JSR Publications

Available Online

J. Sci. Res. **17** (1), 227-234 (2025)

JOURNAL OF SCIENTIFIC RESEARCH

www.banglajol.info/index.php/JSR

Wavelet Based Galerkin Method for the Numerical Solution of Singular Boundary Value Problems using Fibonacci Wavelets

L. M. Angadi*

Department of Mathematics, Shri Siddeshwar Government First Grade College & P. G. Studies Centre, Nargund – 582207, India

Received 7 August 2024, accepted in final revised form 15 October 2024

Abstract

Singular boundary value problems (SBVPs) have become prevalent in scientific applications such as gas dynamics, chemical reactions, and structural mechanics. In review, the numerical approximation of solutions to differential equations serves as a crucial mechanism across various scientific and engineering fields, facilitating the assessment and analysis of complex systems that are not readily solvable through analytical methods. Due to this, the numerical methods are very crucial. As a result, numerical methods are of significant importance. So, the wavelet-based Galerkin method using Fibonacci wavelets for the numerical solution of SBVPs is introduced. The paper also provides illustrative examples to demonstrate the effectiveness and accuracy of the method.

Keywords: Singular boundary value problems; Galerkin method; Fibonacci wavelet.

doi: https://dx.doi.org/10.3329/jsr.v17i1.75341 J. Sci. Res. **17** (1), 227-234 (2025)

1. Introduction

Singular boundary value problems (SBVPs) play a crucial role in various scientific disciplines and are commonly encountered in the mathematical modeling of practical problems such as the theory of three-layer beam, elastic stability, nuclear physics, and more. This study focuses on examining SBVPs in the following form;

$$\left.\begin{array}{l}\dfrac{d^2y}{dx^2}+P(x)\dfrac{dy}{dx}+Q(x)y=f(x)\\ \text{With the boundary conditions}\qquad y(a)=\alpha, y(b)=\beta\end{array}\right\}$$

Where the functions are analytic in $x \in 0,1$and the functions $P(x)$and $Q(x)$are not analytic $x = 0$ (One of the boundary point)i.e. Singularity at $x = 0$.

Lately, a range of numerical methods has been utilized to solve differential equations. These methods encompass the Numerical method [1], Legendre wavelet method [2], Laguerre Wavelet-Galerkin method [3]. and others.

Wavelets have garnered considerable attention because of their robust mathematical properties and diverse applications in a variety of complex physical phenomena. Due to the

* *Corresponding author*: angadi.lm@gmail.com

properties of wavelets, viz., orthogonality, compact support, and ability to provide a precise representation of a variety of functions and operators at different levels of resolution, wavelet methods have attracted a lot of attention in the last three decades for the numerical solution of differential equations [4]. Recently, there has been a surge in interest in wavelet functions among scholars in both theoretical and practical domains. Some of them are the Fibonacci wavelet collocation method [5], Wavelet based lifting schemes [6], and the Fibonacci wavelet operational matrix approach [7].

Anticipate progress in numerical techniques using wavelet bases to attain high spatial and spectral resolutions. A key idea in approximation theory is to represent a smooth function as a series expansion using orthogonal polynomials. This approach serves as the foundation for spectral methods in solving differential equations with functional arguments. The exploration of wavelet function bases is being examined as an alternative to traditional polynomial trail functions in the analysis of differential equations using finite element methods. The Galerkin method is widely recognized in the field of applied mathematics for its convenience and practicality [8,9].

The wavelet-Galerkin method offers significant advantages over both the finite difference and finite element methods, making it a widely utilized approach in various scientific and engineering fields.In some cases, the wavelet technique presents a compelling alternative to the finite element method, offering an effective means of numerically solving differential equations, especially boundary value problems.

This research introduces the FWGM, which is based on Fibonacci wavelets, for numerically solving SBVPs. This method involves representing the solution using Fibonacci wavelets with unspecified coefficients, and using the characteristics of Fibonacci wavelets in combination with the Galerkin method to calculate these coefficients and obtain a numerical solution for the SBVPs.

The organization of the paper is as follows: Section 2 introduces Fibonacci wavelets and their application in function approximation. Section 3 is dedicated to the Galerkin method based on Fibonacci wavelets for addressing SBVPs. Section 4 presents the numerical experiments conducted. Finally, Section 5 offers a discussion on the calculations drawn from the proposed research.

