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Ordering Fuzzy Numbers with Weighted Mean Values and Their Application to Multi-Criteria Decision-Making Problem

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Abstract

The most important aspect of fuzzy numbers is their ordering, which ensures a wide range of their applications in professional life and many academic applied models like linguistic decision-making and fuzzy risk analysis. Even though many researchers have presented various methods, there is still a lot of interest and scope for studies to address the weakness of methods. This paper proposes an approach for ordering generalized fuzzy numbers using weighted mean values (centroid values) of the left and the right fuzziness regions and exponential values of the altitude of the fuzzy number. The proposed method can order two or more fuzzy numbers simultaneously, irrespective of their linear or non-linear membership functions. Furthermore, the proposed method consistently orders the symmetrical fuzzy numbers, the partnered image of the fuzzy number, and the fuzzy numbers that depict the compensation of areas. The advantages of the proposed approach are demonstrated through numerical examples with various types of fuzzy numbers and comparisons with the existing techniques published in the literature. Finally, the proposed method is effectively applied to solve a linguistic multi-criteria decision-making problem related to the stock market.

Keywords: Fuzzy number; Ordering; Weighted mean values; Fuzziness region; Multi-criteria; Decision-making.

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1. Introduction

In practice, the importance of prioritizing fuzzy numbers cannot be overstated because the best choice concept is entirely based on their comparison. Therefore, how to make a preference for fuzzy numbers is a major issue. Many authors have developed fuzzy ordering to resolve the issue of comparing fuzzy numbers, which yields a completely ordered collection or ranking. These approaches include simple, tricky, and intricate techniques from a single fuzzy number attribute to a set of fuzzy numbers attribute. Some ordering or ranking method requires the membership function to be normal; however, in many approaches, the normality restriction on the membership function is inadequate. Many scholars have recently investigated various techniques for ranking fuzzy numbers, and these approaches are used to study a lot of their applications in various fields such as

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Multi-Criteria Decision-Making (MCDM), Risk Analysis (RA), Data Analysis (DA), Control, Optimization, and so on. In the existing techniques, authors have presented either an index or a function based on different constituents related to the fuzzy numbers in different cases. Jain [1] first presented the notion of ranking imprecise quantities represented as fuzzy sets. Dubois and Prade [2] elaborated on the notion of fuzzy numbers and presented associated fuzzy operations. Since then, many scholars have presented a lot of methods for ranking fuzzy numbers. Liou and Wang [3] introduced an indexing technique based on the integral value that also considers decision-makers' attitudes concerning specific purposes. Cheng [4] proposed the distance technique, whereas the area method for ranking fuzzy numbers is presented by Chu and Tsao [5]. Abbasbandy and Asady [6] proposed an approach based on sign distance using the parametric form of the fuzzy number. Asady and Zendehnam [7] presented a method of distance minimization for ranking fuzzy numbers. Wang and Lee [8] suggested a revision in Chu and Tsao [5] and emphasized the importance of the degree of the representative location of fuzzy numbers on the real line. Abbasbandy and Hajjari [9] used the magnitude of the trapezoidal fuzzy numbers for their ranking. Nasseri *et al*. [10] presented a method using the angle between the reference functions of the parametric form of fuzzy numbers to rank them. Yu and Dat [11] presented an improved version of Liou and Wang [3] to overcome the shortcomings of the method for ranking fuzzy numbers with integral values. Rezvani [12] used the probability density function corresponding to the membership functions and presented a ranking of generalized exponential trapezoidal fuzzy numbers based on variance. Patra and Mondal [13,14] studied supplier selection models with fuzzy risk analysis by possibility and necessity constraints and using the balanced solution technique with the soft set. Sen *et al*. [15] presented a Fuzzy risk analysis in familial breast cancer using a similarity measure of interval-valued fuzzy numbers. In the work of Chutia and Chutia [16], the researchers presented a technique based on value and ambiguity with their defuzzifiers at different heights. Nguyen [17] defines a unified index by multiplying two discriminatory components of the fuzzy number and presents comparative reviews. Khorshidi and Nikfalazar [18] proposed a similarity measure between generalized trapezoidal fuzzy numbers containing the geometric distance, the center of gravity (COG), area, perimeter, and height as parameters. Ponnialagan *et al*. [19] presented a new ranking procedure (complete) on the class of trapezoidal fuzzy numbers using the concepts of mid-point, radius, the left and right fuzziness and further introduced a method for solving fuzzy multi-criteria decision-making (Fuzzy MCDM) problem. Chi and Yu [20] proposed the ranking of generalized fuzzy numbers using the integration of centroid point, rank index value, the height of the fuzzy number, and the degree of the decision maker's optimism. Chutia and Gogoi [21] presented fuzzy risk analysis in poultry farming based on a novel similarity measure of fuzzy numbers. Jiang *et al*. [22] studied fuzzy risk analysis based on the ranking score of generalized fuzzy numbers. The ranking score is obtained using the ordered weighted averaging technique. Mao [23] discussed the ranking of fuzzy numbers using weighted distance. Wang [24] presented relative preference relation-based ranking triangular interval-valued fuzzy numbers. Ranking trapezoidal fuzzy numbers using a parametric relation pair is presented in Dombi and Jonas [25]. In the work of Sen *et al*. [26], a similarity measure is used for ranking generalized trapezoidal fuzzy numbers and their application to fuzzy risk analysis. A noble method for ranking generalized fuzzy numbers which take different left and right heights is presented in the work of Barazandeh and Ghazanfari [27] and applied to solve a fuzzy risk analysis problem. Ye *et al*. [28] proposed a novel multi-attribute decision-making method based on fuzzy rough sets. Sen *et al*. [29] suggested the Similarity Measure of Gaussian Fuzzy Numbers and Its Application to the job selection procedure. Patra [30] presented fuzzy risk analysis based on ranking generalized trapezoidal fuzzy numbers using area and perimeter. Ranking fuzzy numbers using the unified integral value that multiplies two different discriminatory components of the fuzzy number is presented in Prasad and Sinha [31]. Firozja *et al.* [32] proposed a method for comparing a real number and a generalized fuzzy number with a degree of accuracy lying between zero and one and then generalized the method to compare two generalized fuzzy numbers.

