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Cosmological Instability in Expanding Universes with Massive Particles in the Context of *f (T)* Gravity

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Abstract

The Bianchi III anisotropic cosmological model has been examined within the context of *f (T)* gravity, where *f (T)* represents Teleparallel gravity. The Main objective is to investigate the instability of the universe by analyzing the derivative of the pressure function with respect to the matter density function. The accelerated expansion of the universe has been examined. The equation of state (EOS) is found to vary with time. To accomplish this derive exact solutions for the space-time field equations in an exponential form and consider a power-law approach with a non-dissipative anisotropic fluid distribution. By analyzing pressures in the spatial directions, obtained a set of energy condition equations, which are then used to formulate the Dark Energy equations.

Keywords: *f(T)* theory; Bianchi III type modal; Perfect Fluid; Energy Condition; Anisotropic Condition.

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1. Introduction

For centuries, human curiosity has grown regarding the many mysteries of the Universe. The large-scale structure of the Universe is regarded as a fascinating area of study by researchers. Although substantial intellectual work has already been undertaken, the exact physical conditions at the Universe's earliest stages remain unknown. According to previous findings, it has been clearly demonstrated that stable topological defects were formed during the initial phase of the Universe. As suggested by Kibble *et al.* [1], the spontaneous breaking of discrete symmetry leads to the creation of topological defects. Among the significant types of topological defects are monopoles, domain walls, and cosmic strings of which domain walls are thought to play an especially active role in the formation of galaxies within the Universe.

Vilenkin [2] and Goetz [3] proposed that galaxies can form with the help of domain walls following the recombination of matter and radiation during a phase transition. One of

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the most significant phase transitions is the conversion from quark-gluon plasma to hadrons gas, which occurs at a cosmic temperature of approximately 200 MeV. It is speculated that at such high temperatures, the color charge becomes screened - a state referred to as the quark-gluon plasma (QGP). Observations suggest a strong interaction between the electric charge and the color charges of quarks and gluons, and that these can interchangeably influence each other.

The Universe is expanding and accelerating - a phenomenon well established by Riess *et al.* [4] and Perlmutter *et al.* [5]. This accelerating expansion of the Universe can be described using modified theories of gravity and dark energy (DE). Among these, one particularly intriguing theory offers an explanation for the current acceleration without invoking DE. Einstein [6] introduced a reformulation of gravitational theory known as Teleparallel Gravity (TG), specifically described as *f (T)* gravity. This is not an alternative to general relativity but rather a different formulation, in which gravity arises not from space-time curvature but from torsion. Böhmer *et al.* [7] investigated spherically symmetric solutions and the nature of relativistic stars within the *f (T)* framework. Setare *et al.* [8] explored real-valued and power-law solutions in *f (T)* gravity. The distinction between ΛCDM and *f (T)* gravity, based on symmetry considerations, was derived by Dong *et al.* [9]. The generalized second law of thermodynamics within *f (T)* gravity was discussed by Bamba [10]. Additionally, Setare *et al.* [11] examined finite-time future singularities in *f (T)* gravity, both with and without viscosity. Astashenok [12] explored effective DE models incorporating cosmological bounce scenarios within the *f (T)* framework.

Furthermore, it has been shown that a combination of two *f (T)* models can replicate outcomes equivalent to those predicted by General Relativity during the pre-inflationary phase of the Universe. Relativistic effects within the context of *f (T)* gravity have been addressed by Tsung *et al.* [13]. Chawla *et al.* [14] examined Bianchi Type I models in the presence of a scattering fluid for time-dependent deceleration parameters. Chirde *et al.* [15] studied a spatially homogeneous and isotropic FRW cosmological model with a viscous baratropic fluid in the *f (T)* framework. This line of research is considered highly motivating for the study of various cosmological models, as experimental results appear to align closely with those predicted by General Relativity, particularly in relation to flat spatial sections. The homogeneous Bianchi model is viewed as one of the most simplified representations of the Universe. The evolution of the early Universe plays a significant role in understanding its overall dynamics. Katore *et al.* [16] analyzed the FRW metric under a constant deceleration parameter and found that domain walls disappeared over time, suggesting that the Universe could have existed during an earlier epoch with observable stability. The study of bulk viscous Bianchi-type cosmological models in *f (T)* gravity has been further explored by Agrawal and Nile [17]. Numerous investigations have been conducted in *f (T)* gravity, including studies on late-time cosmic acceleration [18–21], inflationary models [22], and analyses involving observational constraints, dynamical systems [23–25], and structure formation often assuming *f (T)* follows a power-law form.

In Section 2, we derive the equations of motion for Teleparallel gravity *f (T).* Notably, when $f(T) = (-T)^n$, the theory exhibits dynamics equivalent to those of General Relativity.

In Section 3, we formulate the field equations within the *f (T)* gravity framework, considering the pressures along the spatial directions *x, y,* and *z*. In Section 4, we analyze various cosmological solutions within *f (T)* gravity. These solutions demonstrate remarkable cosmological behavior, with a wide class of *f (T)* models contributing to late-time acceleration. In Section 5, we discuss the energy conditions related to cosmic expansion. In Section 6, we examine the standard stability function of the model. Finally, in Section 7, we summarized our findings and discuss their broader implications.

