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Solution of Non-homogeneous Linear Fractional Differential Equations Involving Conformable Fractional Derivatives

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Abstract

This paper presents a method for solving non-homogeneous linear sequential fractional differential equations (NHLSFDEs) with constant coefficients involving conformable fractional derivatives. For this purpose, the fundamental properties of the conformable derivative and fractional exponential functions are discussed. After this, we determined the particular integrals (PIs) of NHLSFDE in terms of fractional exponential functions, fractional cosine and sine functions. We have demonstrated this developed method with a few examples of NHLSFDEs.

Keywords: Fractional exponential function; Riemann–Liouville derivative; Caputo fractional derivative; Conformable fractional derivative.

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1. Introduction

Fractional differential equations (FDEs) are used to model systems that exhibit non-local behaviours and have applications in various fields such as physics, engineering, finance, and biology. The domain of mathematics that focuses on the integrations and differentiations of arbitrary orders is called fractional calculus. It is a new domain of research in applied sciences [1].

Newton and Leibniz introduced the concept of fractional calculus in 1695. When Hospital questioned Leibniz in the seventeenth century about the meaning of half order derivative of x , he said that that it would be an apparent paradox, $d^{1/2}x$ will be equal to $x\sqrt{dx:x}$ [1]. This was the start of an emerging branch of mathematics "fractional calculus". After Leibniz it was Lacroix who further explained this concept of fractional derivative in 1819 [1].

FDE is just a specialization of the classical differential equation with a non-integer order. The classical differential equation of integer order can be solved using a variety of integral transforms, including the Laplace, Mahgoub, Jafari, Shehu, and others [2-4]. A

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number of authors have conducted in-depth research on the fractional Fourier transform, Sumudu transform, and other integral transform solutions for fractional differential equations. Other techniques for solving FDE include the variation iteration method, the homotopy perturbation method, the exponential-function approach, the fractional sub-equation method, and the differential transform method [5-8]. FDE solution methods and related interpretations are therefore developing areas of practical mathematics. There is no standard approach to solving linear sequential fractional differential equations (LSFDE) to date because the different forms of fractional derivatives give different types of solutions. Now we are giving some definitions of fractional derivatives:

Definition 1.1. The Riemann–Liouville left fractional derivative is defined as [9-12];

$$ {}_aD_x^{\alpha} f(x) = \frac{1}{\Gamma\,(m-\alpha+1)} \left(\frac{d}{dx}\right)^{m+1} \int_a^x (x-u)^{m-\alpha} f(u)\, du \tag{1}$$

where $m \leq \alpha < m+1$, m is positive integer.

when, $0 \leq \alpha < 1$, then $a D_x^{\alpha} f(x) = \frac{1}{\Gamma\,(-\alpha+1)} \frac{d}{dx} \int_a^x (x-u)^{-\alpha} f(u)\, du$ (2)

The right Riemann–Liouville fractional derivative is defined as;

$$ {}_xD_b^{\alpha} f(x) = \frac{1}{\Gamma\,(m-\alpha+1)} \left(-\frac{d}{dx}\right)^{m+1} \int_x^b (u-x)^{m-\alpha} f(u)\, du \tag{3}$$

The classical derivative of a constant is always zero, but Riemann–Liouville definitions (left and right) give a non-zero value for the derivative of a constant.

Definition 1.2. Caputo's definition of fractional derivative goes as follow [13,14]

$$D_x^{\alpha} f(x) = \frac{1}{\Gamma\,(n-\alpha)} \int_a^x (x-u)^{n-\alpha-1} f^n(u)\, du \text{, where } n-1 \leq \alpha < n \tag{4}$$

According to this definition first differentiate f(x), n times and then integrate $n-\alpha$ times. For Caputo fractional derivative of order α the function f(x) must be differentiable n where $n-1 \leq \alpha < n$.

Definition 1.3. The conformable fractional derivative of a function f: $[0,\infty] \rightarrow R$ is defined as [8,9,15-17],

$$\tau^{\alpha}\,[f(x)] = \lim_{\varepsilon \to 0} \frac{f(x+\varepsilon x^{1-\alpha}) - f(x)}{\varepsilon} \text{, for all } x > 0, \alpha \in (0,1). \tag{5}$$

If f is α differentiable in $(0,a)$, $a > 0$ and $\lim_{x \to 0^+} \tau^{\alpha}[f(x)]$ exist then $\tau^{\alpha}\,(0) = \lim_{x \to 0^+} \tau^{\alpha}\,[f(x)]$.

A function f(x) is said to be α-differentiable if it has a conformable fractional derivative of order α. This definition correlates with the Riemann–Liouville definitions of classical derivative. If f(x) is differentiable function for of $x > 0$ and $\alpha \in (0,1)$, then the following results are hold [18];

$$\tau^{\alpha}\,[f(x)] = x^{1-\alpha} \frac{d}{dx}[f(x)] \tag{6}$$

$$\tau^{n\alpha} \left[\exp\left(\frac{x^{\alpha}}{\psi\alpha}\right)\right] = \frac{1}{\psi^n} \exp\left(\frac{x^{\alpha}}{\psi\alpha}\right) \tag{7}$$

where τ^{α} is conformable fractional derivative of order α and ψ is any non-zero constant.

Definition 1.4. If f(t) is a continuous function, then the fractional integral of order α of f(t) is defined as [15]; $I_a^{\alpha} f(t) = I_1^a[t^{\alpha-1} f(t) = \int_a^t \frac{f(x)}{x^{1-\alpha}}\, dx$

where a>0, $\alpha \in$ (0,1) and the integral $\int_a^t \frac{f(x)}{x^{1-\alpha}}\, dx$ is the classical Riemann improper integral. If f(t) is a continuous function, then $\tau^{\alpha}\, I_{\alpha}^{a}[f(t)] = f(t)$, for all $t \geq a$.

2. Non-Homogeneous Linear Fractional Differential Equations

A differential equation of the form

$$\left(a_0\tau^{n\alpha} + a_1\tau^{(n-1)\alpha} + a_2\tau^{(n-2)\alpha} + \cdots . + a_{n-1}\tau^{\alpha} + a_n\right)y = \Omega(x),\ \Omega(x) \neq 0 \tag{8}$$

where and $a_{0,}\ \ a_{1,},\ \ a_{2,} \dots\dots\dots..\ \ a_n$ are constants, is NHLSFDE of order nα with constant coefficients. In this equation $\tau^{n\alpha}$ is conformable fractional derivative of order nα and $\alpha \in (0,1)$, where n is any natural number.

Rewriting "Eq. (8)"

$$f(\tau^{\alpha})y(x) = \Omega(x) \tag{9}$$

where $f(\tau^{\alpha})$ is a linear fractional differential operator. Two separate parts make up the complete solution y(x) of "Eq. (9)", the PI and the complementary function (CF). The solution to "Eq. (9)" is the sum of the CF and the PI. This indicates that the answer to "Eq. (9)" is y(x)= CF+PI.

