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A Discussion on the Solution(s) of the Diophantine Equation $3^x + 15^y = z^2$

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Abstract

The absence of a generalized method for solving Diophantine equations having more unknowns than a number of equations is a challenge for researchers in different fields. The presence of the Diophantine equation is reported in the study of the Hydrogen spectrum, quantum Hall effect, chemistry, cryptography, etc. Some special types of Diophantine equations could be addressed with the help of Catalan's conjecture and Congruence theory. The Diophantine equation $3^x + 15^y = z^2$ is addressed in this paper to find the solution(s) in positive integers. It is found that the equation has only two solutions of (x, y, z) as $(1, 0, 2)$ and $(0, 1, 4)$ in non-negative integers.

Keywords: Modular arithmetic; Catalan's conjecture; Exponential diophantine equation; Congruence.

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1. Introduction

Diophantine equations are studied in the branch of number theory where the solutions of polynomial equations having two or more variables are to be found in terms of integers. In addition, they have a smaller number of equations than a number of unknown variables. Diophantine equations are found in the study of the Hydrogen spectrum [1], and quantum Hall Effect [2]. The applications of Diophantine equations are also associated with different fields of real life [3-8].

The solutions of the nonlinear Diophantine equation $x^m + y^m = z^m$ with $m = 2$ are termed as Pythagorean triple. It has no positive integer solution for $m > 2$. This result is known as Fermat's last theorem [9]. Recently, the Diophantine equations of the form $a^x + b^y = c^z$ are studied at length, where a and b are fixed integers. Keskin and Siar found positive integer solutions of some Diophantine equations in terms of integer sequences [10]. All the positive integer solutions of the Diophantine equation $z^2 = k(k^2 + 3)$ were provided analytically by Islam and Majumder [11]. They also proved that the Diophantine equation $z^2 = k(k^2 + 12)$ has no solution. The numerical solution of the Diophantine equation has been discussed using the concept of artificial neural network [12]. Laipaporn *et al.* showed that the equation $3^x + p5^y = z^2$ has no solution when p is congruent to *5* or

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7 modulo *24* [13]. All the solutions of the Diophantine equation $F_1^P + 2F_2^P + \cdots . + kF_k^p = F_n^q$, an equation on the weighted power terms of the Fibonacci sequence, was obtained by Gueth *et al*. [14]. Asthana and Singh found three positive integer solutions for a special case of the same equation with $p = 113$ [15]. Burshtein studied two Diophantine equations $5^x + 103^y = z^2$ and $5^x + 11^y = z^2$, and established that the equation $5^x + 103^y = z^2$ has no solution whereas the equation $5^x + 11^y = z^2$ has some conditional solutions [16]. Burshtein also found the conditional solution of the Diophantine equation $p^x + (p+4)^y = z^2$ as $(p, x, y, z) = (3, 2, 1, 4)$ [17]. Bakar *et al*. showed that the Diophantine equation $5^x + p^m n^y = z^2$ has the solution for $p > 5$, a prime number and $y = 1, 2$ [18]. All the solutions of the Diophantine equation $\mathrm{P}_\mathrm{n}^\mathrm{x} + \mathrm{P}_{\mathrm{n}+1}^\mathrm{x} + \cdots . + \mathrm{P}_{\mathrm{n}+\mathrm{k}-1}^\mathrm{x} = \mathrm{P}_\mathrm{m}$, where P_i is the i^{th} term for Pell's sequence, were derived by Lucca *et al*. [19].

For several years, researchers have been studying the Diophantine equation of the form $3^x + p^y = z^2$. Two positive integer solutions for (*x*, *y*, *z*) of two Diophantine equations $3^x + 91^y = z^2$ and $3^x + 19^y = z^2$ were found by Rabago [20]. The solution of the Diophantine equation $3^x + 35^y = z^2$ was found to be (1, 0, 2) and (0, 1, 6) as non-negative integers [21]. Asthana and Singh derived four non-negative integer solutions of the Diophantine equation $3^x + 13^y = z^2$ [22]. It was established that a unique solution (1, 0, 2) exists for the Diophantine equations $3^x + 5^y = z^2$ and $3^x + 17^y = z^2$ [23, 24]. Exactly four solutions (1, 0, 2), (3, 1, 12), (7, 1, 48), and (7, 2, 126) were found by Asthana and Singh for the Diophantine equation $3^x + 117^y = z^2$ [25]. Sroysang found that the Diophantine equation $3^x + 85^y = z^2$ has a unique solution (1, 0, 2) for positive integers (*x*, *y*, *z*) [26].