2. Equation of Motion for Teleparallel Gravity *f (T)*

The theory of Teleparallel gravity, along with the derivation of its corresponding field equations, has been presented. In this paper, Latin indices (as subscripts or superscripts) are used to denote components of the tetrad field, while Greek symbols represent space-time coordinates. The line element for a general space-time metric is given as follows:

$$ds^2 = g_{\mu v} dx^\mu dx^v, \quad (2.1)$$

one can transform this line element in Minkowski's description; the respective tetrads are as follows:

$$ds^2 = g_{\mu v} dx^\mu dx^v = \eta_{ij}\theta^i\theta^j \,, dx^\mu = e_i^\mu \theta^i, \text{and } \theta^i = e_\mu^i dx^\mu \,, \quad (2.2)$$

here Minkowski space time metric is η_{ij} and $\eta_{ij} = diag(1, -1, -1, -1)$ and

$$e_i^\mu e_v^i = \delta_v^\mu or\ e_i^\mu e_\mu^j = \delta_i^j \,, \quad (2.3)$$

and $\sqrt{-g} = det[e_\mu^i] = a,$ is the root of metric determinant.

The Weitzen-bocks connection of components for a manifold having Riemann tensor part deprived of torsion term is null and where only non-zero torsion terms exist which are defined as:

$$\Gamma_{\mu v}^\alpha = e_i^\alpha \partial_v e_\mu^i = -e_\mu^i \partial_v e_i^\alpha \,, \quad (2.4)$$

this is with zero curvature yet non zero torsion. So, one may describe various torsion tensors components as:

$$T_{\mu v}^\alpha = \Gamma_{\mu v}^\alpha - \Gamma_{v \mu}^\alpha = e_i^\alpha (\partial_\mu e_v^i - \partial_\mu e_\mu^i). \quad (2.5)$$

Con-Tensor is a space time tensor which is a difference between the Weitzen-bock connections and Levi-Civita are given by:

$$K_\alpha^{\mu v} = \left(-\frac{1}{2}\right)\left(T^{\mu v}{}_\alpha + T^{v\mu}{}_\alpha - T_\alpha^{\mu v}\right). \quad (2.6)$$

Now another new tensor $S_\alpha^{\mu v}$ is defined from the components of the torsion and con-torsion tensor as:

$$S_\alpha^{\mu v} = \left(\frac{1}{2}\right)(K^{\mu v}{}_\alpha + \delta_\alpha^\mu T^{\beta v}{}_\beta - \delta_\alpha^v T_\beta^{\beta\mu}, \quad (2.7)$$

and

$$T = T_{\mu v}^\alpha S_\alpha^{\mu v} \text{ , or } T = \frac{1}{4}\, T^{\rho\mu v}\, T_{\rho\mu v} + \frac{1}{2}\, T^{\rho\mu v}\, T_{v\mu\rho} - T^\rho{}_{\rho\mu}\, T^{v\mu}{}_v \,, \quad (2.8)$$

here *T* is a torsion scalar.

Define the Teleparallel i.e., $f(T)$ theory as follows:

$$S = \int [f(T) + L_{matter}] a d^4x \,, \quad (2.9)$$

here $f(T)$ is an algebraic function of torsion scalar *T* with respect to tetrads for $f(T)$ gravity the equation of motion is given as follows:

$$4\left[a^{-1}\partial_{\mu}\left(aS_{i}^{\mu\upsilon}\right)-h_{i}^{\lambda}T_{\mu\lambda}^{\rho}S_{\rho}^{\upsilon\mu}\right]f_{T}+4S_{i}^{\mu\upsilon}\partial_{\mu}(T)f_{TT}-h_{i}^{\upsilon}f(T)=4\pi h_{i}^{\rho}T_{\rho}^{\upsilon}, \quad (2.10)$$

here $S_{i}^{\mu\upsilon}=h_{i}^{\rho}S_{\rho}^{\mu\upsilon}$, $f_{T}=\frac{df}{dT}$, $f_{TT}=\frac{d^{2}f}{dT^{2}}$, $a=\sqrt{-g}$,
T_{μ}^{υ} is energy momentum tensor. In terms of tetrads and their partial derivatives above equation of motion appears different from Einstein's equation, whereas for the value $f(T)=(-T)^{n}$ above equation behaves dynamically same as General Relativity.

3. Basic Equations

Assume Bianchi III type metric as:

$$ds^{2}=dt^{2}-D^{2}dx^{2}-E^{2}e^{-2\alpha x}dy^{2}-F^{2}dz^{2}, \quad (3.1)$$

here *D, E,* and *F* are the cosmic time functions and α is a constant.
Consider the matter energy momentum tensor as:

$$T_{\rho}^{\upsilon}=diag\left(-p_{x},-p_{y},-p_{z},\rho_{M}\right), \quad (3.2)$$

here p_{x}, p_{y}, and p_{z} are defined the pressures along spatial directions x, y, z.

$$e_{\rho}^{\mu}=diag(D,\mathrm{e}^{-\alpha x}E,F,1),\ a=\sqrt{-g}=\mathrm{e}^{-\alpha x}DEF. \quad (3.3)$$

Substitute the value in correspondingly value in torsion tensor tetrad (2.5), the contortion tensor (2.6) and super-potential tensor $S_{\rho}^{\mu\upsilon}$ (2.7) obtained the non-vanishing components of torsion tensor from (2.5) as:

$$T_{14}^{1}=-\frac{\dot{D}}{D},T_{24}^{2}=-\frac{\dot{E}}{E},T_{34}^{3}=-\frac{\dot{F}}{F},T_{41}^{1}=\frac{\dot{D}}{D},T_{42}^{2}=\frac{\dot{E}}{E},T_{43}^{3}=\frac{\dot{F}}{F}. \quad (3.4)$$