The CF is the solution of homogeneous linear sequential fractional differential equation (HLSFDE) $f(\tau^{\alpha})y = 0$ corresponding to NHLSFDE with constant coefficients which is $f(\tau^{\alpha})y = \Omega(x)$. In a previous research paper [18], we found the solutions of HLSFDEs with constant coefficients having fractional exponential functions and fractional type of sine and cosine functions as follows [18];

(i) If $\sigma_1, \sigma_2, \sigma_{3,\dots\dots\dots\dots}, \sigma_n$ are n real and distinct roots of AE of homogeneous LSFDE with constant coefficients, then its solution is $y(x) = C_1 \exp\left(\sigma_1 \frac{x^{\alpha}}{\alpha}\right) + C_2 \exp\left(\sigma_2 \frac{x^{\alpha}}{\alpha}\right) + C_3 \exp\left(\sigma_3 \frac{x^{\alpha}}{\alpha}\right) + \cdots + C_n \exp\left(\sigma_n \frac{x^{\alpha}}{\alpha}\right)$

(ii) If the roots of AE of homogeneous LSFDE with constant coefficients has r repeated roots $(\sigma_1 = \sigma_2 = \sigma_3 = \cdots \dots = \sigma_r)$ for $1 \leq r \leq n$,then its solution is $y(x) = C_1 \exp\left(\sigma_1 \frac{x^{\alpha}}{\alpha}\right) + x^{\alpha} C_2 \exp\left(\sigma_1 \frac{x^{\alpha}}{\alpha}\right) + x^{2\alpha} C_3 \exp\left(\sigma_1 \frac{x^{\alpha}}{\alpha}\right) + \cdots + x^{(r-1)\alpha} C_r \exp\left(\sigma_1 \frac{x^{\alpha}}{\alpha}\right)$

(iii) If the roots of AE of homogeneous LSFDE are complex $\gamma \pm i\delta$,then its solution is $y(x) = \exp\left(\gamma \frac{x^{\alpha}}{\alpha}\right) [A \cos_{\alpha}\left(\left(\frac{\delta}{\alpha}\right) x^{\alpha}\right) + B \sin_{\alpha}(\left(\frac{\delta}{\alpha}\right) x^{\alpha})$ where A and B are constants.

3. Particular Integral

The PI for "Eq. (9)" is $PI = \frac{1}{f(\tau^{\alpha})}\ \Omega(x)$

The PI depending on the nature of the function $\Omega(x)$. The function $\Omega(x)$can be in various forms. In this section we will find the PI for different forms of $\Omega(x)$ which are fractional exponential function, fractional cosine, and sine functions.

Lemma 3.1. $\frac{1}{\tau^{\alpha}-a}\mu(x) = \exp\left(a\ \frac{x^{\alpha}}{\alpha}\right)\ \tau^{-\alpha}\left[\exp\left(-a\ \frac{x^{\alpha}}{\alpha}\right)\mu(x)\ \right]$

Proof: Let $\frac{1}{\tau^{\alpha}-a}\mu(x) == y$

Operating by $(\tau^{\alpha} - a)$, $(\tau^{\alpha} - a)\frac{1}{\tau^{\alpha}-a}\mu(x) = (\tau^{\alpha} - a)y$

$\mu(x) = (\tau^{\alpha} - a)y$

On multiplying by $\exp\left(-a\ \frac{x^{\alpha}}{\alpha}\right)$, $\exp\left(-a\ \frac{x^{\alpha}}{\alpha}\right)(\tau^{\alpha} - a)y = \exp\left(-a\ \frac{x^{\alpha}}{\alpha}\right)\mu(x)$

$$\exp\left(-a\ \frac{x^{\alpha}}{\alpha}\right)\tau^{\alpha}y - a\,y\exp\left(-a\ \frac{x^{\alpha}}{\alpha}\right) = \exp\left(-a\ \frac{x^{\alpha}}{\alpha}\right)\mu(x)$$

Using "Eq. (7)", $\exp\left(-a\ \frac{x^\alpha}{\alpha}\right)\tau^\alpha y + \ y\,\tau^\alpha[\exp\left(-a\ \frac{x^\alpha}{\alpha}\right)] = \exp\left(-a\ \frac{x^\alpha}{\alpha}\right)\mu(x)$

$\tau^\alpha\left[y\exp\left(-a\ \frac{x^\alpha}{\alpha}\right)\right] = \exp\left(-a\ \frac{x^\alpha}{\alpha}\right)\mu(x)$

Operating $\tau^{-\alpha}$, $\tau^{-\alpha}\tau^\alpha\left[y\exp\left(-a\ \frac{x^\alpha}{\alpha}\right)\right] = \tau^{-\alpha}[\exp\left(-a\ \frac{x^\alpha}{\alpha}\right)\mu(x)]$

$\left[y\exp\left(-a\ \frac{x^\alpha}{\alpha}\right)\right] = \tau^{-\alpha}[\exp\left(-a\ \frac{x^\alpha}{\alpha}\right)\mu(x)]$

$y = \exp\left(a\ \frac{x^\alpha}{\alpha}\right)\ \tau^{-\alpha}[\exp\left(-a\ \frac{x^\alpha}{\alpha}\right)\mu(x)]$

$\frac{1}{\tau^\alpha - a}\mu(x) = \exp\left(a\ \frac{x^\alpha}{\alpha}\right)\ \tau^{-\alpha}[\exp\left(-a\ \frac{x^\alpha}{\alpha}\right)\mu(x)]$

Theorem 3.1. If $\Omega(x) = \exp\left(\sigma\frac{x^\alpha}{\alpha}\right)$,then $\frac{1}{f(\tau^\alpha)}\Omega(x) = \frac{1}{f(\sigma)}\exp\left(\sigma\frac{x^\alpha}{\alpha}\right)$, provided $f(\sigma) = 0$.

Proof: The conformable fractional functional fractional derivative is

$f(\tau^\alpha) \equiv \left(a_0\tau^{n\alpha} + a_1\tau^{(n-1)\alpha} + a_2\tau^{(n-2)\alpha} + \cdots \ldots\ldots + a_{n-1}\tau^\alpha + a_n\right)$

operating $f(\tau^\alpha)$on fractional exponential function $\exp\left(\sigma\frac{x^\alpha}{\alpha}\right)$

$f(\tau^\alpha)\exp\left(\sigma\frac{x^\alpha}{\alpha}\right) = \left(a_0\tau^{n\alpha} + a_1\tau^{(n-1)\alpha} + a_2\tau^{(n-2)\alpha} + \cdots + a_{n-1}\tau^\alpha + a_n\right)\exp\left(\sigma\frac{x^\alpha}{\alpha}\right)$

using "Eq. (7)",

$f(\tau^\alpha)\exp\left(\sigma\frac{x^\alpha}{\alpha}\right) = \left(a_0\ \sigma^n\ \exp\left(\sigma\frac{x^\alpha}{\alpha}\right) + \cdots \ldots + a_{n-1}\ \sigma\ \exp\left(\sigma\frac{x^\alpha}{\alpha}\right) + a_n\exp\left(\sigma\frac{x^\alpha}{\alpha}\right)\right)$

$f(\tau^\alpha)\exp\left(\sigma\frac{x^\alpha}{\alpha}\right) = (a_0\ \sigma^n\ + a_1\sigma^{n-1}\ + \cdots \ldots\ldots + a_{n-1}\ \sigma\ + a_n)\exp\left(\sigma\frac{x^\alpha}{\alpha}\right)$

$f(\tau^\alpha)\exp\left(\sigma\frac{x^\alpha}{\alpha}\right) = f(\sigma)\ \exp\left(\sigma\frac{x^\alpha}{\alpha}\right)$

