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FRW Cosmological Model with Bulk Viscosity and Phantom Dark Energy in $f(R)$ Gravity

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Abstract

In this paper, The Friedmann-Robertson-walker (FRW) cosmological model with bulk viscosity is investigated in the $f(R)$ theory of gravitation. For the Power and Exponential expansion, the field equations are solved. The functional form of the function $f(R)$ such as $f(R) = R + \alpha R^2$ is chosen for investigation. It is found that the bulk viscosity coefficient reacts similarly to the energy density; the model is viable only for $k = -1$, which represents an open universe. The Phantom field potential and scalar field are obtained.

Keywords: FRW; $f(R)$ gravity; Bulk Viscosity.

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1. Introduction

The most remarkable achievement of cosmology is the accelerated expansion of the universe confirmed through various cosmological observations. After a century, Perlmutter *et al*. [1] have shown that the expansion is accelerating. Their significant work awarded them the Nobel Prize. It is thought that dark energy (DE), a type of unexplained energy with negative pressure may be responsible for the universe's accelerating expansion. To investigate and evaluate the cosmic behavior of the DE, a number of theoretical models has been developed. The simplest DE candidate is the cosmological constant. Quintessence [2], Phantom [3], Chaplygin gas [4] are models from the Bamba *et al*. [5], studied crossing of the phantom divide in modified gravity. The DE's equation of state (EoS) has not yet been precisely calculated because it is unknown. According to the observational data, the DE's EoS parameter (ω) ranges from -1.46 to -0.78. Therefore, greater explanation is needed to comprehend the DE's nature. The Brans-Dicke theory of gravitation has solutions for dark energy, as shown by Katore *et al*. [6]. Recently Pawar *et al*. [7] have investigated accelerating expansion of dark matter and holographic DE in $f(T)$ gravity. Mete *et al*. [8] studied qualitative behavior of cosmological model with cosmic strings and minimally interacting dark energy.

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A key alternative theory that can be used to explain the universe's accelerated expansion is the modification of the Einstein-Hilbert action of general relativity. When the $f(R)$, an arbitrary function of scalar curvature R, is replaced in the E-H action, we get the $f(R)$. Phantom crossing, equation-of-state singularities and local gravity constraints in $f(R)$ models studied by Amendola and Tsujikawa [9]. Hatkar and Katore [10] studied dark energy scenario in metric $f(R)$ formalism. Nozari and Azizi [11] investigated phantom-like behavior in $f(R)$ gravity. Aktaş *et al.* [12] studied behaviors of dark energy and mesonic scalar field for anisotropic universe in $f(R)$ gravity. The FRW universe with two fluids in the $f(R)$ theory of gravitation has been examined by Katore and Hatkar [13]. Guarnizo *et al.* [14] have studied the geodesic deviation equation in the $f(R)$ gravity. Nzioki [15] has developed a new covariant formalism to treat spherically symmetric space–time in the metric $f(R)$ theory.

Viscosity is basically a measurement of the fluid's resistance to flow and is classified into two types namely bulk and shear viscosity. Usually, bulk viscosity is related with isotropic universe while shear viscosity corresponds to anisotropic universe. Bulk viscosity plays an important role because it supports the universe's inflationary phase. Inflationary pressure caused by the gravity of the universe's matter distribution is defeated by negative bulk viscous pressure. Potential causes of the viscosity in the universe include the decoupling of neutrinos during the radiation era, the production of superstring during the quantum era, the origin of galaxies, particle collisions involving gravitation and the particle creation process. Above 10^k, neutrinos viscosity reduces the anisotropy. Therefore, bulk viscosity can be used to analyze the kinetics of dissipation. In another word, the presence of viscosity in the fluid introduces many interesting pictures in dynamics of homogeneous cosmological models, which is used to study the evolution of universe. Cosmologist study the viscous model to find an alternative model of the universe. The idea of viscous DE models has been presented in different ways to understand evolution of the universe. Singh [16] has investigated Observation on the Role of Bulk Viscosity in Present Scenario of the Evolution in FRW Model Universe. Mete *et al.* [17] have studied Bianchi type IX magnetized bulk viscous string cosmological Model in general relativity. Bhoyar *et al.* [18] explore accelerating universe with viscous cosmic string in quadratic form of teleparallel Gravity. Anisotropic LRS Bianchi type-V Cosmological models with bulk viscous string within the framework of saez-ballester theory in five-dimensional space-time shown by Daimary & Baruah [19]. Bianchi type-I cosmological model with perfect fluid in modified $f(T)$ gravity studied by Mete *et al.* [20]. Five-dimensional Bianchi type- III metrics in the framework of Lyra geometry with matter source as a bulk viscous fluid with one-dimensional cosmic string has been studied by Mete *et al.* [21]. Pawar and Dabre [22] investigated the Bianchi type VI0 space-time in the presence of string of clouds coupled with perfect fluid within the context of $f(R,T)$ gravity. Thick domain wall coupled with bulk viscous fluid in $(n+2)$ dimensional flat FRW universe was initiated by Mete *et al.* [23]. Regular black holes universes without singularities and phantom scalar field transitions are discussed by Chataignier and Kamenshchik [24]. Sakti and Sulaksono [25] studied Dark energy stars with a phantom field. Gaikwad and Mule [26] have investigated Bianchi type-V Dark Energy Cosmological Model in $f(R,T)$ Theory of Gravitation.