The non-vanishing components of super-potential tensor from (2.7) as:

$$S_{1}^{14}=-\frac{1}{2}\left(\frac{\dot{E}}{E}+\frac{\dot{F}}{F}\right),S_{1}^{41}=\frac{1}{2}\left(\frac{\dot{E}}{E}+\frac{\dot{F}}{F}\right),S_{2}^{24}=-\frac{1}{2}\left(\frac{\dot{D}}{D}+\frac{\dot{F}}{F}\right),S_{2}^{24}=\frac{1}{2}\left(\frac{\dot{D}}{D}+\frac{\dot{F}}{F}\right),$$
$$S_{3}^{34}=-\frac{1}{2}\left(\frac{\dot{D}}{D}+\frac{\dot{E}}{E}\right),S_{3}^{43}=\frac{1}{2}\left(\frac{\dot{D}}{D}+\frac{\dot{E}}{E}\right),S_{1}^{22}=\frac{\alpha}{2D^{2}}, \quad (3.5)$$

using these values in torsion scalar (2.8) obtained as:

$$T=-2\left(\frac{\dot{D}\dot{E}}{DE}+\frac{\dot{D}\dot{F}}{DF}+\frac{\dot{E}\dot{F}}{EF}\right), \quad (3.6)$$

using these values in the equations of motion (2.10) yields as:

$$f+4f_{T}\left[\frac{\dot{D}\dot{E}}{DE}+\frac{\dot{D}\dot{F}}{DF}+\frac{\dot{E}\dot{F}}{EF}-\frac{\alpha^{2}}{2D^{2}}\right]=16\pi\rho_{M}, \quad (3.7)$$

$$f+2f_{T}\left[\frac{\ddot{E}}{E}+\frac{\ddot{F}}{F}+\frac{\dot{D}\dot{E}}{DE}+\frac{\dot{D}\dot{F}}{DF}+2\frac{\dot{E}\dot{F}}{EF}\right]+2\dot{T}f_{TT}\left[\frac{\dot{E}}{E}+\frac{\dot{F}}{F}\right]=-16\pi p_{x}, \quad (3.8)$$

$$f+2f_{T}\left[\frac{\ddot{D}}{D}+\frac{\ddot{F}}{F}+\frac{\dot{D}\dot{E}}{DE}+2\frac{\dot{D}\dot{F}}{DF}+\frac{\dot{E}\dot{F}}{EF}\right]+2\dot{T}f_{TT}\left[\frac{\dot{D}}{D}+\frac{\dot{F}}{F}\right]=-16\pi p_{y}, \quad (3.9)$$

$$f+2f_{T}\left[\frac{\ddot{D}}{D}+\frac{\ddot{E}}{E}+2\frac{\dot{D}\dot{E}}{DE}+\frac{\dot{D}\dot{F}}{DF}+\frac{\dot{E}\dot{F}}{EF}-\frac{\alpha^{2}}{D^{2}}\right]+2\dot{T}f_{TT}\left[\frac{\dot{D}}{D}+\frac{\dot{E}}{E}\right]=-16\pi p_{z}, \quad (3.10)$$

$$\frac{\alpha^{2}}{2D^{2}}\left[\left(\frac{\dot{D}}{D}-\frac{\dot{E}}{E}\right)f_{T}-\dot{T}f_{TT}\right]=0, \quad (3.11)$$

$$\alpha\left[\frac{\dot{D}}{D}-\frac{\dot{E}}{E}\right]f_{T}=0. \quad (3.12)$$

4. Solution of the Field Equation

The solution of Bianchi III type has observed complications. The complication has occurred from the constraint (3.11) and (3.12), because the Eq. (3.12) as:

$$\alpha\left[\frac{\dot{D}}{D}-\frac{\dot{E}}{E}\right]f_{T}=0,$$

$\frac{\dot{D}}{D} = \frac{\dot{E}}{E}\ so\ E = mD$, and $f_T \neq 0$, (4.1)

using this condition in Eq. $\frac{\alpha^2}{2D^2}\left[\ \left(\frac{\dot{D}}{D} - \frac{\dot{E}}{E}\right) f_T - \ \dot{T} f_{TT}\right] = 0$,

$\dot{T} f_{TT} = 0$, this implies that either,

$\dot{T} = 0\ or\ f_{TT} = 0.$ (4.2)

Case I: If $\dot{T} = 0$, and $f_{TT} \neq 0$, then

so $T = K,$ (4.3)

here *K* is a constant.

Using this relation in torsion scalar (3.6) obtained as:

$$K = \left(\frac{\dot{D}^2}{D^2} + 2\frac{\dot{D}\dot{F}}{DF}\right), \quad (4.4)$$

consider $D = F^r,$ (4.5)

using *D* in Eq. (4.4) yields as:

$$F = c_1\, e^{\left(\pm\sqrt{\frac{K}{r(r+2)}}\, t\right)},\ \text{for}\ r > 0\ and\ r < -2, \quad (4.6)$$

here c_1 is a constant.