Operating $\frac{1}{f(\tau^\alpha)}$, we get $\frac{1}{f(\tau^\alpha)}\ f(\tau^\alpha)\exp\left(\sigma\frac{x^\alpha}{\alpha}\right) = \frac{1}{f(\tau^\alpha)}\ f(\sigma)\ \exp\left(\sigma\frac{x^\alpha}{\alpha}\right)$

$$\frac{1}{f(\tau^\alpha)}\ \exp\left(\sigma\frac{x^\alpha}{\alpha}\right) = \frac{1}{f(\sigma)}\ \exp\left(\sigma\frac{x^\alpha}{\alpha}\right),\text{ provided } f(\sigma) \neq 0$$

Lemma 3.2. If $f(\sigma) = 0$, then $\frac{1}{f(\tau^\alpha)}\ \exp\left(\sigma\frac{x^\alpha}{\alpha}\right) = \frac{x^\alpha}{\alpha}\frac{1}{f'(\sigma)}\ \exp\left(\sigma\frac{x^\alpha}{\alpha}\right)$ provided $f'(\sigma) \neq 0$

Proof: If $f(\sigma) = 0$, then σ is the root of AE of homogeneous FDE $f(\tau^\alpha) = 0$

Therefore $\tau^\alpha - \sigma$ is a factor of $f(\tau^\alpha)$. Let $f(\tau^\alpha) = (\tau^\alpha - \sigma)\mu(\tau^\alpha)$, where $\mu(\sigma) \neq 0$, then

$f'(\tau^\alpha) = (\tau^\alpha - \sigma)\mu'(\tau^\alpha) + \mu(\tau^\alpha).1,$

Putting $\tau^\alpha = \sigma$, we get $f'(\sigma) = (0)\mu'(\sigma) + \mu(\sigma) = \mu(\sigma)$

$\frac{1}{f(\tau^\alpha)}\exp\left(\sigma\frac{x^\alpha}{\alpha}\right) = \frac{1}{(\tau^\alpha - \sigma)\mu(\tau^\alpha),}\ \exp\left(\sigma\frac{x^\alpha}{\alpha}\right) = \frac{1}{\mu(\tau^\alpha)}.\frac{1}{(\tau^\alpha - \sigma)}\ \exp\left(\sigma\frac{x^\alpha}{\alpha}\right)$

Using lemma 3.1, $\frac{1}{f(\tau^\alpha)}\exp\left(\sigma\frac{x^\alpha}{\alpha}\right) = \frac{1}{\mu(\sigma)}\exp\left(\sigma\frac{x^\alpha}{\alpha}\right)\ \tau^{-\alpha}[1]$

By Definition 1.4

$\frac{1}{f(\tau^\alpha)}\exp\left(\sigma\frac{x^\alpha}{\alpha}\right) = \frac{1}{\mu(\sigma)}\exp\left(\sigma\frac{x^\alpha}{\alpha}\right).\ \frac{x^\alpha}{\alpha} = \frac{x^\alpha}{\alpha}.\frac{1}{f'(\sigma)}\ \ \exp\left(\sigma\frac{x^\alpha}{\alpha}\right)$

Theorem 3.2. If $\Omega(x) = \cos_\alpha\left(a\frac{x^\alpha}{\alpha}\right)$, then $\frac{1}{f(\tau^{2\alpha})}\Omega(x) = \frac{1}{f(-a^2)}\cos_\alpha\left(a\frac{x^\alpha}{\alpha}\right)$, provided $f(-a^2) \neq 0$.

Proof: Using "Eq. (6)"

$$\tau^\alpha[\cos_\alpha\left(a\frac{x^\alpha}{\alpha}\right)] = -a\ x^{1-\alpha}\sin_\alpha\ \left(a\frac{x^\alpha}{\alpha}\right)\ (x^{\alpha-1})\ = -a\ \tau^\alpha[\sin_\alpha\ \left(a\frac{x^\alpha}{\alpha}\right)]$$

$\tau^{2\alpha}[\cos_\alpha ax^\alpha] = -a\, x^{1-\alpha}\ \cos_\alpha\left(a\frac{x^\alpha}{\alpha}\right)\ (a\ x^{\alpha-1}) = -a^2 \cos_\alpha\left(a\frac{x^\alpha}{\alpha}\right)$

$\tau^{3\alpha}[\cos_\alpha ax^\alpha] = a^3 \sin_\alpha\ \left(a\frac{x^\alpha}{\alpha}\right)$

$\tau^{4\alpha}\ [\cos_\alpha ax^\alpha] = a^3\ \tau^\alpha\ \sin_\alpha\ \left(a\frac{x^\alpha}{\alpha}\right) = a^3 x^{1-\alpha}\ \cos_\alpha\left(a\frac{x^\alpha}{\alpha}\right)\ (a\, x^{\alpha-1}) = a^4\ \cos_\alpha\left(a\frac{x^\alpha}{\alpha}\right)$

$\tau^{2(2\alpha)}[\cos_\alpha ax^\alpha] = a^4 \cos_\alpha\left(a\frac{x^\alpha}{\alpha}\right)$

$\tau^{2(n\alpha)}[\cos_\alpha ax^\alpha] = (-a^2)^n \cos_\alpha\left(a\frac{x^\alpha}{\alpha}\right)$

Hence $f(\tau^{2\alpha})\cos_\alpha\left(a\frac{x^\alpha}{\alpha}\right) = f(-a^2)\cos_\alpha\left(a\frac{x^\alpha}{\alpha}\right)$

Operating $\frac{1}{f(\tau^{2\alpha})}$, $\frac{1}{f(\tau^{2\alpha})} f(\tau^{2\alpha})\cos_\alpha\left(a\frac{x^\alpha}{\alpha}\right) = \frac{1}{f(\tau^{2\alpha})} f(-a^2)\ \cos_\alpha\left(a\frac{x^\alpha}{\alpha}\right)$

$\frac{1}{f(\tau^{2\alpha})}\cos_\alpha\left(a\frac{x^\alpha}{\alpha}\right) = \frac{1}{f(-a^2)}\ \cos_\alpha\left(a\frac{x^\alpha}{\alpha}\right)$, provided $f(-a^2)\ \neq 0$

Lemma 3.3. If $f(-a^2) \neq 0$, then $\frac{1}{f(\tau^{2\alpha})}\cos_\alpha ax^\alpha = \frac{x^\alpha}{\alpha}\frac{1}{f'(-a^2)}\cos_\alpha ax^\alpha$, provided $f'(-a^2) \neq 0$

Proof: we have $\exp(iax^\alpha) = \cos_\alpha ax^\alpha + i\sin_\alpha ax^\alpha$

$\frac{1}{f(\tau^{2\alpha})}\cos_\alpha ax^\alpha = \text{Real part of } \frac{1}{f(\tau^{2\alpha})}\exp(iax^\alpha)$

Using Lemma 3.2, $\frac{1}{f(\tau^{2\alpha})}\cos_\alpha ax^\alpha = \text{Real part of } \frac{x^\alpha}{\alpha}\frac{1}{f'((ia)^2)}\exp(iax^\alpha)$

$\frac{1}{f(\tau^{2\alpha})}\cos_\alpha ax^\alpha = \text{Real part of } \frac{x^\alpha}{\alpha}\frac{1}{f'(-a^2)}[\cos_\alpha ax^\alpha + i\sin_\alpha ax^\alpha]$

$\frac{1}{f(\tau^{2\alpha})}\cos_\alpha ax^\alpha = \frac{x^\alpha}{\alpha}\frac{1}{f'(-a^2)}\cos_\alpha ax^\alpha$, provided $f'(-a^2) \neq 0$

Theorem 3.3. If $\Omega(x) = \sin_\alpha\left(a\frac{x^\alpha}{\alpha}\right)$,then $\frac{1}{f(\tau^{2\alpha})}\Omega(x) = \frac{1}{f(-a^2)}\sin_\alpha\left(a\frac{x^\alpha}{\alpha}\right)$, provided that $f(-a^2)\ \neq 0$.

