) and (12), can be written as

$$2\frac{\ddot{B}}{B} + \frac{\dot{B}^2}{B^2} + \frac{3}{4}\beta^2 = -8\pi\phi^{-1}p_f, \tag{14}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} + \frac{3}{4}\beta^2 = -8\pi\phi^{-1}p_f, \tag{15}$$

$$2\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}^2}{B^2} - \frac{3}{4}\beta^2 = 8\pi\phi^{-1}\left(\rho_m + \rho_f\right), \tag{16}$$

$$\ddot{\phi} + \dot{\phi}\left(\frac{\dot{A}}{A} + 2\frac{\dot{B}}{B}\right) = \frac{8\pi}{3}\eta\left(\rho_m + \rho_f - 3p_f\right). \tag{17}$$

The energy conservation equation (13) for matter and dark energy is given as

$$\dot{\rho}_m + \left(\frac{\dot{A}}{A} + 2\frac{\dot{B}}{B}\right)\rho_m + \dot{\rho}_f + \left(\frac{\dot{A}}{A} + 2\frac{\dot{B}}{B}\right)\left(\rho_f + p_f\right) = 0, \tag{18}$$

where overhead dot stands for ordinary differentiation with respect to t.

The continuity equation of the matter is

$$\dot{\rho}_m + \left(\frac{\dot{A}}{A} + 2\frac{\dot{B}}{B}\right)\rho_m = 0. \tag{19}$$

The continuity equation of the holographic dark energy is

$$\dot{\rho}_f + \left(\frac{\dot{A}}{A} + 2\frac{\dot{B}}{B}\right)\left(\rho_f + p_f\right) = 0. \tag{20}$$

Using equations (19) and (20) and the barotropic equation of state $p_f = \omega_f \rho_f$, the equation of state Barrow HDE parameter is obtained as

$$\omega_f = -1 - \frac{(3\gamma - 2)(2\ell_1 H\dot{H} + \ell_2 \ddot{H})}{6\gamma H(\ell_1 H^2 + \ell_2 \dot{H})}. \tag{21}$$

Now the average scale factor and the volume of the universe are defined as

$$a(t) = \left(AB^2\right)^{\frac{1}{3}}. \tag{22}$$

The spatial volume

$$V = a^3(t) = AB^2. \tag{23}$$

The directional Hubble parameters and average Hubble parameter

$$H_x = \frac{\dot{A}}{A},\ H_y = \frac{\dot{B}}{B}. \tag{24}$$

And

$$H = \frac{1}{3}\left(\frac{\dot{A}}{A} + 2\frac{\dot{B}}{B}\right). \tag{25}$$

The dynamical scalar expansion θ and shear scalar σ^2 are

$$\theta = 3H$$

$$\sigma^2 = \frac{1}{2}\left(\frac{\dot{A}}{A} - \frac{\dot{B}}{B}\right)^2. \tag{26}$$

The average anisotropic parameter is

$$\Delta = \frac{1}{3}\sum_{i=1}^{3}\left(\frac{H_i - H}{H}\right)^2. \tag{27}$$

Here H_i represents the directional Hubble parameters $(i = 1,2,3)$
The deceleration parameter is

$$q = -1 + \frac{d}{dt}\left(\frac{1}{H}\right) = -\left[\frac{a\ddot{a}}{\dot{a}^2}\right]. \tag{28}$$

3. Solution of the Field Equations

Now equations (14)-(17) are a system of five independent equations in six unknowns $A, B, \phi, \rho_m, \rho_f$ and p_f. Hence, the following physically reasonable conditions are required to obtain a determinate solution (with over determinacy resolved by the field equations).
(i)

$$T + \overline{T} = \left(\rho_m + \rho_f - 3p_f\right) = 0. \tag{29}$$

which physically corresponds the vanishing of trace of both matter and dark energy tensors. This is analogous to the disordered radiation condition of general relativity.
(ii) Consider the relation between *H* and *a*, which was proposed by Berman [32].

$$H = na^{\frac{-1}{n}}. \tag{30}$$

where $n \geq 0$ are constants.
From equations (28) and (30) leads to

$$q = -1 + \frac{1}{n} . \tag{31}$$

Now, using equations (30) and (31), the solution of equation (28) gives the law of variation of the average scale factor of the form

$$a(t) = (t+c)^n, n \neq 0 . \tag{32}$$

Now, from equations (14) and (15) yields

$$\frac{\ddot{B}}{B} - \frac{\ddot{A}}{A} + \frac{\dot{B}^2}{B^2} - \frac{\dot{A}\dot{B}}{AB} = 0 . \tag{33}$$

On integration gives,

$$\frac{\dot{B}}{B} - \frac{\dot{A}}{A} = \frac{c_1}{AB^2} . \tag{34}$$

where c_1 is constant of integration.
Substituting equation (22) into equation (34) and integrating again leads to

$$\frac{B}{A} = c_2 \exp\left(\int \frac{c_1}{a^3} dt \right) .$$

The metric functions A and B in terms of average scale factor $a(t)$ are given by

$$A(t) = c_2^{\frac{-2}{3}} a \exp\left(\frac{-2c_1}{3} \int a^{-3} dt \right), \quad B(t) = c_2^{\frac{1}{3}} a \exp\left(\frac{c_1}{3} \int a^{-3} dt \right). \tag{35}$$