Taken +ve sign in Eq. (4.6) and using *F* in Eq. (4.5) and (4.1) obtained as:

$$F = c_1\, e^{\left(\sqrt{\frac{K}{r(r+2)}}\, t\right)},\ D = c_1{}^r\, e^{\left(r\sqrt{\frac{K}{r(r+2)}}\, t\right)},\ \text{and}\ E = m c_1{}^r\, e^{\left(r\sqrt{\frac{K}{r(r+2)}}\, t\right)}, \quad (4.7)$$

assume that $\xi = \sqrt{\frac{K}{r(r+2)}}$,

$$D = c_1{}^r\, e^{r\,\xi t}\ , E = m c_1{}^r\, e^{r\,\xi t}, F = c_1\, e^{\xi t}\ . \quad (4.8)$$

Evaluate the values from Eq. (4.8) as:

$$\frac{\dot{D}}{D} = \frac{\dot{E}}{E} = r\xi\ , \frac{\dot{F}}{F} = \ \xi\ ,\ \text{and}\ \frac{\ddot{D}}{D} = \frac{\ddot{E}}{E} = r^2\xi^2\ , \frac{\ddot{F}}{F}\ =\ \xi^2. \quad (4.9)$$

Consider $f(T) = \ (-T)^n$, and $f_T = \ n(-T)^{n-1}$. (4.10)

Rest energy density:

Using the values of *D, E,* and *F* from Eq. (4.8) and *f (T)* from Eq. (4.8) in the Eq. (3.7) obtained as:

$$\rho_M = \frac{1}{16\pi}\ \left[-K + 4n\left(r^2\xi^2 + 2r\xi^2 - \frac{\alpha^2}{2{c_1}^{2r}} e^{-2r\xi t}\right)\right]. \quad (4.11)$$

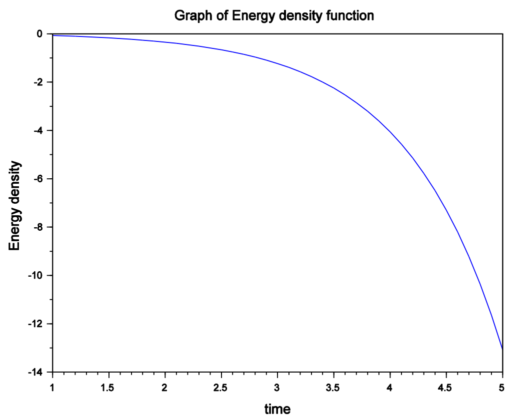

Fig. 1. Energy density with time.

As illustrated in Fig. 1, the energy density decreases over time, indicating that the Universe had a higher density in its early stages, which diminishes as it, continues to expand.

Pressure P:

Using the values of *D, E,* and *F* from Eq. (4.8) and *f (T)* from Eq. (4.8) in the Eqⁿ's (3.8), (3.9), and (3.10) found as:

$$p_x = \frac{-1}{16\pi}\ [-K + 2n(2r^2\xi^2 + 3r\xi^2 + \xi^2)]\,, \tag{4.12}$$

$$p_y = \frac{-1}{16\pi}\ [-K + 2n(2r^2\xi^2 + 3r\xi^2 + \xi^2)], \tag{4.13}$$

$$p_z = \frac{-1}{16\pi}\ \left[-K + 2n\left(4r^2\xi^2 + 2r\xi^2 - \frac{\alpha^2}{{c_1}^{2r}} e^{-2r\xi t}\right)\right]. \tag{4.14}$$

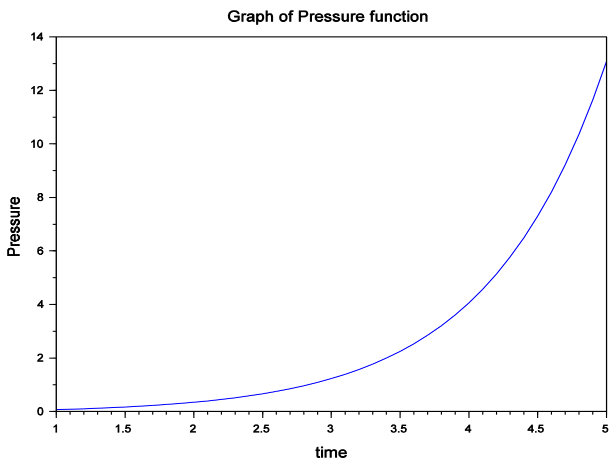

Fig. 2. Variation of pressure against time.

Equations (4.12) and (4.13) show that the pressure along the spatial directions p_x and p_y remains constant, indicating no directional change in pressure along the *x* and *y* axes. As illustrated in Fig. 2, p_z increases with time, representing anisotropic pressure along the *z*-axis.

Spatial volume:

$$V = \sqrt{-g}\,,$$

$$V = DEF\, e^{-\alpha x}\,, \tag{4.15}$$

using the values of *D, E,* and *F* from Eq. (4.8) yield as:

$$V = {c_1}^{(2r+1)}\, e^{((2r+1)\xi t - \alpha x)}. \tag{4.16}$$

As shown in Fig. 3, the spatial volume decreases over time, indicating that the Universe is contracting rather than expanding a scenario commonly referred to as the *Big Crunch*.