2. Fibonacci Wavelets and Function Approximation

Fibonacci Polynomials: The standard description of Fibonacci polynomials [10,11] is outlined as follows:

$$\tilde{F}_m(x) = \begin{cases} 1, m = 0 \\ x, m = 1 \\ x\tilde{F}_{m-1}(x) + \tilde{F}_{m-2}(x), m > 1 \end{cases} \tag{2.1}$$

Furthermore, these polynomials can be represented in the form of powers as demonstrated:

$$\tilde{F}_m(x) = \sum_{i=0}^{\frac{m}{2}} \binom{m-i}{i} x^{m-2i}, m > 0 \tag{2.2}$$

Also, if $\tilde{F}_m(x), m = 0,1, \ldots . M - 1$ are Fibonacci polynomials, then

$$\int_0^1 \tilde{F}_m(x)\tilde{F}_n(x)dx = \sum_{i=0}^{\frac{m}{2}}\sum_{j=0}^{\frac{m}{2}}\binom{m-i}{i}\binom{m-i}{j}\left(\frac{1}{m+n-2i-2j+1}\right) \quad (2.3)$$

Fibonacci wavelets: Fibonacci wavelets [6,7] are defined in the following manner:

$$\psi_{n,m}(x) = \begin{cases} 2^{\frac{k-1}{2}}\dfrac{\hat{F}_m(2^{k-1}x-n)}{\sqrt{W_m}}, & \dfrac{n}{2^{k-1}} \le x < \dfrac{n+1}{2^{k-1}}, \\ 0, & \text{otherwise}, \end{cases} \quad (2.4)$$

In which $\hat{F}_m(x) = \frac{1}{\sqrt{W_m}}\tilde{F}_m(x)$ with $W_m(x) = \int_0^1 \{\tilde{F}_m(x)\}^2 dx$

where W_m, for $m = 0,1,2,...,M-1$ are obtained by equation (2.3), and m denotes the order of the Fibonacci polynomials and , $n = 1,2,...,2^{k-1}$, $k \in N$.

For instance, for $k = 1$ and $M = 3$, the Fibonacci wavelet bases as given below:

$\psi_{1,0}(x) = 1$,

$\psi_{1,1}(x) = \sqrt{3}x$,

$\psi_{1,2}(x) = \frac{1}{2}\sqrt{\frac{15}{7}}(1 + x^2)$ and so on.

Function approximation:

Let us consider $y(x) \in L^2(0, 1]$ that can be represented through Fibonacci wavelets in the subsequent way:

$$y(x) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\,\psi_{n,m}(x) \quad (2.5)$$

By truncating the infinite series referenced earlier, we have

$$y(x) = \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{n,m}\,\psi_{n,m}(x) \quad (2.6)$$

3. Method of Solution

Consider the SBVP in the following form,

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)\,y = f(x) \quad (3.1)$$

With boundary conditions $y(a) = \alpha, \; y(b) = \beta$ (3.2)

Where the functions $P(x), Q(x)$and $f(x)$are analytic in $x \in 0,1$and the functions $P(x)$ and $Q(x)$are not analytic $x = 0$ i.e. Singularity at $x = 0$.

Rewrite the Eq. (3.1) as

$$R(x) = \frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)\,y - f(x) \quad (3.3)$$

In cases where $R(x)$the residual of Eq. (3.1) equals zero, the exact solution is identified, and the boundary conditions are satisfied.

The trial series solution of Eq. (3.1), within the range of $(0\ ,\ 1]$ meets the specified boundary conditions and can be expanded to a modified Fibonacci wavelet by introducing unknown parameters in the process as follows:

$$y(x) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m}\, \psi_{n,m}(x) \tag{3.4}$$

The unknown coefficients $c_{n,m}$'s, which are to be determined,

The precision of the solution is improved by choosing higher-degree Fibonacci wavelet polynomials.

Now, differentating Eq. (3.4) w.r.t. x twice in order to obtain the values of $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}$ and substitute these values in Eq. (3.3). Solve the unknown parameters $c_{n,m}$'s by using weight functions as the assumed basis elements and integrating the boundary values along with the residual to achieve zero [12].

i.e. $\int_0^1 \psi_{1,m}(x)\, R(x)dx = 0,\ \ m = 0,1,2,\ldots\ldots..$

on solving the system of linear algebraic equations, which can be solved by the unknown parameters and then the unkwon parameters are obtained. Substitute these into the trial solution, referred to as Eq. (3.4), to calculate the numerical solution for Eq. (3.1).
In order to assess the accuracy of the FWGM in the test cases, the maximum absolute error is considered to calculate the error. The maximum absolute error is defined as:

$$E_{\max} = \max\left| y(x)_e - y(x)_n \right|,$$

where $y(x)_e$ and $y(x)_n$ are exact and numerical solution.