Proof: Using "Eq. (6)" $\tau^\alpha\left(\sin_\alpha\ \left(a\frac{x^\alpha}{\alpha}\right)\right) = a\ x^{1-\alpha}\cos_\alpha\left(a\frac{x^\alpha}{\alpha}\right)\ (x^{\alpha-1})\ = a\ \cos_\alpha\left(a\frac{x^\alpha}{\alpha}\right)$

$\tau^{2\alpha}\left[\sin_\alpha\ \left(a\frac{x^\alpha}{\alpha}\right)\right] = -a^2\sin_\alpha\ \left(a\frac{x^\alpha}{\alpha}\right)$

$\tau^{3\alpha}\left[\sin_\alpha\ \left(a\frac{x^\alpha}{\alpha}\right)\right] = -a^3\cos_\alpha\ \left(a\frac{x^\alpha}{\alpha}\right)$

$\tau^{4\alpha}\ \left[\sin_\alpha\ \left(a\frac{x^\alpha}{\alpha}\right)\right] = -a^3\ \tau^\alpha\ (\cos_\alpha\ \left(a\frac{x^\alpha}{\alpha}\right) = a^4\ \sin_\alpha\ \left(a\frac{x^\alpha}{\alpha}\right)$

$\tau^{2(2\alpha)}\left[\sin_\alpha\ \left(a\frac{x^\alpha}{\alpha}\right)\right] = (-a\,)^2\sin_\alpha\ \left(a\frac{x^\alpha}{\alpha}\right)$

$\tau^{2(n\alpha)}\left[\sin_\alpha\ \left(a\frac{x^\alpha}{\alpha}\right)\right] = (-a^2\,)^n\sin_\alpha\ \left(a\frac{x^\alpha}{\alpha}\right)$

Hence $f(\tau^{2\alpha})\sin_\alpha\ \left(a\frac{x^\alpha}{\alpha}\right) = f(-a^2)\sin_\alpha\ \left(a\frac{x^\alpha}{\alpha}\right)$

Operating $\frac{1}{f(\tau^{2\alpha})}$, $\frac{1}{f(\tau^{2\alpha})} f(\tau^{2\alpha})\sin_\alpha\ \left(a\frac{x^\alpha}{\alpha}\right) = \frac{1}{f(\tau^{2\alpha})} f(-a^2)\sin_\alpha\ \left(a\frac{x^\alpha}{\alpha}\right)$

$\frac{1}{f(\tau^{2\alpha})}\sin_\alpha\ \left(a\frac{x^\alpha}{\alpha}\right) = \frac{1}{f(-a^2)}\ \sin_\alpha\ \left(a\frac{x^\alpha}{\alpha}\right)$, provided $f(-a^2) \neq 0$

Lemma 3.4. If $f(-a^2) \neq 0$,then $\frac{1}{f(\tau^{2\alpha})}\sin_\alpha ax^\alpha = \frac{x^\alpha}{\alpha}\frac{1}{f'(-a^2)}\sin_\alpha ax^\alpha$, provided $f'(-a^2) \neq 0$

Proof: we have $\exp(\mathrm{iax}^{\alpha}) = \cos_{\alpha} a\mathrm{x}^{\alpha} + \mathrm{i}\sin_{\alpha} a\mathrm{x}^{\alpha}$

$\frac{1}{\mathrm{f}(\tau^{2\alpha})}\sin_{\alpha}\mathrm{ax}^{\alpha} = $ Imaginary part of $\frac{1}{\mathrm{f}(\tau^{2\alpha})}\exp(\mathrm{iax}^{\alpha})$

Using Lemma 3.2, $\frac{1}{\mathrm{f}(\tau^{2\alpha})}\sin_{\alpha}\mathrm{ax}^{\alpha} = $ Imaginary part of $\frac{\mathrm{x}^{\alpha}}{\alpha}\frac{1}{f'((\tau^{\alpha})^2)}\exp(\mathrm{iax}^{\alpha})$

$$\frac{1}{\mathrm{f}(\tau^{2\alpha})}\sin_{\alpha}\mathrm{ax}^{\alpha} = \frac{\mathrm{x}^{\alpha}}{\alpha}\frac{1}{f'(-a^2)}\sin_{\alpha} a\mathrm{x}^{\alpha}, \text{ provided } f'(-\mathrm{a}^2) \neq 0$$

Theorem 3.4. If $\Omega(\mathrm{x}) = \exp\left(\sigma\frac{\mathrm{x}^{\alpha}}{\alpha}\right)\psi(x)$, then $\frac{1}{\mathrm{f}(\tau^{\alpha})}\Omega(\mathrm{x}) = \exp\left(\sigma\frac{\mathrm{x}^{\alpha}}{\alpha}\right)\frac{1}{(\tau^{\alpha}+\sigma)}\psi(\mathrm{x})$

Proof: Let $V(x)$ is a function of variable x, then by differentiation

$$\tau^{\alpha}\left[\exp\left(\sigma\frac{\mathrm{x}^{\alpha}}{\alpha}\right)V(x)\right] = \exp\left(\sigma\frac{\mathrm{x}^{\alpha}}{\alpha}\right)\tau^{\alpha}V(x) + V(x)\,\tau^{\alpha}\exp\left(\sigma\frac{\mathrm{x}^{\alpha}}{\alpha}\right)$$

using "Eq. (7)" $\tau^{\alpha}\left[\exp\left(\sigma\frac{\mathrm{x}^{\alpha}}{\alpha}\right)V(x)\right] = \exp\left(\sigma\frac{\mathrm{x}^{\alpha}}{\alpha}\right)(\tau^{\alpha} + \sigma)V(x) =$

$$\tau^{\alpha}\tau^{\alpha}\left[\exp\left(\sigma\frac{\mathrm{x}^{\alpha}}{\alpha}\right)V(x)\right]$$

$$= \tau^{\alpha}\left[\exp\left(\sigma\frac{\mathrm{x}^{\alpha}}{\alpha}\right)\tau^{\alpha}V(x) + V(x)\sigma\exp\left(\sigma\frac{\mathrm{x}^{\alpha}}{\alpha}\right)\right]$$

$$= \tau^{\alpha}\left[\exp\left(\sigma\frac{\mathrm{x}^{\alpha}}{\alpha}\right)\tau^{\alpha}V(x)\right] + \sigma\,\tau^{\alpha}[V(x)\exp\left(\sigma\frac{\mathrm{x}^{\alpha}}{\alpha}\right)]$$