The above discussion and the investigation have motivated us to examine FRW space-time with bulk viscosity within the framework of $f(R)$ gravity. As k in the FRW space-time represents a closed ($k = 1$), flat ($k = 0$) and open ($k = -1$) universe. In the present work, results are obtained in both the cases for $k = -1$, which shows open universe. The paper is organized as follows: Preliminary definitions of $f(R)$ gravity are discussed in Section 2. Sections 3 and 4 deals with the derivation and solutions of the field equations. The conclusions are given in Section 5.

2. Field Equations for FRW Model

The current universe is isotropic and homogeneous, yet it is not static. There are only three prospective space-time metrics for an isotropic and homogeneous universe that are consistent with cosmological theories, as defined by Friedman-Robertson-Walker (FRW). The FRW line element represented by the following metric:

$$ds^2 = dt^2 - a^2(t)\left\{\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right\}, \tag{1}$$

where the $0 \le \theta \le \pi$ and $0 \le \varphi \le \pi$ are the azimuthal and polar angles of the spherical co-ordinate system. The curvature of the space is represented by k. Positive values correspond to real finite radius, while negative values suggest infinite radius and zero values indicate imaginary radius. The bulk viscous dark fluid is represented by the energy momentum tensor of the cosmic fluid, which is written [27] as

$$T_{\mu\nu} = \left(\rho + \overline{P}\right)u_\mu u_\nu - \overline{P} g_{\mu\nu} \ . \tag{2}$$

Together with $u_\mu u^\mu = 1$ and $\overline{P} = P - \eta u^\mu_{;\mu}$ is the effective pressure where η is the coefficient of bulk viscosity, ρ is the energy density. In co-moving coordinate from Eq. (2), we have

$$T_1^1 = T_2^2 = T_3^3 = -\overline{P},\ \ T_4^4 = \rho \ . \tag{3}$$

The most common easy method for explaining the relationship between energy density and pressure has been utilized. Here, we are interested in examining how bulk viscosity affects dark energy potential. The scalar field and potential are obtained by considering the relation $P = \rho$ in the EoS of Chaplygin gas, follow Chaubey [28]. So, implemented the technique with another dark energy field (Phantom field).

Now consider the action of the $f(R)$ theory of gravitation

$$S = \frac{1}{l^2}\int d^4x\sqrt{-g}\, f(R) + S_m\left(g_{\mu\nu}, \psi\right) , \tag{4}$$

where $l^2 = 8\pi G$, ψ refers collectively denotes the matter fields. The field equations of theory of gravitation are

$$F(R)R_{\mu\nu} - \frac{1}{2} f(R) g_{\mu\nu} - \nabla_\mu \nabla_\nu F(R) + g_{\mu\nu}\nabla^\mu \nabla_\nu F(R) = l^2 T_{\mu\nu} , \tag{5}$$

where, $\nabla^\mu \nabla_\nu F = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g} g^{\mu\nu}\partial_\nu\right)$, $F$is the derivative of f with respect to the real argument. It is generally known that when the Lagrangian is linear in R, *i.e.* when $f(R) = R$, the General Relativity can be obtained from the $f(R)$. As a result of being of fourth order, the field equation of the $f(R)$ is more complex than that of general relativity. A higher-order curvature effect is referred to as extra gravitational stress in the third and fourth terms in the left side of Eq. (5). The field Eq. (3) has the following form for the line element (1) using (5)