4. Numerical Experiment

Problem 4.1 First, consider the SBVP$\Big($In Eq. (3.1) $P(x) = \frac{1}{x}, Q(x) = 1,$ $and\ f(x) = x^2 - x^3 - 9x + 4)$i.e.

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + y = x^2 - x^3 - 9x + 4,\quad 0 \le x \le 1 \tag{4.1}$$

With boundary conditions: $y(0) = 0,\ \ y(1) = 0$ (4.2)

The Eq. (4.1) is implemented according to the procedure outlined in Section 3 in the following manner:
The residual of Eq. (4.1) can be written as:

$$R(x) = x\frac{d^2y}{dx^2} + \frac{dy}{dx} + xy - (x^3 - x^4 - 9x^2 + 4x) \tag{4.3}$$

Subsequently, the appropriate weight function $w(x) = x(1-x)$ must be chosen for Fibonacci wavelet bases to satisfy the prescribed boundary conditions in Eq. (4.2).

i.e. $\psi(x) = w(x) \times \psi(x)$

$$\psi_{1,0}(x) = \psi_{1,0}(x) \times x(1-x) = x(1-x)$$

$$\psi_{1,1}(x) = \psi_{1,1}(x) \times x(1-x) = \left(\sqrt{3}x\right)x(1-x)$$

$$\psi_{1,2}(x) = \psi_{1,2}(x) \times x(1-x) = \frac{1}{2}\sqrt{\frac{15}{7}}(1+x^2)x(1-x)$$

The trail solution of Eq. (4.1) for $k = 1$ and $m = 2$ is given by

$$y(x) = c_{1,0}\psi_{1,0}(x) + c_{1,1}\psi_{1,1}(x) + c_{1,2}\psi_{1,2}(x) \tag{4.4}$$

Now, Eq. (4.4) becomes

$$y(x) = c_{1,0}\{x(1-x)\} + c_{1,1}\{(\sqrt{3}x)x(1-x)\} + c_{1,2}\left\{\frac{1}{2}\sqrt{\frac{15}{7}}(1+x^2)x(1-x)\right\} \tag{4.5}$$

By differentiating Eq. (4.5) twice with respect to the specified variable and substituting the corresponding values into Eq. (4.3), we obtain the residual of Eq. (4.1). The "weight functions" utilized are identical to the basis functions.

Subsequently, employing the weighted Galerkin method, we examine the following:

$$\int_0^1 \psi_{1,j}(x)\, R(x)dx = 0,\ j = 0,1,2 \tag{4.6}$$

For $j = 0,1,2$ in Eq. (4.6),

$$\text{i.e.} \left.\begin{aligned} \int_0^1 \psi_{1,0}(x)\, R(x)dx = 0 \\ \int_0^1 \psi_{1,1}(x)\, R(x)dx = 0 \\ \int_0^1 \psi_{1,2}(x)\, R(x)dx = 0 \end{aligned}\right\} \tag{4.7}$$

From Eq. (4.7), a system of algebraic equations with unknown coefficients, namely $c_{1,0}$, $c_{1,1}$ and $c_{1,2}$ can be determined. By applying the Gauss elimination method, the values of $c_{1,0} = -0.0269$, $c_{1,1} = 0.5516$, and $c_{1,2} = 0.0526$ can be obtained. Substituting these values in Eq. (4.5) yields the numerical solution. The comparison of the numerical solution and the absolute errors is displayed in Table 1, while the numerical solution is depicted in Fig. 1 alongside the exact solution of Eq. (4.1) $y(x) = x^2 - x^3$ [3].