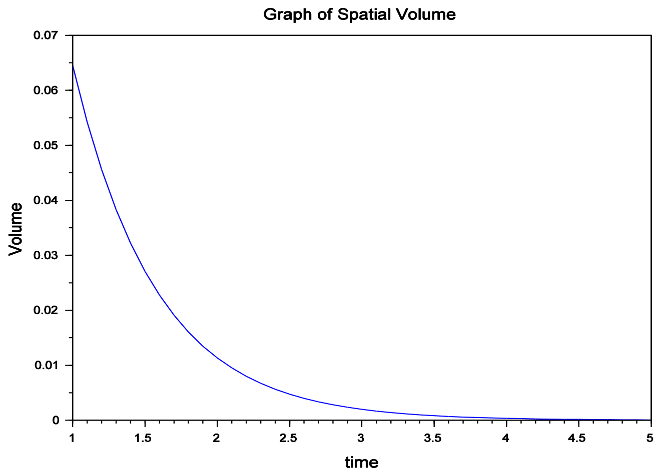

Fig. 3. Change in volume with time.

Average scale factor:

$$H = \frac{1}{3}\left[\frac{\dot{D}}{D} + \frac{\dot{E}}{E} + \frac{\dot{F}}{F}\right], \tag{4.17}$$

using the values of *D, E,* and *F* from Eq. (4.8) occurred as:

$$H = \frac{(2r+1)\xi}{3}. \tag{4.18}$$

Since ξ is constant in Equation (4.18), the average scale factor also remains constant, indicating that the Universe is neither expanding nor contracting.

Expansion tensor:

$$\theta = U^i_j + U^i \Gamma^i_{ki},$$

$$\theta = \left[\frac{\dot{D}}{D} + \frac{\dot{E}}{E} + \frac{\dot{F}}{F}\right] = 3H, \tag{4.19}$$

using the values of *H* from Eq. (4.15) found as:

$$\theta = (2r+1)\xi. \tag{4.20}$$

Since ξ is constant in Equation (4.20), the expansion tensor also remains constant, signifying a static Universe.

Anisotropic parameter:

$$A_m = \frac{1}{3}\sum_{i=1}^{3}\left(\frac{H_i - H}{H}\right)^2,$$

$$A_m = \frac{1}{27H^2}\left[\left(\frac{2\dot{D}}{D} - \frac{\dot{E}}{E} - \frac{\dot{F}}{F}\right)^2 + \left(\frac{2\dot{E}}{E} - \frac{\dot{D}}{D} - \frac{\dot{F}}{F}\right)^2 + \left(\frac{2\dot{F}}{F} - \frac{\dot{D}}{D} - \frac{\dot{E}}{E}\right)^2\right], \tag{4.21}$$

using the values of *D, E,* and *F* from Eq. (4.8) and *H* from Eq. (4.18) got as:

$$A_m = \frac{2(r-1)^2}{(2r+1)^2}. \tag{4.22}$$

Since r is constant in Equation (4.22), the anisotropic parameter also remains constant, indicating that the Universe is isotropic.

Shear scalar:

$$\sigma^2 = \frac{3}{2}H^2 A_m \,, \tag{4.23}$$

using the values of H from Eq. (4.18) and A_m from Eq. (4.22) obtained as:

$$\sigma^2 = \frac{1}{3}\,\xi^2 (r-1)^2. \tag{4.24}$$

Since r and ξ are constant in Equation (4.24), the shear scalar also remains constant, indicating that the Universe is isotropic.

Deceleration parameter:

Akarsu and Dereli [26] suggested a deceleration parameter in linear time with a negative slope.

$$q = -\frac{\ddot{R}R}{\dot{R}^2} = -1 + \frac{d}{dt}\left(\frac{1}{H}\right). \tag{4.25}$$

Using the values of H from Eq. (4.15) yielded as:

$$q = -1, \tag{4.26}$$

A negative value of the deceleration parameter (q) indicates that the expansion of the Universe is accelerating.

When, using the values of θ from Eq. (4.20) and the values of σ from Eq. (4.24) obtained as:

$$\lim_{t\to\infty} \frac{\sigma}{\theta} = \frac{1}{\sqrt{3}}\frac{(r-1)}{(2r+1)} \neq 0. \tag{4.27}$$

Equation (4.27) indicates that the model is anisotropic.

Case II: If $\dot{T} \neq 0$, and $f_{TT} = 0$, then

$$f_T = K_1, \text{ and } f(T) = K_1 T + K_2. \tag{4.28}$$

Consider the power solution as:

$$D = E = D_0\, t^l, F = F_0\, t^m \,, \tag{4.29}$$

here *l, m, n* and D_0, E_0, F_0 are constants.

Evaluate the values from Eq. (4.29) as:

$$\frac{\dot{D}}{D} = \frac{\dot{E}}{E} = \frac{l}{t},\ \frac{\dot{F}}{F} = \frac{m}{t}, \text{ and } \frac{\ddot{D}}{D} = \frac{\ddot{E}}{E} = \frac{l(l-1)}{t^2},\ \frac{\ddot{F}}{F} = \frac{m(m-1)}{t^2}.$$

Using these relations in torsion scalar Eq. (2.8) found as:

$$T = -2\left(\frac{l^2}{t^2} + \frac{2ml}{t^2}\right). \tag{4.30}$$

So, it is clear that the Torsion scalar is a function of cosmic time.