$$3F\left(\frac{\ddot{a}}{a}\right)+3\left(\frac{\dot{a}}{a}\right)\dot{F}-\frac{1}{2}f=l^2\rho \quad , \tag{6}$$

$$F\left(-\frac{\ddot{a}}{a}-2\frac{\dot{a}^2}{a^2}-2\frac{k}{a^2}\right)-3\left(\frac{\dot{a}}{a}\right)\ \dot{F}+\frac{1}{2}f-\ddot{F}=l^2\overline{P}\ , \tag{7}$$

where, the differentiation with respect to t denoted by overdot. There are four unknowns $(a, f, \overline{P},\ \rho)$ and two equations. So, there need more conditions. As a first condition, we proceed by assuming the functional relation of the $f(R)$.The accelerated expansion of the universe can be achieved with the following function that was proposed by Starobinsky [29]. Temperature anisotropies in the CMB are acceptable with this possible inflationary idea. It is given by $f(R)=R+\alpha R^2$ where $\alpha>0$and in the second condition; consider the value of the scale factor.

3. Case I Power Law Model

In this section, the power law model of the following form is used to solve the field equations. The scale factor increases as time passes on. The universe is accelerating and expanding.

$$a=c_1t^n\ , \tag{8}$$

where $c_1$and n are positive constant.

Using Eqns. (1) and (8), we get

$$ds^2=dt^2-c_1^2t^{2n}\left\{\frac{dr^2}{1-kr^2}+r^2\left(d\theta^2+\sin^2\theta\, d\phi^2\right)\right\}. \tag{9}$$

In the review of the $f(R)$ theory, Katore and Hatkar [6] have discussed some interesting features of this model and derived solution for the $f(R)$ gravity. Using Eqns. (6) – (8), we have expressions for energy density and pressure

$$\rho=\frac{1}{l^2}\left[\frac{k_1k}{t^{2n+2}}-\frac{k_2}{t^4}-\frac{k_3}{t^2}-\frac{k_4k}{t^{2n}}-\frac{k_5k^2}{t^{4n}}\right], \tag{10}$$

$$\overline{P}=\frac{1}{l^2}\left[\frac{k_6}{t^2}-\frac{k_7}{t^{n+2}}+\frac{k_8k}{t^{2n}}+\frac{k_9}{t^4}-\frac{k_{10}}{t^{n+4}}+\frac{k_{11}k}{t^{2n+2}}-\frac{k_{12}}{t^{3n+2}}k-\frac{k_{13}k^2}{t^{4n}}\right], \tag{11}$$

where,

$$k_1=\frac{24\alpha n-132\alpha n^2}{c_1^2}\ ,\ k_2=36\alpha n^3+54\alpha n^2+72\alpha n^3\ ,\ k_3=3n^2\ ,\ k_4=\frac{3}{c_1}\ ,\ k_5=\frac{18\alpha}{c_1^4}\ ,$$

$$k_6=5n^2-2n\ ,\ k_7=2(n^2-n)\ ,\ k_8=\frac{1}{c_1^2}\ ,\ k_9=72n\alpha-114n^2\alpha+108n^3\alpha-24n^4\alpha\ ,$$

$$k_{10}=\frac{4\alpha(n^2-n)(12n^2-6n)}{c_1}\ ,\ k_{11}=\frac{138\alpha n^2+(24n^2\alpha-24n\alpha)c_1}{c_1^2}\ ,\ k_{12}=\frac{24\alpha(n^2-n)}{c_1^3}\ ,$$

$$k_{13}=\frac{6\alpha}{c_1^4}.$$

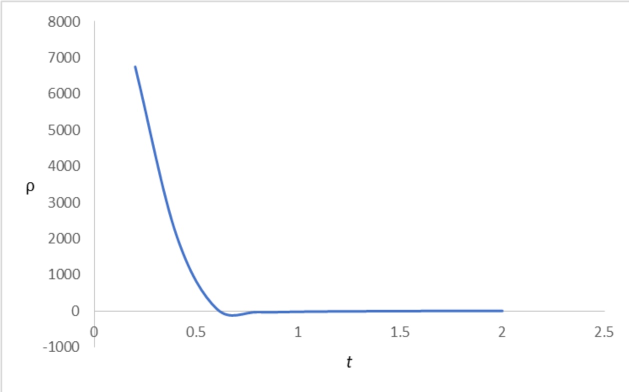

Fig. 1. Energy density v/s time.