$$= \exp\left(\sigma\frac{\mathrm{x}^{\alpha}}{\alpha}\right)\tau^{\alpha}\tau^{\alpha}V(x) + \{\tau^{\alpha}V(x)\}\tau^{\alpha}V(x)\exp\left(\sigma\frac{\mathrm{x}^{\alpha}}{\alpha}\right)] + \sigma[\exp\left(\sigma\frac{\mathrm{x}^{\alpha}}{\alpha}\right)\tau^{\alpha}V(x) + V(x)\sigma\exp\left(\sigma\frac{\mathrm{x}^{\alpha}}{\alpha}\right)]$$

$$= \exp\left(\sigma\frac{\mathrm{x}^{\alpha}}{\alpha}\right)\tau^{2\alpha}V(x) + \sigma\exp\left(\sigma\frac{\mathrm{x}^{\alpha}}{\alpha}\right)\{\tau^{\alpha}V(x)\}] + \sigma\exp\left(\sigma\frac{\mathrm{x}^{\alpha}}{\alpha}\right)\tau^{\alpha}V(x) + V(x)\sigma^2\exp\left(\sigma\frac{\mathrm{x}^{\alpha}}{\alpha}\right)]$$

$$= \exp\left(\sigma\frac{\mathrm{x}^{\alpha}}{\alpha}\right)[\tau^{2\alpha}V(x) + \sigma\,\tau^{\alpha}V(x) + \sigma\,\tau^{\alpha}V(x) + \sigma^2V(x)]$$

$$= \exp\left(\sigma\frac{\mathrm{x}^{\alpha}}{\alpha}\right)[\tau^{2\alpha}V(x) + 2\sigma\,\tau^{\alpha}V(x) + \sigma^2V(x)]$$

$$\tau^{2\alpha}\left[\exp\left(\sigma\frac{\mathrm{x}^{\alpha}}{\alpha}\right)V(x)\right] = \exp\left(\sigma\frac{\mathrm{x}^{\alpha}}{\alpha}\right)(\tau^{\alpha} + \sigma)^2V(x)$$

Similarly, $\tau^{3\alpha}\left[\exp\left(\sigma\frac{\mathrm{x}^{\alpha}}{\alpha}\right)V(x)\right] = \exp\left(\sigma\frac{\mathrm{x}^{\alpha}}{\alpha}\right)(\tau^{\alpha} + \sigma)^3V(x)$

In general, $\tau^{n\alpha}\left[\exp\left(\sigma\frac{\mathrm{x}^{\alpha}}{\alpha}\right)V(x)\right] = \exp\left(\sigma\frac{\mathrm{x}^{\alpha}}{\alpha}\right)(\tau^{\alpha} + \sigma)^nV(x)$

$$\mathrm{f}(\tau^{\alpha})\left[\exp\left(\sigma\frac{\mathrm{x}^{\alpha}}{\alpha}\right)V(x)\right] = \exp\left(\sigma\frac{\mathrm{x}^{\alpha}}{\alpha}\right)f(\tau^{\alpha} + \sigma)V(x)$$

Operating $\frac{1}{\mathrm{f}(\tau^{\alpha})}$, $\frac{1}{\mathrm{f}(\tau^{\alpha})}\mathrm{f}(\tau^{\alpha})\left[\exp\left(\sigma\frac{\mathrm{x}^{\alpha}}{\alpha}\right)V(x)\right] = \frac{1}{\mathrm{f}(\tau^{\alpha})}\exp\left(\sigma\frac{\mathrm{x}^{\alpha}}{\alpha}\right)f(\tau^{\alpha} + \sigma)V(x)$

$$\exp\left(\sigma\frac{\mathrm{x}^{\alpha}}{\alpha}\right)V(x) = \frac{1}{\mathrm{f}(\mathrm{T}_{\alpha})}\exp\left(\sigma\frac{\mathrm{x}^{\alpha}}{\alpha}\right)f(\tau^{\alpha} + \sigma)V(x) \tag{10}$$

Now let $f(\tau^{\alpha} + \sigma)V(x) = \psi(x)$, i.e., $V(x) = \frac{1}{f(\tau^{\alpha}+\sigma)}\psi(x)$

From "Eq. (10)"

$$\frac{1}{\mathrm{f}(\tau^{\alpha})}\left[\exp\left(\sigma\frac{\mathrm{x}^{\alpha}}{\alpha}\right)\psi(x)\right] = \exp\left(\sigma\frac{\mathrm{x}^{\alpha}}{\alpha}\right)\frac{1}{f(\tau^{\alpha}+\sigma)}\psi(x) \tag{11}$$

4. Results and Discussion

This section presents analytical and graphical solutions of some NHLSFDEs involving conformable fractional derivatives. The solutions so obtained are compared with graphs of solutions of the differential equations of integer order. We discovered that the graphical solution correlates with the solution of the classical differential equation as α moves from a non-integer to an integer value.

Example 4.1. Solve the NHLSFDE $(\tau^{2\alpha} + 5\,\tau^{\alpha} + 6)y(x) = \exp\left(\frac{x^{\alpha}}{\alpha}\right)$ (12)

Solution: The AE of "Eq. (12)" is $\sigma^2 + 5\,\sigma + 6 = 0$ and root of AE are $\sigma = -2\,, -3$

The CF is $C_1 \exp\left(-2\frac{x^{\alpha}}{\alpha}\right) + C_2 \exp\left(-3\,\frac{x^{\alpha}}{\alpha}\right)$ and PI is $\frac{1}{\tau^{2\alpha}+5\,\tau^{\alpha}+6} \exp\left(\frac{x^{\alpha}}{\alpha}\right)$

Using "Theorem 3.1" $PI = \frac{1}{(1)^2+5(1)+6} \exp\left(\frac{x^{\alpha}}{\alpha}\right) = \frac{1}{12} \exp\left(\frac{x^{\alpha}}{\alpha}\right)$

The complete solution of "Eq. (12)" for $\alpha = 0.25, 0.5, 0.75$ and $\alpha = 1$ are

$$y(x) = C_1 \exp\left(-2\frac{x^{\alpha}}{\alpha}\right) + C_2 \exp\left(-3\,\frac{x^{\alpha}}{\alpha}\right) + \frac{1}{12} \exp\left(\frac{x^{\alpha}}{\alpha}\right)$$
$$y(x) = C_1 \exp(-4\,x^{0.5}) + C_2 \exp(-6\,x^{0.5}) + \frac{1}{12} \exp\left(\frac{x^{0.5}}{0.5}\right)$$
$$y(x) = C_1 \exp\left(-2\frac{x^{0.75}}{0.75}\right) + C_2 \exp\left(-3\,\frac{x^{0.75}}{0.75}\right) + \frac{1}{12} \exp\left(\frac{x^{0.75}}{0.75}\right)$$

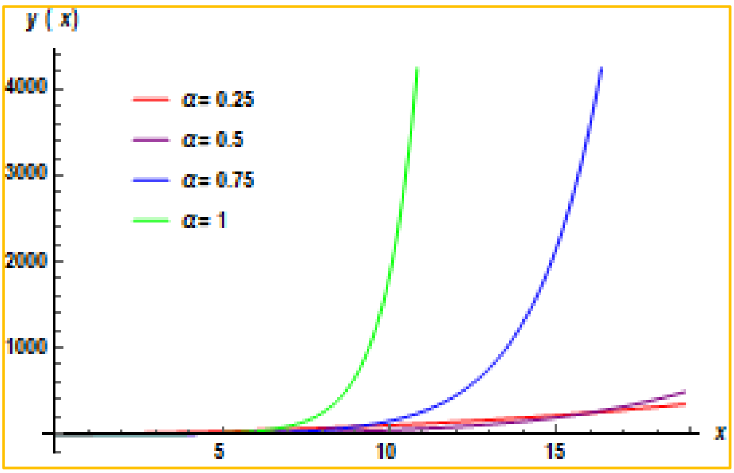

Fig. 1. Solution of "equation (12)".