Most of the approaches in the literature investigate linear membership functions like triangular and trapezoidal fuzzy numbers, leaving plenty of potential for more investigation into non-linear membership functions of fuzzy numbers. Many approaches in the literature have inconsistencies when they come to ordering the symmetrical fuzzy numbers and the fuzzy numbers that depict the compensation of areas. Numerical illustrations are demonstrated in Examples. The weighted mean values of the left and the right fuzziness regions are expressed in terms of integrals. The average of the two weighted mean values multiplied by the exponential value of the altitude of the fuzzy number is referred to as the weight score of fuzziness for the fuzzy number and is used as a discriminatory tool for discriminating the fuzzy numbers. To use this technique, the membership functions of the fuzzy numbers need not be normal and linear. The suggested technique shows noticeable consistency, intuitiveness, and computational easiness. The advantages of the suggested ordering technique are illustrated through several numerical examples and comparisons with the published approaches. Further, an effort has been made to apply the proposed method to solve a multi-criteria decision-making problem.

Apart from the introduction, the rest of the paper is divided into five sections. Section 2 briefly reviews the basic concept of the fuzzy number under the heading "Preliminaries". The proposed weight score of fuzziness and the ordering procedure are defined in section 3. Also, some attributes of the weight score of fuzziness are discussed with proof. Section 4 presents comparative studies in contrast to various commonly used ranking approaches using several model fuzzy numbers which are available in the literature. The application of the proposed method to solve a multi-criteria decision-making problem is demonstrated in section 5. Section 6 conveys conclusions.

2. Preliminaries

In this section, we review some basic definitions and notations relating to the present study. Liou and Wang [3] followed.

2.1. *Generalized fuzzy number*

A generalized fuzzy number is a fuzzy subset A of the real line R with a membership function $\mathcal{M}_A(x)$, possesses the following conditions for $a, b, c, d \in R, (a \leq b \leq c \leq d)$:

(i) $\mathcal{M}_A(x)$ is a piece-wise continuous function from the real line R to the closed interval $[0, \omega]$ where ω is constant and $0 \leq \omega \leq 1$.

(ii) $\mathcal{M}_A(x) = 0$, for all $x \in]-\infty, a]$.

(iii) $\mathcal{M}_A(x)$ is strictly increasing on $[a, b]$.

(iv) $\mathcal{M}_A(x) = \omega$, for all $x \in [b, c]$.

(v) $\mathcal{M}_A(x)$ is strictly decreasing on $[c, d]$.

(vi) $\mathcal{M}_A(x) = 0$, for all $x \in [d, \infty[$.