For $P = \rho$ in the relation $\overline{P} = P - \eta u^{\mu}_{;\mu}$ and using Eqns. (10) – (11), we obtain

$$\eta = \frac{1}{3l^2}\left\{\begin{matrix} \frac{96k-588nk+2(n-1)t^n}{t^{2n+1}} + \frac{48n^3-432n^2+180n+144}{t^3} - \frac{2(4n^2-1)}{t} \\ -\frac{4tk}{nt^{2n}} - \frac{24k^2t}{nt^{4n}} + \frac{48n(1+2n^2-3n)}{t^{n+3}} - \frac{48k(n-1)}{t^{3n+1}} \end{matrix}\right\}. \tag{12}$$

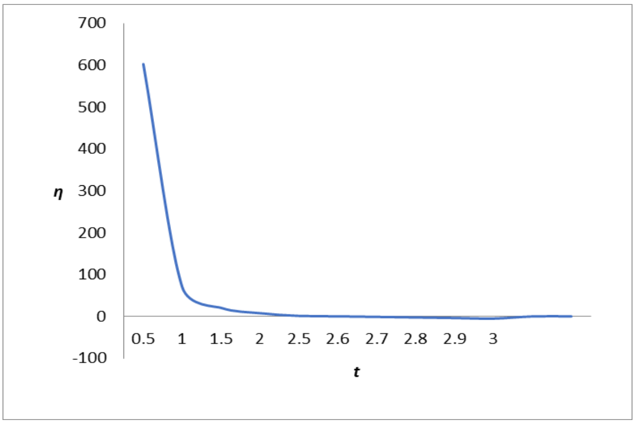

Fig. 2. Coefficient of bulk viscosity v/s time.

There are solutions of the field equations for $c_1 = 1$ and $\alpha = 2$. When $n = 2$ the energy density is negative for $k = 0$ and $k = 1$ and it is positive only for $k = -1$ (Fig. 1). The coefficient of bulk viscosity is initially positive but quickly tends to be negative as the duration of time passes for $n = 2, k = -1$ (Fig. 2). As k in the FRW space-time represents a closed ($k = 1$), flat ($k = 0$) and open ($k = -1$) universe. Negative energy density values are not of relevance to us. So, for $n = 2$, the model is an open universe.

Phantom Field: The energy density and pressure of the Phantom field ϕ are respectively given by

$$\rho_\varphi = -\frac{1}{2}\dot{\varphi}^2 + V(\varphi)\ , \quad (13)$$

$$P_\varphi = -\frac{1}{2}\dot{\varphi}^2 - V(\varphi)\ , \quad (14)$$

where $V(\varphi)$ is the Phantom field potential. To determine the scalar field and phantom field potential values, following Chaubey [28]. We consider the equation of state as $\omega\rho_\varphi = P_\varphi$ in which we put$\rho_\varphi = \rho$. So that from Eqns. (10), (13) – (14) and these relationships allow it to be easier to get the following expressions

$$V(\varphi) = -\frac{1}{2}(P_\varphi - \rho_\varphi)$$

$$V(\varphi) = -\frac{1}{2}(\omega\rho_\varphi - \rho_\varphi)$$

$$V(\varphi) = -\frac{1}{2}(\omega - 1)\rho \quad (15)$$

$$V(\varphi) = -\frac{1}{2}(\omega - 1)\frac{1}{l^2}\left[\frac{k_1 k}{t^{2n+2}} - \frac{k_2}{t^4} - \frac{k_3}{t^2} - \frac{k_4 k}{t^{2n}} - \frac{k_5 k^2}{t^{4n}}\right] . \quad (16)$$

From Eqns. (13) and (14)

$$\dot{\varphi}^2 = -(P_\varphi + \rho_\varphi)$$

$$\dot{\varphi}^2 = -(\omega\rho_\varphi + \rho_\varphi)$$

$$\dot{\varphi}^2 = -(1 + \omega)\rho$$

$$\dot{\varphi}^2 = -\frac{(1 + \omega)}{l^2}\left[\frac{k_1 k}{t^{2n+2}} - \frac{k_2}{t^4} - \frac{k_3}{t^2} - \frac{k_4 k}{t^{2n}} - \frac{k_5 k^2}{t^{4n}}\right]$$

$$\phi = \frac{\sqrt{-(1+\omega)}}{l}\int\left[\frac{k_1 k}{t^{2n+2}} - \frac{k_2}{t^4} - \frac{k_3}{t^2} - \frac{k_4 k}{t^{2n}} - \frac{k_5 k^2}{t^{4n}}\right]^{\frac{1}{2}} dt \quad (17)$$

The changing potential of the Phantom field across time is shown in Fig. 3. It is decreasing in time. When $k = -1$ i.e. for the open universe, $V(\varphi)$ is extremely large at $t = 0$ which show that there was inflation during the universe's initial stages. Katore and Hatkar [10] have obtained the variation of phantom field potential with time.