The graphical solution of "eq. (12)" for α = 0.25, 0.5, and 0.75 is shown in Fig. 1 to be correlated with the solution of the classical differential equation for α = 1.

Example 4.2. Solve the NHLSFDE $(\tau^{\alpha}+3)y(x)=\cos_{\alpha}\left(\frac{x^{\alpha}}{\alpha}\right)$ (13)

Solution: The AE is $\sigma+3=0$ and CF is $C_1\exp\left(-3\frac{x^{\alpha}}{\alpha}\right)$

The PI is $PI=\frac{1}{\tau^{\alpha}+3}\cos_{\alpha}\left(\frac{x^{\alpha}}{\alpha}\right)=\frac{(\tau^{\alpha}-3)}{(\tau^{\alpha}+3)(\tau^{\alpha}-3)}\cos_{\alpha}\left(\frac{x^{\alpha}}{\alpha}\right)=\frac{(\tau^{\alpha}-3)}{(\tau^{\alpha})^2-9}\cos_{\alpha}\left(\frac{x^{\alpha}}{\alpha}\right)$

Using Theorem 3.2, $PI=-\frac{(\tau^{\alpha}-3)}{10}\cos_{\alpha}\left(\frac{x^{\alpha}}{\alpha}\right)=-\frac{1}{10}\left[\tau^{\alpha}\cos_{\alpha}\left(\frac{x^{\alpha}}{\alpha}\right)-3\cos_{\alpha}\left(\frac{x^{\alpha}}{\alpha}\right)\right]$

$PI=-\frac{1}{10}\left[-x^{1-\alpha}\ x^{\alpha-1}\sin_{\alpha}\left(\frac{x^{\alpha}}{\alpha}\right)-3\cos_{\alpha}\left(\frac{x^{\alpha}}{\alpha}\right)\right]=\frac{1}{10}\left[\sin_{\alpha}\left(\frac{x^{\alpha}}{\alpha}\right)+3\cos_{\alpha}\left(\frac{x^{\alpha}}{\alpha}\right)\right]$

The complete solution of "Eq. (13)" for $\alpha=0.25, 0.5, 0.75$ *and* $\alpha=1$ are

$$y(x)=C_1\exp\left(-3\frac{x^{0.25}}{0.25}\right)+\frac{1}{10}\left[\sin_{0.25}\left(\frac{x^{0.25}}{0.25}\right)+3\cos_{0.25}\left(\frac{x^{0.25}}{0.25}\right)\right]$$

$$y(x)=C_1\exp\left(-3\frac{x^{0.5}}{0.5}\right)+\frac{1}{10}\left[\sin_{0.5}\left(\frac{x^{0.5}}{0.5}\right)+3\cos_{0.5}\left(\frac{x^{0.5}}{0.5}\right)\right]$$

$$y(x)=C_1\exp\left(-3\frac{x^{0.75}}{0.75}\right)+\frac{1}{10}\left[\sin_{0.75}\left(\frac{x^{0.75}}{0.75}\right)+3\cos_{0.75}\left(\frac{x^{0.75}}{0.75}\right)\right]$$

$$y(x)=C_1e^{-3x}+\frac{1}{10}[\sin x+3\cos x]$$

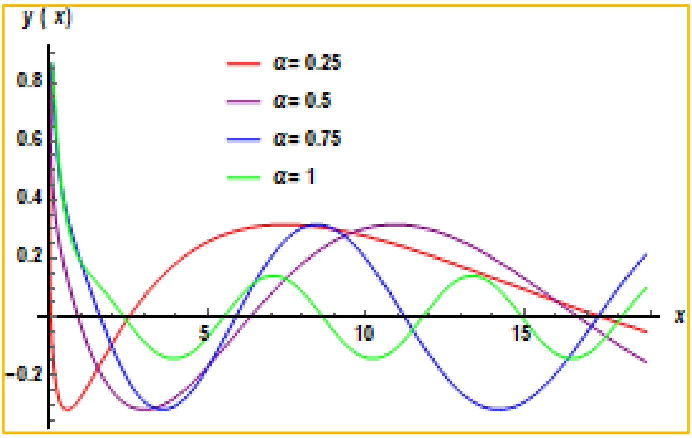

Fig. 2. Solution of "equation (13)".

Fig. 3 represents that the graphical solution of "eq. (13)" for $\alpha=0.25, 0.5, 0.75$ correlates with the solution of the classical differential equation for $\alpha=1$.

Example 4.3. Solve NHLSFDE $(\tau^{2\alpha}+6\tau^{\alpha}+9)y(x)=\exp\left(-3\frac{x^{\alpha}}{\alpha}\right)\cos_{\alpha}\left(\frac{x^{\alpha}}{\alpha}\right)$ (14)

Solution: The AE is $\sigma^2+6\sigma+9=0$ and roots are $\sigma=-3,-3$.

The CF is $C_1\exp\left(-3\frac{x^{\alpha}}{\alpha}\right)+x^{\alpha}C_2\exp\left(-3\frac{x^{\alpha}}{\alpha}\right)$

The PI is $\frac{1}{\tau^{2\alpha}+6\tau^{\alpha}+9}\exp\left(-3\frac{x^{\alpha}}{\alpha}\right)\cos_{\alpha}\left(\frac{x^{\alpha}}{\alpha}\right)=\frac{1}{(\tau^{\alpha}+3)^2}\exp\left(-3\frac{x^{\alpha}}{\alpha}\right)\cos_{\alpha}\left(\frac{x^{\alpha}}{\alpha}\right)$

Using "Theorem 3.4" $PI=\exp\left(-3\frac{x^{\alpha}}{\alpha}\right)\frac{1}{(\tau^{\alpha}-3+3)^2}\cos_{\alpha}\left(\frac{x^{\alpha}}{\alpha}\right)=$

$-\exp\left(-3\frac{x^{\alpha}}{\alpha}\right)\cos_{\alpha}\left(\frac{x^{\alpha}}{\alpha}\right)$

The complete solution of "Eq. (14)" for $\alpha = 0.25, 0.5, 0.75$ *and* $\alpha = 1$ are

$y(x) = C_1 \exp\left(-3\frac{x^{0.25}}{0.25}\right) + +x^{0.25}C_2\exp\left(-3\frac{x^{0.25}}{0.25}\right) - \exp\left(-3\frac{x^{0.25}}{0.25}\right)\cos_{0.25}\left(\frac{x^{0.25}}{0.25}\right)$