Conveniently, the generalized fuzzy number is represented by $A = (a, b, c, d; \omega)$, and its membership function $\mathcal{M}_A(x)$ is expressed as

$$\mathcal{M}_A(x) = \begin{cases} \mathcal{M}_A^L(x)\,; & x \in [a, b] \\ \omega; & x \in [b, c] \\ \mathcal{M}_A^R(x)\,; & x \in [c, d] \\ 0; & Otherwise \end{cases}, \tag{1}$$

Where $\mathcal{M}_A^L(x): [a, b] \longrightarrow [0, \omega]$ and $\mathcal{M}_A^R(x): [c, d] \longrightarrow [0, \omega]$ are, respectively, known as the left and the right membership functions of the fuzzy number A. $\mathcal{M}_A^L(x)$ is continuous and strictly increasing on $[a, b]$, whereas $\mathcal{M}_A^R(x)$ is also continuous but strictly decreasing on $[c, d]$.

2.2. *Image of the generalized fuzzy number*

The image of a generalized fuzzy number $A = (a, b, c, \mathrm{d};\ \omega)$ with respect to the membership, axis is denoted by A' and defined as $A' = (-d, -c, -b, -a;\ \omega)$, where, $0 \leq \omega \leq 1$. Its membership function $\mathcal{M}_{A'}(x)$ can be expressed by

$$\mathcal{M}_{A'}(x) = \begin{cases} \mathcal{M}_{A'}^L(x)\,; & x \in [-d, -c] \\ \omega\,; & x \in [-c, -b] \\ \mathcal{M}_{A'}^R(x)\,; & x \in [-b, -a] \\ 0; & Otherwise \end{cases}, \tag{2}$$

Where $\mathcal{M}_{A'}^L(x): [-d, -c] \longrightarrow [0, \omega]$ and $\mathcal{M}_{A'}^R(x): [-b, -a] \longrightarrow [0, \omega]$ are respectively known as the left and the right membership functions of A'. $\mathcal{M}_{A'}^L(x)$ is continuous and strictly increasing on $[-d, -c]$, whereas $\mathcal{M}_{A'}^R(x)$ is also continuous but strictly decreasing on $[-b, -a]$.

2.3. *Generalized trapezoidal fuzzy number*

A generalized fuzzy number $A = (a, b, c, d; \omega)$ is said to be a generalized trapezoidal fuzzy number if its membership function $\mathcal{M}_A(x)$ is given by

$$\mathcal{M}_A(x) = \begin{cases} \mathcal{M}_A^L(x) = \omega \frac{x-a}{b-a}\ ; & x \in [a, b] \\ \omega\ ; & x \in [b, c] \\ \mathcal{M}_A^R(x) = \omega \frac{x-d}{c-d}\ ; & x \in [c, d] \\ 0; & Otherwise \end{cases} \tag{3}$$

Remark 1. The notation of fuzzy numbers with linear or non-linear membership functions and generalized (normalized and non-normalized) fuzzy numbers are the same. But they are characterized and identified differently by their respective membership functions.

2.4. *Generalized triangular fuzzy number*

A generalized fuzzy number $A = (a, b, c, d; \omega)$ is said to be a generalized triangular fuzzy number if $b = c$ and its membership function $\mathcal{M}_A(x)$is represented by

$$\mathcal{M}_A(x) = \begin{cases} \mathcal{M}_A^L(x) = \omega \frac{x-a}{b-a}\ ; & x \in [a, b] \\ \omega\ ; & x = b \\ \mathcal{M}_A^R(x) = \omega \frac{x-d}{b-d}\ ; & x \in [b, d] \\ 0; & otherwise \end{cases} \tag{4}$$

2.5. *Arithmetic operations on fuzzy number*

The arithmetic operations for any two fuzzy numbers $A_i = (a_i, b_i, c_i, d_i;\ \omega_i)$ and $A_j = \left(a_j, b_j, c_j, d_j;\ \omega_j\right)$, $0 \leq \omega_i, \omega_j \leq 1$ are defined as follows:

(i) Addition of fuzzy numbers

$$A_i \oplus A_j = (a_i, b_i, c_i, d_i;\ \omega_i) \oplus \left(a_j, b_j, c_j, d_j;\ \omega_j\right),$$
$$= \left(a_i + a_j, b_i + b_j, c_i + c_j, d_i + d_j;\ \min\{\omega_i, \omega_j\}\right).$$