Table 1. Comparison of numerical solution and absolute error in relation to the exact solution for problem 4.1.

x	Numerical solution			Exact solution	Absolute error		
	FDM	Ref [3]	FWGM		FDM	Ref [3]	FWGM
0.1	-0.014709	0.010673	0.009677	0.009000	2.37e-02	1.67e-03	6.77e-04
0.2	-0.013726	0.033159	0.032675	0.032000	4.57e-02	1.16e-03	6.75e-04
0.3	-0.002584	0.063290	0.063354	0.063000	6.56e-02	2.90e-04	3.54e-04
0.4	0.015387	0.095881	0.095981	0.096000	8.06e-02	1.19e-04	1.90e-05
0.5	0.036564	0.125034	0.124731	0.125000	8.84e-02	3.40e-05	2.69e-04
0.6	0.056572	0.144429	0.143688	0.144000	8.74e-02	4.29e-04	3.12e-04
0.7	0.070066	0.147623	0.146841	0.147000	7.69e-02	6.23e-04	1.59e-04
0.8	0.070568	0.128350	0.128089	0.128000	5.74e-02	3.50e-04	8.90e-05
0.9	0.050294	0.080816	0.080862	0.081000	3.07e-02	1.84e-04	1.38e-04

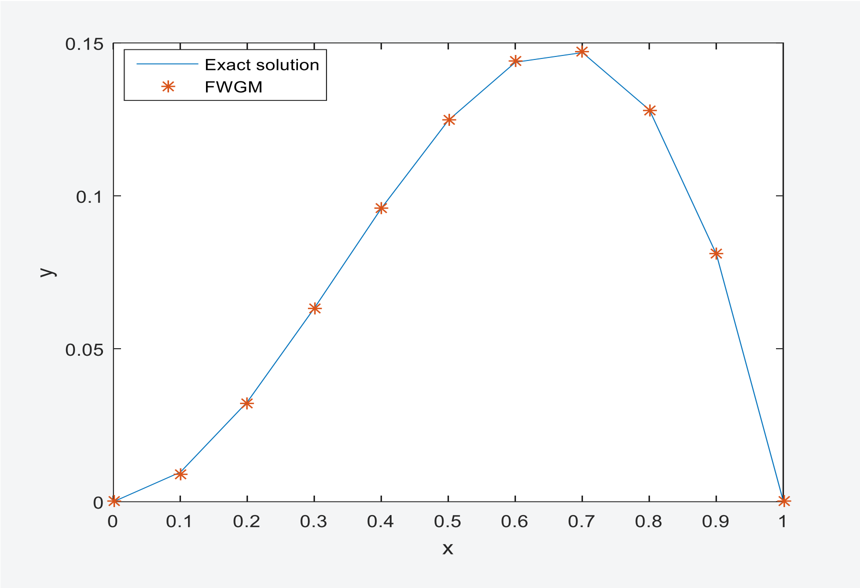

Fig. 1. Comparison of numerical solution with exact solution of the problem 4.1.

Problem 4.2 Next, consider another SBVP$\left(\text{In Eq. (3.1) } P(x) = \frac{8}{x}, Q(x) = x \text{ and } f(x) = x^5 - x^4 + 44x^2 - 30x\right)$i.e.

$$\frac{d^2y}{dx^2} + \frac{8}{x}\frac{dy}{dx} + xy = x^5 - x^4 + 44x^2 - 30x, \quad 0 \le x \le 1 \tag{4.8}$$

With boundary conditions: $y(0) = 0, \quad y(1) = 0$ (4.9)

In section and in the preceding problem, the values of $c_{1,0} = 1.0036$, $c_{1,1} = 0.0022$ and $c_{1,2} = -1.3722$ are derived. By substituting these values into Eq. (4.5), we arrive at the numerical solution. The comparison between the numerical solution and the absolute errors is outlined in Table 2, and the numerical solution with the exact solution of Eq. (4.8) is $y(x) = x^4 - x^3$ [3] depicted in Fig. 2.

Table 2. Comparison of numerical solution and absolute error in relation to the exact solution for problem 4.2.

x	Numerical solution			Exact solution	Absolute error		
	FDM	Ref [3]	FWGM		FDM	Ref [3]	FWGM
0.1	0.024647	-0.000823	-0.000937	-0.000900	2.55e-02	7.70e-05	3.70e-05
0.2	0.024538	-0.004844	-0.006426	-0.006400	3.09e-02	1.56e-03	2.60e-05
0.3	0.016024	-0.016861	-0.018899	-0.018900	3.40e-02	2.04e-03	1.00e-06
0.4	-0.000072	-0.037304	-0.038381	-0.038400	3.83e-02	1.10e-03	1.90e-05
0.5	-0.022021	-0.062986	-0.062482	-0.062500	4.05e-02	4.86e-04	1.80e-05
0.6	-0.045926	-0.087854	-0.086406	-0.086400	4.05e-02	1.45e-03	6.00e-06
0.7	-0.065532	-0.103744	-0.102944	-0.102900	3.74e-02	8.44e-04	4.40e-05
0.8	-0.072190	-0.101131	-0.102477	-0.102400	3.02e-02	1.27e-03	7.70e-05
0.9	-0.054840	-0.069880	-0.072976	-0.072900	1.81e-02	3.02e-03	7.60e-05