Rest energy density:

Using the values of *D, E,* and *F* from Eq. (4.29) and *f (T)* from Eq. (4.28) into the Eq. (3.7) occurred as:

$$\rho_M = \frac{1}{16\pi}\left[K_2 + K_1\left(\frac{2l^2}{t^2} + \frac{4lm}{t^2} - \frac{2\alpha^2 t^{-2l}}{{D_0}^2}\right)\right]. \tag{4.31}$$

As shown in Fig. 4, the energy density decreases over time, indicating that the Universe is expanding.

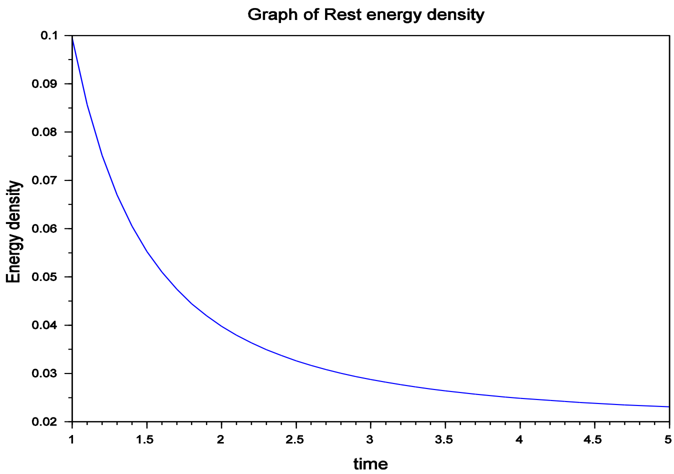

Fig. 4. Change in energy density with time.

Pressure:

Using the values of *D, E,* and *F* from Eq. (4.29) and *f (T)* from Eq. (4.28) in the Eq. (3.8), (3.9), and (3.10) obtained as:

$$p_x = \frac{-1}{16\pi} \left[K_2 + K_1\left(\frac{2l(l-1)}{t^2} + \frac{2m(m-1)}{t^2} + \frac{2lm}{t^2}\right)\right], \tag{4.32}$$

$$p_y = \frac{-1}{16\pi} \left[K_2 + K_1\left(\frac{2l(l-1)}{t^2} + \frac{2m(m-1)}{t^2} + \frac{2lm}{t^2}\right)\right], \tag{4.33}$$

$$p_z = \frac{-1}{16\pi} \left[K_2 + K_1\left(\frac{4l(l-1)}{t^2} + \frac{2l^2}{t^2} - \frac{\alpha^2 t^{-2l}}{{D_0}^2}\right)\right]. \tag{4.34}$$

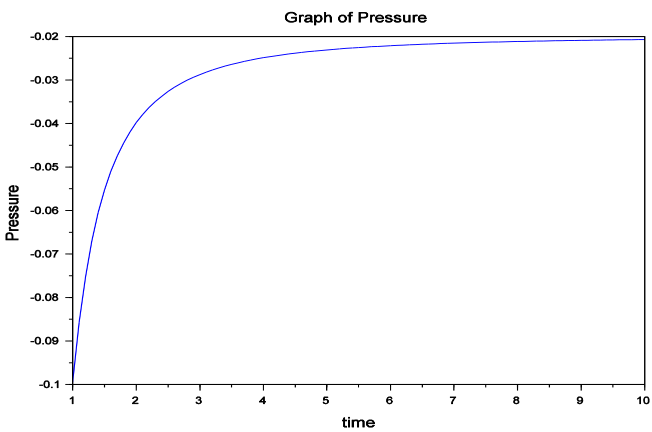

Fig. 5. Pressure along the spatial direction with time.

As illustrated in Fig. 5, the pressure along the spatial directions x, y and z decrease over time, indicating isotropic behaviour and supporting a dust-filled model of the Universe.

Spatial volume:

Using the values of *D, E,* and *F* from Eq. (4.29) into the Eq. (4.15) found as:

$$V = {D_0}^2 \ F_0 \, t^{2l+m} \, e^{-\alpha x} \, . \tag{4.35}$$

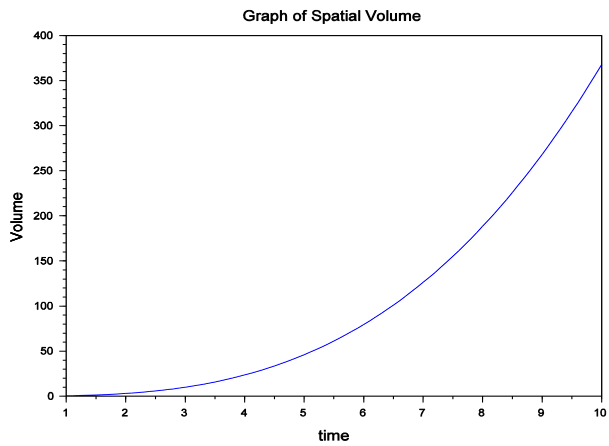

Fig. 6. Change in spatial volume with time

As shown in Fig. 6, the spatial volume increases over time, indicating that the Universe is undergoing continuous expansion.