One such Diophantine equation is the subject of this article. The purpose of this article is to discuss whether (or not) a solution to the non-linear Diophantine equation $3^x + 15^y = z^2$, where *x*, *y*, and *z* are non-negative integers, exists. The paper is organized as follows: In sec. 2, Catalan's conjecture and two lemmas would be considered. The main theorem along with its proof is discussed in sec. 3. Conclusion is given in sec. 4.

2. Preliminaries

In this section, Catalan's conjecture [27,28] is used to prove the Lemmas 2.2 and 2.3.

2.1. *Proposition*

$(3, 2, 2, 3)$ is a unique solution of (p, q, x, y) for the Diophantine equation $p^x - q^y = 1$, where *p*, *q*, *x*, and *y* are integers with $min\,\{p, q, x, y\} > 1$ [27, 28].

2.2. *Lemma*

The Diophantine equation $1 + 15^y = z^2$, where *y* and *z* are non-negative integers, has a unique solution of (y, z) as $(1, 4)$.

Proof: Let *y* and z be non-negative integers such that $1 + 15^y = z^2$. For $y = 0$,$15^0 + 1 = 2 = z^2$, that is impossible. It follows that $y \geq 1$. Thus, $z^2 = 15^y + 1 \geq 15^1 + 1 =$

16. Hence, $z \geq 4$. Again, the equation $15^y + 1 = z^2$ can be written as $z^2 - 15^y = 1$. By proposition 2.1, $z = 4$ for $y = 1$. Therefore, a unique solution of the Diophantine equation $1 + 15^y = z^2$ for (y, z) is $(1, 4)$.

2.3. *Lemma*

The Diophantine equation $3^x + 1 = z^2$, where *x* and *z* are non-negative integers, has a unique solution of (x, z) as $(1, 2)$**.**

Proof: Let *x* and z be non-negative integers such that $3^x + 1 = z^2$. For $x = 0$, $3^0 + 1 = 2 = z^2$, that is impossible. It shows that $x \geq 1$. Thus, $z^2 = 3^x + 1 \geq 3^1 + 1 = 4$. Hence, $z \geq 2$. Again, the equation $3^x + 1 = z^2$ can be expressed as $z^2 - 3^x = 1$. By proposition 2.1, $z = 2$ for $x = 1$. Therefore, a unique solution of the Diophantine equation $3^x + 1 = z^2$ for (x, z) is $(1, 2)$.

3. Result and Discussion

This section proves that the Diophantine equation $3^x + 15^y = z^2$ has two unique non-negative integer solutions.

Theorem 3.1: The non-linear Diophantine equation $3^x + 15^y = z^2$ has only two non-negative integer solutions of (x, y, z) as $(1, 0, 2)$ and $(0, 1, 4)$, where *x*, *y*, *z* are non-negative integers.

***Proof*:** Here three cases will be considered.

Case – I: For $x = 0$, it can be concluded from Lemma 2.2 that the solution of the Diophantine equation $3^x + 15^y = z^2$ for (x, y, z) is $(0, 1, 4)$.

Case – II: For $y = 0$, it can be concluded from Lemma 2.3 that the solution of the Diophantine equation $3^x + 15^y = z^2$ for (x, y, z) is $(1, 0, 2)$.

Case – III: If $x, y \geq 1$, then 3^x and 15^y both are odd. Thus z^2 is even. So, *z* is even. Now it can be shown that $3^x \equiv 3\ (mod\ 4)$ for the odd values of *x*, and $3^x \equiv 1\ (mod\ 4)$ for the even values of *x*. Similarly, $15^y \equiv 3\ (mod\ 4)$ for the odd values of *y*, and $15^y \equiv 1\ (mod\ 4)$ for the even values of *y*. Thus, $3^x + 15^y \equiv 2\ (mod\ 4)$ for even and odd values of, y ; and $Z^2 \not\equiv 2$. Therefore, it can be concluded that the Diophantine equation has no solution for $x, y \geq 1$.

4. Conclusion

There is no general process to find out all the solutions (if exist) of different types of Diophantine equations. Based on forms, different methods are suitable to search the solutions of different types of Diophantine equations. In this communication, Catalan's conjecture and Congruence theory are employed to find solutions of the Diophantine equation $3^x + 15^y = z^2$. It is found that the Diophantine equation has solutions for non-negative integers *x*, *y*, and *z* as $(1, 0, 2)$ and $(0, 1, 4)$. The result may be useful for researchers of diversified fields. Additionally, it might also motivate mathematicians to focus on solving more $3^x + p^y = z^2$ types of Diophantine equations.

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