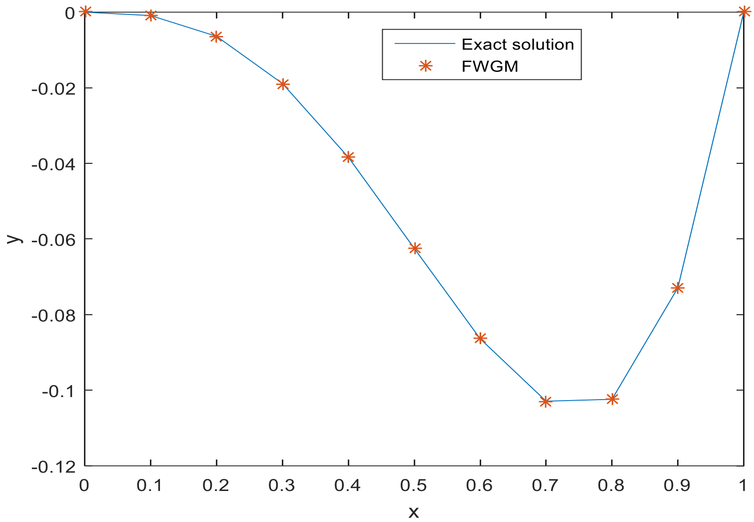

Fig. 2. Comparison of numerical solution with exact solution of the problem 4.2.

5. Conclusions

This paper explores the Fibonanci wavelets-based Galerkin method for the solution of SBVPs numerically. The advancement of new research in numerical analysis is significantly enhanced by this, proving advantageous for emerging researchers. The method introduced has been applied to some examples, yielding results that are notably satisfactory when compared to other established numerical methods.The data presented in the tables and figures above indicate that the presented method yields numerical solutions that surpass those obtained by finite difference method (FDM) and other existing methods, approaching the exact solution more closely. And, the margin of error resulting from this approach is notably reduced in comparison to FDM and the existing method. Therefore, the use of Fibonacci wavelets in the Galerkin method has proven to be highly effective in solving singular boundary value problems (SBVPs).

References

1. A. Hamad, M. Tadi, and M. Radenkovic, J. Appl. Math. and Phys. **2**, 882 (2014). https://10.4236/jamp.2014.29100
2. J. Iqbal, R. Abass, and P. Kumar, Ital. J. Pure Appl. Math. **40**, 311 (2018).
3. L. M. Angadi, Int. J. Modern Math. Sci. **19**, 34 (2021).
4. S. Kumbinarasaiah and R. Yeshwanth, J. Comput. Nonlin. Dyan. **19**, ID 091003 (2024). https://doi.org/10.1115/1.4065843.
5. G. Manohara, and S. Kumbinarasaiah, Appl. Num. Math. **201**, 347 (2024). https://doi.org/10.1016/j.apnum.2024.03.016
6. L. M. Angadi, J. Sci. Res. **16**, 31 (2024). https://doi.org/10.3329/jsr.v16i1.63085
7. G. Manohara and S. Kumbinarasaiah, Math. Comp. Simul. **221**, 358 (2024) https://doi.org/10.1016/j.matcom.2024.02.021

8. K. Amaratunga and J. R. William, Inter. J. Num. Meth. Eng. **37**, 2703 (1994). https://doi.org/10.1002/nme.1620371602
9. J. W. Mosevic, Math. Comp. **31**, 139 (1977).https://www.ams.org/journals/mcom/1977-31-137
10. S. C. Shiralashetti and Lata Lamani, Elect. J. Math. Anal. Appl. **9**, 88 (2021).
11. S. Kumbinarasaiah and M. Mulimani, J. Taib. Univ. Sci. **16**, 1112 (2022). https://doi.org/10.1080/16583655.2022.2143636
12. J. E. Cicelia, Ind. J. Sci. Tech. **7**, 52 (2014). https://doi.org/10.17485/ijst/2014/v7sp3.3