Average scale factor:

Using the values of *D, E,* and *F* from Eq. (4.29) into the Eq. (4.14) occurred as:

$$H = \frac{1}{3}\left(\frac{(2l+m)}{t}\right). \tag{4.36}$$

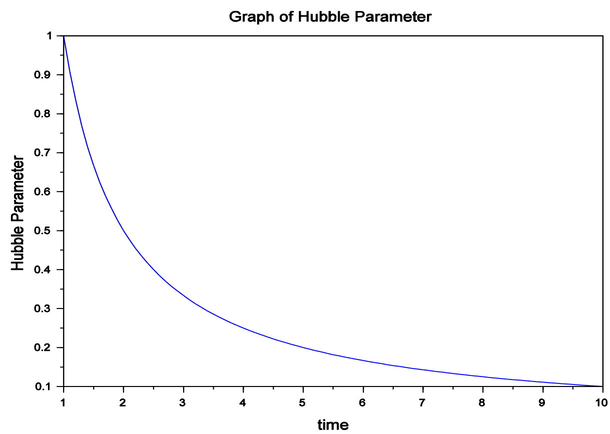

Fig. 7. Variation of Hubble parameter with time.

As shown in Fig. 7, the average scale factor decreases over time, indicating a static Universe.

Expansion tensor:
Using the values of H from Eq. (4.36) into the Eq. (4.19) found as:

$$\theta = \frac{(2l+m)}{t}. \quad (4.37)$$

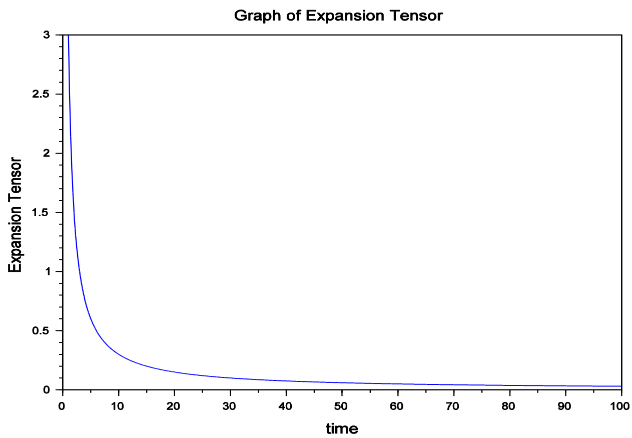

Fig. 8. Change in expansion tensor with time.

As shown in Fig. 8, the expansion tensor decreases over time, suggesting a transition towards a static Universe.

Anisotropic parameter:
Using the values of D, E, and F from Eq. (4.29) and H from Eq. (4.36) into the Eq. (4.21) occurred as:

$$A_m = \frac{2(l-m)^2}{(2l+m)^2}. \quad (4.38)$$

Since l and m are constant in Equation (4.38), the anisotropic parameter also remains constant, indicating that the Universe is isotropic.

Shear scalar:
Using the values of H from Eq. (4.36) and A_m from Eq. (4.38) into the Eq. (4.23) found as:

$$\sigma^2 = \frac{(l-m)^2}{3t^2}. \quad (4.39)$$

Since l and m are constant in Equation (4.39), the shear scalar tends to zero as time becomes large, indicating that the Universe approaches isotropy.

The deceleration parameter:
Using the values of H from Eq. (4.36) into the Eq. (4.25) yielded as:

$$q = -1 + \frac{3}{(2l+m)} = -ve. \quad (4.40)$$

A negative value of the deceleration parameter (q) indicates that the expansion of the Universe is accelerating.

When, using the values of θ from Eq. (4.37) and the values of σ from Eq. (4.39) occurred as:

$$\lim_{t\to\infty} \frac{\sigma}{\theta} = \frac{1}{\sqrt{3}} \frac{(l-m)}{(2l+m)} \neq 0. \quad (4.41)$$

Equation (4.41) indicates that the model is anisotropic.

5. Energy Conditions of the Universe

All normal and Newtonian matter has strong energy condition between the particles but in vacuum, it cannot obey. The linear baratropic equation state classified as:

$$p = \omega\, \rho\,,$$

where ρ is the matter energy density, p is the matter pressure, and ω is a constant. The strong energy condition involves $\omega \geq -\frac{1}{3}$, but as a false vacuum for the state as $\omega = -1$.

Case I:

Using the values of ρ_M from Eq. (4.11), p_x, p_y, and p_z from Eqs. (4.12), (4.13), and (4.14) obtained:

$$\omega_x = \frac{p_x}{\rho_M} = -\frac{-K+2n(2r^2\xi^2+3r\xi^2+\xi^2)}{-K+4n\left(r^2\xi^2+2r\xi^2-\frac{\alpha^2}{2{c_3}^{2r}}e^{-2r\xi t}\right)}, \quad (5.1)$$

$$\omega_y = \frac{p_y}{\rho_M} = -\frac{-K+2n(2r^2\xi^2+3r\xi^2+\xi^2)}{-K+4n\left(r^2\xi^2+2r\xi^2-\frac{\alpha^2}{2{c_3}^{2r}}e^{-2r\xi t}\right)}, \quad (5.2)$$

so $\omega_x = \omega_y$.

$$\omega_z = \frac{p_z}{\rho_M} = -\frac{-K+2n\left(4r^2\xi^2+2r\xi^2-\frac{\alpha^2}{{c_3}^{2r}}e^{-2r\xi t}\right)}{-K+4n\left(r^2\xi^2+2r\xi^2-\frac{\alpha^2}{2{c_3}^{2r}}e^{-2r\xi t}\right)}. \quad (5.3)$$