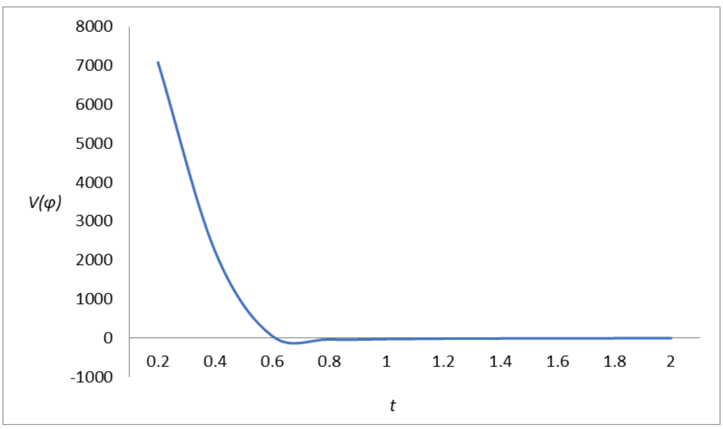

Fig. 3. Phantom field potential v/s time.

4. Case II Exponential Expansion Law Model

In this section, the exponential expansion law of the scale factor is used to solve the field equations.

$$a = c_2 e^{nt} \ . \tag{18}$$

Using Eqns. (1) and (18), we get

$$ds^2 = dt^2 - c_2^2 e^{2nt}\left\{\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right\} . \tag{19}$$

In this model, the universe is expanding and accelerating. The rate or expansion is constant. Singularities do not exist. We consider the functional form, as discussed in case I,

$f(R) = R + \alpha R^2$

$$\rho = \frac{1}{l^2}\left[-3n^2 - \frac{(3+108\alpha n^2)k}{c_2^2 e^{2nt}} - \frac{6\alpha k^2}{c_2^4 e^{4nt}}\right] , \tag{20}$$

$$\overline{P} = \frac{1}{l^2}\left[k_1 - \frac{k_2}{e^{nt}} + \frac{k_3 k}{e^{2nt}} - \frac{k_4 k}{e^{3nt}} - \frac{k_5 k^2}{e^{4nt}}\right] , \tag{21}$$

where,

$$k_1 = (5n^2 + 48\alpha n^4) \ , \ k_2 = \frac{2n^2 + 48\alpha n^4}{c_2} \ , \ k_3 = \frac{(1 + 36\alpha n^2)}{c_2^2} \ , \ k_4 = \frac{24\alpha n^2}{c_2^3} \ , \ k_5 = \frac{18\alpha}{c_2^4}.$$

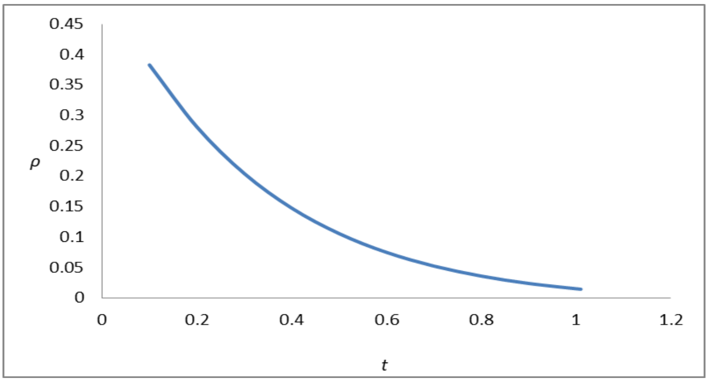

Fig. 4. Energy density v/s time.

For $P = \rho$ in the relation $\overline{P} = P - \eta u^{\mu}_{;\mu}$ and using Eqns. (20) and (21), we get coefficient of bulk viscosity η.

$$\eta = \frac{1}{3l^2}\left\{-8n\ (1+12n^2) - \frac{4(1+72n^2)k}{ne^{2nt}} + \frac{24k^2}{ne^{4nt}} + \frac{2n\ (1+48n^2)}{e^{nt}} + \frac{48kn}{e^{3nt}}\right\} \tag{22}$$

For $k = 0$, density is constant and negative. For $k = -1$, it is positive and decreasing function of time. For $k = 1$, it is negative. Thus, only for $k = -1$ the model is viable, which shows an open universe (Fig. 4). The bulk viscosity coefficient behaves similarly to the energy density (Fig. 5).