Now, using equation (32) in equation (35) yields

$$A(t) = c_2^{\frac{-2}{3}} (t+c)^n \exp\left(\frac{-2c_1}{3(1-3n)} (t+c)^{1-3n} \right), \quad B(t) = c_2^{\frac{1}{3}} (t+c)^n \exp\left(\frac{c_1}{3(1-3n)} (t+c)^{1-3n} \right). \tag{36}$$

4. Kinematical Parameters of the Model

In this section, the kinematical parameters of model (8) are evaluated, as they play a significant role in the discussion of the cosmological model of the universe.

The spatial volume of the metric is

$$V = a^3(t) = (t+c)^{-3n} . \tag{37}$$

The directional and average Hubble parameter

$$H_x = \frac{n}{(t+c)} - \frac{2c_1}{3}(t+c)^{-3n},$$
$$H_y = \frac{n}{(t+c)} + \frac{c_1}{3}(t+c)^{-3n}. \tag{38}$$

And

$$H = \frac{n}{(t+c)}. \tag{39}$$

The dynamical scalar expansion θ and shear scalar σ^2 are

$$\theta = 3\frac{n}{(t+c)}. \tag{40}$$

$$\sigma^2 = \frac{c_1^2}{2}(t+c)^{-6n}. \tag{41}$$

The average anisotropic parameter is

$$\Delta = \frac{2c_1^2}{9}(t+c)^{2-6n}. \tag{42}$$

Applying the conservation condition for the left-hand side of equation (9) yields

$$\beta\left(\dot{\beta} + \beta\left[\frac{\dot{A}}{A} + 2\frac{\dot{B}}{B}\right]\right) = 0. \tag{43}$$

After integrating equation (43) gives

$$\beta = \beta_0 (t+c)^{-3n}. \tag{44}$$

From equations (17) and (29), the scalar field ϕ satisfies the equation

$$\phi = \frac{\phi_0}{(1-3n)}(t+c)^{1-3n}. \tag{45}$$

Using equation (40) in equation (35), the pressure of FHDE is given by

$$8\pi\phi^{-1} p_f = \frac{n(2-3n)}{(t+c)^2} - \left(\frac{c_1^2}{3} - \frac{3\beta_0}{4}\right)(t+c)^{-6n}. \tag{46}$$

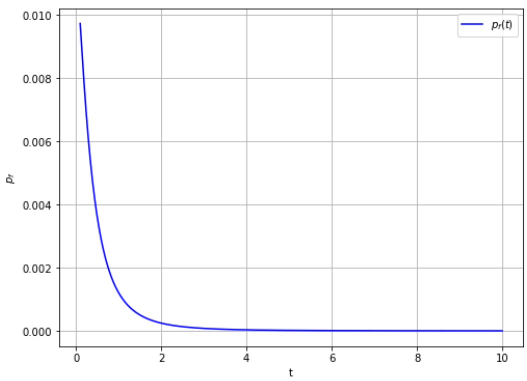

Fig. 1. Plot of pressure (p_f) of FHDE vs time (t).

Using equations (43), (44) and (49) in equation (13), the energy density of matter is

$$8\pi\phi^{-1}\rho_f = \frac{3n^2}{(t+c)^2} + \left(\frac{c_1^2}{3} - \frac{3\beta_0}{4}\right)(t+c)^{-6n} - 24\pi\phi^{-1}\left(\frac{\ell_1 n^2 - \ell_2 n}{(t+c)^2}\right)^{\frac{3\gamma-2}{2\gamma}} . \tag{47}$$

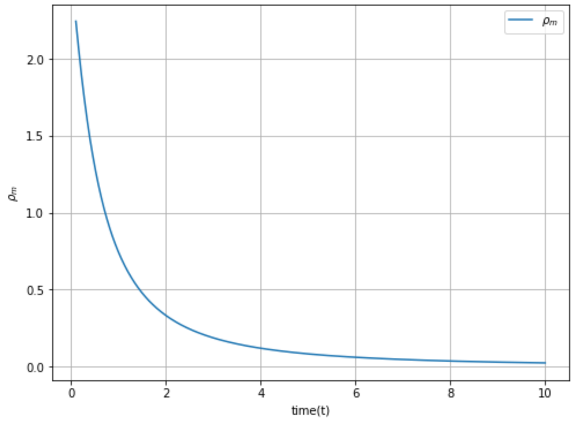

Fig. 2. Plot of energy density (ρ_m) of matter vs time (t).