$y(x) = C_1 \exp\left(-3\frac{x^{0.5}}{0.5}\right) + +x^{0.5}C_2\exp\left(-3\frac{x^{0.5}}{0.5}\right) - \exp\left(-3\frac{x^{0.5}}{0.5}\right)\cos_{0.5}\left(\frac{x^{0.5}}{0.5}\right)$

$y(x) = C_1 \exp\left(-3\frac{x^{0.75}}{0.75}\right) + +x^{0.75}C_2\exp\left(-3\frac{x^{0.75}}{0.75}\right) - \exp\left(-3\frac{x^{0.75}}{0.75}\right)\cos_{0.75}\left(\frac{x^{0.75}}{0.75}\right)$

$y(x) = C_1 e^{-3x} + x\,C_2 e^{-3x} - e^{-3x}\cos x$

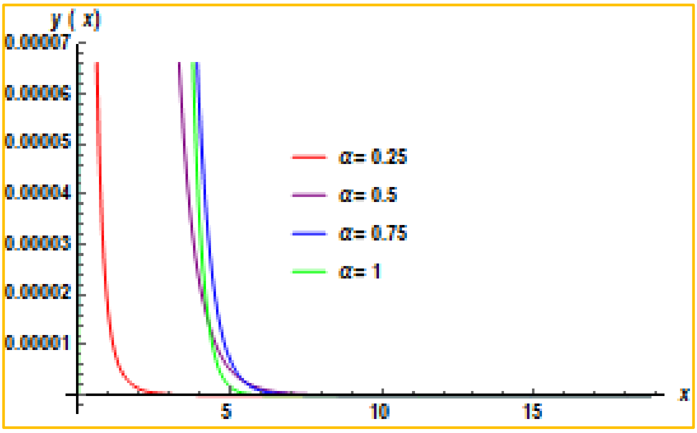

Fig. 4. Solution of "equation (14)".

The solution of the classical differential equation for α=1 coincides with the graphical solution of "Eq. (14)" for α=0.25, 0.5, and 0.75, as shown in Fig. 3.

Example 4.4. Solve NHLSFDE $(\tau^{2\alpha} + 1)y(x) = \sin_{\alpha}\left(2\frac{x^{\alpha}}{\alpha}\right)$ (15)

Solution: The AE of "Eq. (15)" is $\sigma^2 + 1 = 0$ and roots are $\sigma = \pm i$

The CF is $y(x) = A\cos_{\alpha}\left(\left(\frac{1}{\alpha}\right)x^{\alpha}\right) + B\sin_{\alpha}(\left(\frac{1}{\alpha}\right)x^{\alpha})$

The PI is $PI = \frac{1}{\tau^{2\alpha}+1}\sin_{\alpha}\left(2\frac{x^{\alpha}}{\alpha}\right)$

Using Theorem 3.3, $PI = \frac{1}{-4+1}\sin_{\alpha}\left(2\frac{x^{\alpha}}{\alpha}\right) = -\frac{1}{3}\sin_{\alpha}\left(2\frac{x^{\alpha}}{\alpha}\right)$

The solution of "Eq. (15)" is $y(x) = A\cos_{\alpha}\left(\left(\frac{1}{\alpha}\right)x^{\alpha}\right) + B\sin_{\alpha}(\left(\frac{1}{\alpha}\right)x^{\alpha}) - \frac{1}{3}\sin_{\alpha}\left(2\frac{x^{\alpha}}{\alpha}\right)$

For $\alpha = 0.25, 0.5, 0.75$ and $\alpha = 1$ the complete solution of "Eq. (15)" are

$$y(x) = A\cos_{0.25}\left(\left(\frac{1}{0.25}\right)x^{0.25}\right) + B\sin_{0.25}(\left(\frac{1}{0.25}\right)x^{0.25}) - \frac{1}{3}\sin_{0.25}\left(2\frac{x^{0.25}}{0.25}\right)$$

$$y(x) = A\cos_{0.5}\left(\left(\frac{1}{0.5}\right)x^{0.5}\right) + B\sin_{0.5}(\left(\frac{1}{0.5}\right)x^{0.5}) - \frac{1}{3}\sin_{0.5}\left(2\frac{x^{0.5}}{0.5}\right)$$

$$y(x) = A\cos_{0.75}\left(\left(\frac{1}{0.75}\right)x^{0.75}\right) + B\sin_{0.75}(\left(\frac{1}{0.75}\right)x^{0.75}) - \frac{1}{3}\sin_{0.75}\left(2\frac{x^{0.75}}{0.75}\right)$$

$$y(x) = A\cos x + B\sin x - \frac{1}{3}\sin x$$

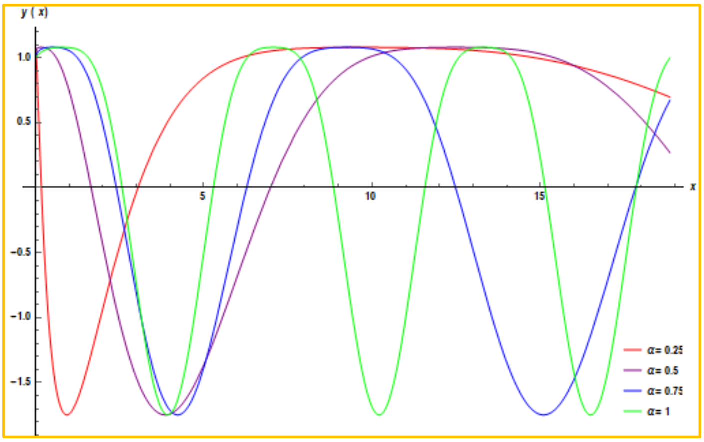

Fig. 5. Solution of “equation (15)”.

Fig. 4 represents that the graphical solution of “Eq. (15)” for $\alpha = 0.25, 0.5, 0.75$ correlates with the solution of the classical differential equation for $\alpha = 1$.

Example 4.5. Solve NHLSFDE $(\tau^{2\alpha} - 3\,\tau^{\alpha} + 2)y(x) = \exp\left(2\frac{x^{\alpha}}{\alpha}\right)$ (16)

Solution: The AE is $\sigma^2 - 3\sigma + 2 = 0$ and roots are $\sigma = 1,\ 2$.

The CF is $C_1 \exp\left(\frac{x^{\alpha}}{\alpha}\right) + C_2\exp\left(2\frac{x^{\alpha}}{\alpha}\right)$

PI is $\frac{1}{\tau^{2\alpha}-3\,\tau^{\alpha}+2} \exp\left(2\frac{x^{\alpha}}{\alpha}\right)$

Using Lemma 3.2, $PI = \frac{x^{\alpha}}{\alpha}\frac{1}{2\,\tau^{\alpha}-3} \sin_{\alpha}\left(2\frac{x^{\alpha}}{\alpha}\right) =$
$\frac{x^{\alpha}}{\alpha}\frac{1}{2(2)-3} \exp\left(2\frac{x^{\alpha}}{\alpha}\right) = -\frac{x^{\alpha}}{\alpha} \exp\left(2\frac{x^{\alpha}}{\alpha}\right)$

The solution of “Eq. (16)” is $y(x) = C_1 \exp\left(\frac{x^{\alpha}}{\alpha}\right) + \ + C_2\exp\left(2\frac{x^{\alpha}}{\alpha}\right) - \frac{x^{\alpha}}{\alpha} \exp\left(2\frac{x^{\alpha}}{\alpha}\right)$