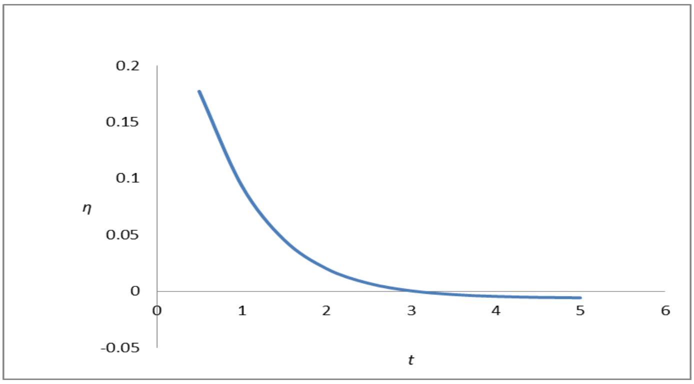

Fig. 5. Coefficient of bulk viscosity v/s time.

Phantom field: $V(\varphi)$ is the potential of the Phantom field. For determining the value of the scalar field and phantom field potential, follow Chaubey [28]. For that we anticipate equation of state as $\omega\rho_\varphi = P_\varphi$ in which we put $\rho_\varphi = \rho$. So that from Eqns. (20), (13) – (14), and We can easily obtain the following expressions using these relations

$$V(\varphi) = -\frac{1}{2}(\omega - 1)\frac{1}{l^2}\left[-3n^2 - \frac{(3+108\alpha n^2)k}{c_2^2 e^{2nt}} - \frac{6\alpha k^2}{c_2^4 e^{4nt}}\right] . \tag{23}$$

$$\varphi = \frac{\sqrt{-(1+\omega)}}{l}\int\left[-3n^2 - \frac{(3+108\alpha n^2)k}{c_2^2 e^{2nt}} - \frac{6\alpha k^2}{c_2^4 e^{4nt}}\right]^{\frac{1}{2}} dt . \tag{24}$$

For $k = -1$, $V(\varphi)$ is positive decreasing with increasing of time (Fig.6). The model is acceptable only for $k = -1$, therefore for this open universe, the $V(\varphi)$ dominates for $t = 0$ i.e. there is inflation near $t = 0$. Katore and Hatkar [10] have obtained the variation of phantom field potential with time.

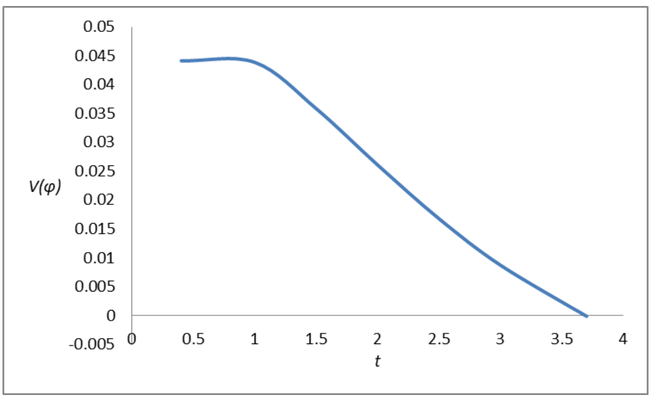

Fig. 6. Phantom field potential v/s time.

6. Conclusion

In this paper, we have studied Friedmann Robertson Walker space-time with bulk viscosity in the framework of the $f(R)$ theory of gravitation. Considering functional form $f(R) = R + \alpha R^2$ to solve field equations, the phantom field potential and scalar fields are obtained. In the case I of the power law model, when $n = 2$, the energy density is negative for $k = 0$ and $k = 1$, but positive only for $k = -1$. We found that the universe is open, and the nature of potential is decreasing. In case II of the exponential expansion model, if $k = 0$ density is constant and negative, for $k = -1$, it is a positive and decreasing function of time; for $k = 1$, it is negative. Thus, the model is viable only for $k = -1$, which represents an open universe; here also, the nature of potential is decreasing. The bulk viscosity coefficient reacts similarly to the energy density in both cases.

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