Using equation (39) in equation (7), the energy density of FHDE is given by

$$\rho_f = 3\left(\frac{\ell_1 n^2 - \ell_2 n}{(t+c)^2}\right)^{\frac{3\gamma-2}{2\gamma}} . \tag{48}$$

The density ρ_f of fractional HDE model in Granda-Oliveros cut-off increases uniformly with increasing red-shift at all times.

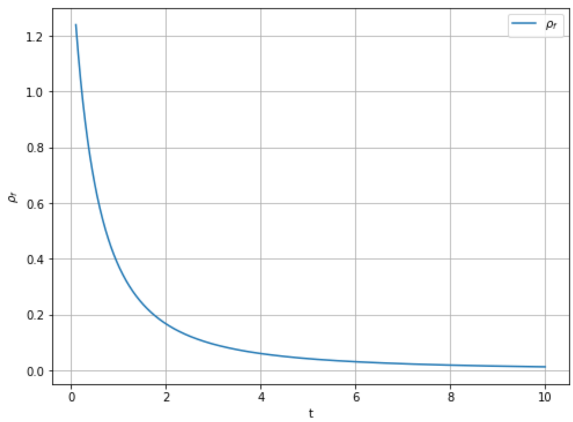

Fig. 3. Plot of energy density (ρ_f) of Fractional holographic dark energy vs time (t).

The Figs. 2 and 3 represent the plots of energy density (ρ_m) of matter and FHDE with the Granda-Oliveros cutoff decreases respectively. It is observed that both ρ_m and ρ_f are positive and decrease as universe evolves.

Using equations (43), (44) and (49) in equation (15), the EoS parameter for FHDE is given by

$$\omega_f = -1 + \frac{2}{n}\left(\frac{\ell_1 n^2 - \ell_2 n}{(t+c)^2}\right)^{\frac{3\gamma-2}{2\gamma}} . \tag{49}$$

Equation (49) shows the ω_f of fractional HDE model in Granda-Oliveros cut-off gives -1, it is typically depend upon the vales of ℓ_1 and ℓ_2 associated with dark energy or a cosmological constant, which leads to the accelerated expansion of the universe.

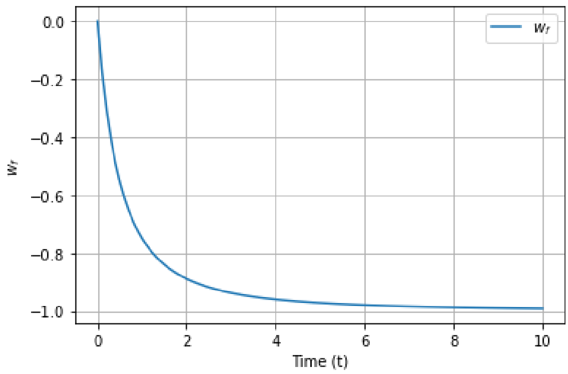

Fig. 4. Plot of EoS parameter (ω_f) of Fractional HDE vs time (t).

The above figure shows the EoS parameter of the fractional HDE model with the Granda–Oliveros cut-off approaching –1, indicating that the behavior of the field resembles that of a cosmological constant, characterized by constant energy density and negative pressure.

Matter density parameter Ω_m and the holographic dark energy parameter Ω_f are given by

$$\Omega_m = \frac{\rho_m}{3H^2} \text{ and } \Omega_f = \frac{\rho_f}{3H^2} , \tag{50}$$

Using equations (39), (46), (47) and (50), the overall density parameter is

$$\Omega = \Omega_m + \Omega_f = 1 + \frac{1}{3}\left(\frac{c_1^2}{3} - \frac{3\beta_0}{4}\right)(t+c)^{-6n+2} . \tag{51}$$

Therefore, the total energy density parameter (Ω) of Fractional HDE model in Granda-Oliveros cut-off as a function of time t suggests that the universe could initially have a higher density (indicating a closed or high-density universe) but asymptotically approaches a flat universe as time progresses. This evolution could reflect the dominance of different energy components over time (such as matter, radiation, and dark energy), with the universe eventually stabilizing toward a flat state.

5. Conclusion

The LRS Bianchi type-I cosmological model was studied within the framework of FHDE in SCT and Lyra geometry. Initially, the model exhibits a point singularity at $t = 0$, where the spatial volume $V \to 0$ and physical parameters such as the Hubble parameter H, energy densities ρ_m, ρ_f and pressure p_f are infinite. As time progresses, V increases while these parameters decrease and tend to zero as $t \to \infty$, indicating an expanding universe. The anisotropy parameter Δ is zero at $t = 0$, showing isotropy initially, but the ratio $\frac{\sigma^2}{\theta^2}$ does not vanish at late times, implying persistent anisotropy. The fractional holographic dark energy density ρ_f decreases over time, and the average density parameter approaches a constant, consistent with a flat universe supported by observations. The EoS parameter ω_f evolves between $-1 < \omega_f < 0$, indicating quintessence-like behavior driving accelerated expansion. As $\omega_f \to -1$, the model mimics a cosmological constant; as $\omega_f \to 0$, it suggests a possible transition from dark energy to matter domination.

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