For $\alpha = 0.25,\ 0.5, 0.75$ and $\alpha = 1$,the complete solutions of “Eq. (16)” are

$$y(x) = C_1 \exp\left(\frac{x^{0.25}}{0.25}\right) + C_2\exp\left(2\frac{x^{0.25}}{0.25}\right) - \frac{x^{0.25}}{0.25} \exp\left(2\frac{x^{0.25}}{0.25}\right)$$

$$y(x) = C_1 \exp\left(\frac{x^{0.5}}{0.5}\right) + C_2\exp\left(2\frac{x^{0.5}}{0.5}\right) - \frac{x^{0.5}}{0.5} \exp\left(2\frac{x^{0.5}}{0.5}\right)$$

$$y(x) = C_1 \exp\left(\frac{x^{0.75}}{0.75}\right) + C_2\exp\left(2\frac{x^{0.75}}{0.75}\right) - \frac{x^{0.75}}{0.75} \exp\left(2\frac{x^{0.75}}{0.75}\right)$$

$$y(x) = C_1\, e^{x} + C_2\, e^{2x} - x\, e^{2x}$$

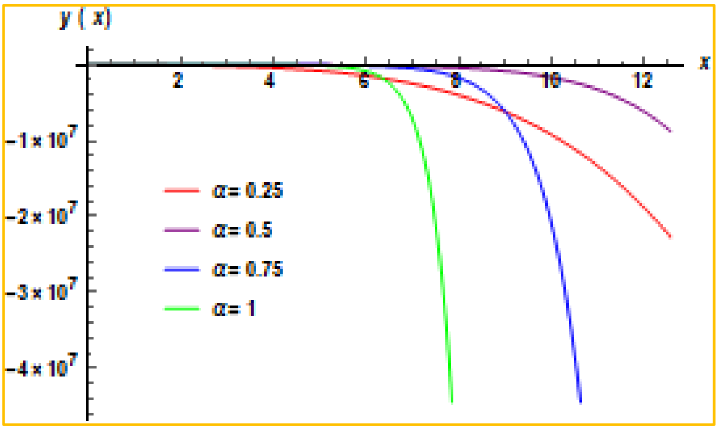

Fig. 6. Solution of "equation (16)".

The solution of the classical differential equation for α=1 coincides with the graphical solution of "Eq. (16)" for α=0.25, 0.5, and 0.75, as shown in Fig. 5.

Example 4.6. Find the solution of NHLSFDE $(\tau^{2\alpha}+4)\mathrm{y}(\mathrm{x}) = \cos_{\alpha}\left(2\frac{x^{\alpha}}{\alpha}\right)$ (17)

Solution: The AE of "Eq. (17)" is $\sigma^2+4=0$ and roots are $\sigma=\pm 2i$

The CF is $y(x) = A\cos_{\alpha}\left(\left(\frac{2}{\alpha}\right)x^{\alpha}\right) + B\sin_{\alpha}(\left(\frac{2}{\alpha}\right)x^{\alpha})$

The PI is $PI = \frac{1}{\tau^{2\alpha}+4}\cos_{\alpha}\left(2\frac{x^{\alpha}}{\alpha}\right)$

Using Lemma 3.3, $PI = \frac{x^{\alpha}}{\alpha}\frac{1}{2\,\tau^{\alpha}}\cos_{\alpha}\left(2\frac{x^{\alpha}}{\alpha}\right) = \frac{x^{\alpha}}{\alpha}\frac{\tau^{\alpha}}{2\,\tau^{2\alpha}}\cos_{\alpha}\left(2\frac{x^{\alpha}}{\alpha}\right)$

Using "Theorem 3.3", $PI = -\frac{x^{\alpha}}{8\,\alpha}\tau^{\alpha}\cos_{\alpha}\left(2\frac{x^{\alpha}}{\alpha}\right) = \frac{x^{\alpha}}{4\alpha}\sin_{\alpha}\left(2\frac{x^{\alpha}}{\alpha}\right)$

The solution of "Eq. (17)" is $y(x) = A\cos_{\alpha}\left(\left(\frac{2}{\alpha}\right)x^{\alpha}\right) + B\sin_{\alpha}\left(\frac{2}{\alpha}\right)x^{\alpha} + \frac{x^{\alpha}}{4\alpha}\sin_{\alpha}\left(2\frac{x^{\alpha}}{\alpha}\right)$

For $\alpha = 0.25$, 0.5, 0.75 and $\alpha = 1$,the complete solutions of "Eq. (17)" are

$$y(x) = A\cos_{0.25}\left(\left(\frac{2}{0.25}\right)x^{0.25}\right) + B\sin_{0.25}\left(\frac{2}{0.25}\right)x^{0.25} + \frac{x^{0.25}}{4(0.25)}\sin_{0.25}\left(2\frac{x^{0.25}}{0.25}\right)$$

$$y(x) = A\cos_{0.5}\left(\left(\frac{2}{0.5}\right)x^{0.5}\right) + B\sin_{0.5}\left(\frac{2}{0.5}\right)x^{0.5} + \frac{x^{0.5}}{4(0.5)}\sin_{0.5}\left(2\frac{x^{0.5}}{0.5}\right)$$

$$y(x) = A\cos_{0.75}\left(\left(\frac{2}{0.75}\right)x^{0.75}\right) + B\sin_{0.75}\left(\frac{2}{0.75}\right)x^{0.75} + \frac{x^{0.75}}{4(0.75)}\sin_{0.75}\left(2\frac{x^{0.75}}{0.75}\right)$$

$$y(x) = A\cos 2x + B\sin 2x + \frac{x}{4}\sin 2x$$

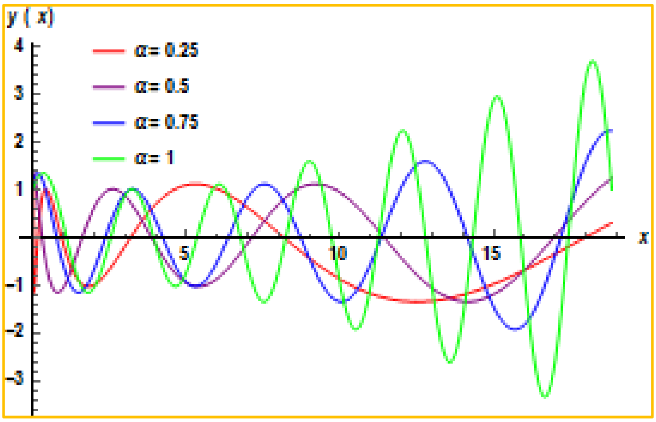

Fig. 7. Solution of “equation (17)”.

Fig. 6 demonstrates the correlation between the solution of the classical differential equation for $\alpha = 1$ and the graphical solution of "Eq. (17)" for $\alpha = 0.25, 0.5$, and 0.75.

5. Conclusion

An analytical approach to solving NHLSFDE with constant coefficients is provided in this research work. This approach provides an association with the solutions of classical differential equations and is based on finding complementary functions and particular integrals of FDE. To demonstrate the validity of the developed method, we applied this method to six different problems of NHLSFDE with constant coefficients. The solution of the NHLSFDE with constant coefficients and the linear differential equation with constant coefficients of integer order were therefore found to be correlated. This approach is more precise and simpler.

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