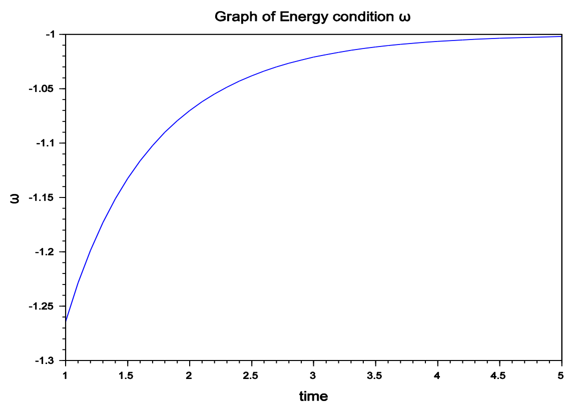

Fig. 9. Change in energy (ω) with time

As shown in Fig. 9, the energy parameters $\omega_x = \omega_y = \omega_z = -1$ at large time, indicating that the Universe is dominated by a cosmological constant or dark energy with isotropic negative pressure.

Case II:
Using the values of ρ_M from Eq. (4.31), p_x, p_y, and p_z from Eqs. (4.32), (4.33), and (4.34) obtained:

$$\omega_x = \frac{p_x}{\rho_M} = -\frac{K_2+K_1\left(\frac{2l(l-1)}{t^2}+\frac{2m(m-1)}{t^2}+\frac{2lm}{t^2}\right)}{K_2+K_1\left(\frac{2l^2}{t^2}+\frac{4lm}{t^2}-\frac{2\alpha^2 t^{-2l}}{{D_0}^2}\right)}, \tag{5.4}$$

$$\omega_y = \frac{p_y}{\rho_M} = -\frac{K_2+K_1\left(\frac{2l(l-1)}{t^2}+\frac{2m(m-1)}{t^2}+\frac{2lm}{t^2}\right)}{K_2+K_1\left(\frac{2l^2}{t^2}+\frac{4lm}{t^2}-\frac{2\alpha^2 t^{-2l}}{{D_0}^2}\right)}, \tag{5.5}$$

$$\omega_z = \frac{p_z}{\rho_M} = -\frac{K_2+K_1\left(\frac{4l(l-1)}{t^2}+\frac{2l^2}{t^2}-\frac{\alpha^2 t^{-2l}}{{D_0}^2}\right)}{K_2+K_1\left(\frac{2l^2}{t^2}+\frac{4lm}{t^2}-\frac{2\alpha^2 t^{-2l}}{{D_0}^2}\right)}. \tag{5.6}$$

From Equations (5.4), (5.5), and (5.6), it is observed that $\omega_x = \omega_y = \omega_z = -1$ as t becomes large, signifying that the Universe is dominated by a cosmological constant or dark energy with isotropic negative pressure.

6. Stability

The model stability is dependent upon the function $c_s^2 = \frac{dP}{d\rho}$. If the function c_s^2 is greater than zero then the model is stable otherwise the model is unstable.

Case I:

$$c_{sx}^2 = \frac{dp_x}{d\rho_M} = 0\,, c_{sy}^2 = \frac{dp_y}{d\rho_M} = 0, c_{sz}^2 = \frac{dp_z}{d\rho_M} = -1 < 0. \tag{6.1}$$

From Equation (6.1), it is evident that the model is unstable along the *z*-direction.

Case II:

$$c_{sx}^2 = \frac{dp_x}{d\rho_M} = -\frac{1}{2} < 0, c_{sy}^2 = \frac{dp_y}{d\rho_M} = -\frac{1}{2} < 0, c_{sz}^2 = \frac{dp_z}{d\rho_M} = -\frac{1}{2} < 0. \tag{6.2}$$

From Equation (6.2), it follows that the model is unstable.

7. Conclusion

The results of the model are summarized as follows: In Case I, the energy density decreases over time, indicating a higher density in the early Universe and a declining trend as the Universe expand. The pressure along the spatial directions p_x and p_y remains constant, indicating no directional change in these directions, whereas p_z increases with time, suggesting anisotropic pressure along the *z*-axis. The spatial volume decreases over time, implying that the Universe is contracting rather than expanding. Both the average scale factor and the expansion tensor remain constant, suggesting a static Universe. The anisotropic parameter and shear scalar are also constant, demonstrating that the Universe behaves isotropically. In Case II, the energy density again decreases with time, indicating that the Universe is expanding. The pressure along the spatial directions *x, y* and *z* are decreases with time, reflecting isotropic behaviour and a dust-filled Universe. The spatial volume increases over time, signifying continuous expansion. However, both the average scale factor and the expansion tensor decrease with time, which paradoxically implies a static Universe. The anisotropic parameter and shear scalar remain constant, again indicating isotropic behaviour. In both cases presented here, the deceleration parameter is negative, indicating that the Universe is undergoing accelerated expansion. Furthermore, the condition $\lim_{t\to\infty} \frac{\sigma}{\theta} \neq 0$ suggests that the model remains anisotropic at late times. In both cases studied, it is found that $\omega_x = \omega_y = \omega_z = -1$, indicating the Universe is dominated by a cosmological constant or dark energy with isotropic negative pressure. At large values of time, the Universe is accelerating and remains anisotropic. In both scenarios, the value of the sound speed squared c_s^2 is found to be negative, implying that the Universe is unstable. The stability of the Universe plays a significant role in *f (T)* theory, and this aspect represents a novel contribution of the present work. According to the cosmological interpretation, the proposed model is not stable.

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