(ii) Subtraction of fuzzy numbers

$$A_i \ominus A_j = (a_i, b_i, c_i, d_i;\ \omega_i) \ominus \left(a_j, b_j, c_j, d_j;\ \omega_j\right),$$
$$= \left(a_i - a_j, b_i - b_j, c_i - c_j, d_i - d_j;\ \min\{\omega_i, \omega_j\}\right).$$

(iii) Multiplication of fuzzy numbers

$$A_i \otimes A_j = (a_i, b_i, c_i, d_i;\ \omega_i) \otimes \left(a_j, b_j, c_j, d_j;\ \omega_j\right),$$
$$= \left(a_i \times a_j, b_i \times b_j, c_i \times c_j, d_i \times d_j;\ \min\{\omega_i, \omega_j\}\right).$$

(iv) Division of fuzzy numbers

$$A_i \oslash A_j = (a_i, b_i, c_i, d_i;\ \omega_i) \oslash \left(a_j, b_j, c_j, d_j;\ \omega_j\right),$$
$$= \left(\frac{a_i}{d_j}, \frac{b_i}{c_j}, \frac{c_i}{b_j}, \frac{d_i}{a_j};\ \min\{\omega_i, \omega_j\}\right).$$

(v) Multiplication by a scalar $'k'$

$$kA_i = \begin{cases} (ka_i, kb_i, kc_i, kd_i;\ \omega_i)\ ; & if\ k \geq 0 \\ (kd_i, kc_i, kb_i, ka_i;\ \omega_i)\ ; & if\ k < 0 \end{cases}$$

3. Weight Score of Fuzziness and Ordering Procedure

In this section, the proposed weight score of fuzziness and the ordering procedure are presented after a quick overview of the notion of weighted mean values of the left and the right fuzziness regions of a generalized fuzzy number.

3.1. *Weighted mean values of the fuzziness regions*

Let $\overline{w}_A^L$ and $\overline{w}_A^R$ denote the weighted mean values of the left and the right fuzziness regions of a generalized fuzzy number $A = (a, b, c, \mathrm{d};\ \omega)$ with its membership functions as defined in Eq. (1). Then, by Cheng [4]

$$\overline{w}_A^L = \frac{\int_a^b x\, \mathcal{M}_A^L(x)\, dx}{\int_a^b \mathcal{M}_A^L(x)\, dx}, \tag{5}$$

$$\overline{w}_A^R = \frac{\int_c^d x\, \mathcal{M}_A^R(x)\, dx}{\int_c^d \mathcal{M}_A^R(x)\, dx}\ . \tag{6}$$

Similarly, if $\overline{w}_{A'}^L$ and $\overline{w}_{A'}^R$ denote the weighted mean values of the left-right fuzziness areas of the image $A' = (-\mathrm{d}, -c, -b, -a;\ \omega)$, then

$$\overline{w}_{A'}^L = \frac{\int_{-d}^{-c} x\, \mathcal{M}_{A'}^L(x)\, dx}{\int_{-d}^{-c} \mathcal{M}_{A'}^L(x)\, dx}\ , \tag{7}$$

$$\overline{w}_{A'}^R = \frac{\int_{-b}^{-a} x\, \mathcal{M}_{A'}^R(x)\, dx}{\int_{-b}^{-a} \mathcal{M}_{A'}^R(x)\, dx}\ . \tag{8}$$

The visual representation of the weighted mean values of the fuzziness regions of the fuzzy number $A = (a, b, c, \mathrm{d};\ \omega)$ and its partnered image $A' = (-d, -c, -b, -a;\ \omega)$, are shown in Fig. 1.

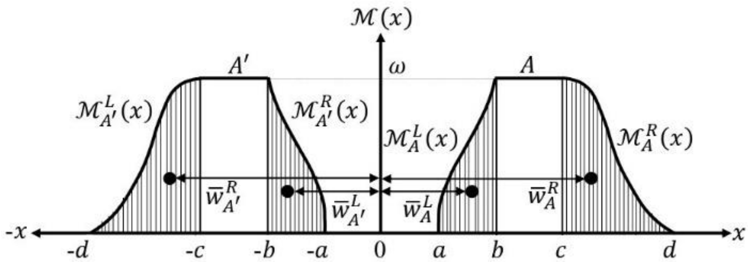

Fig. 1. Representation of the weighted mean values of the fuzzy number and its partnered image.

3.2. *Weight score of fuzziness*

Let $W_T(A)$ denotes the weight score of fuzziness of a generalized fuzzy number $A = (a, b, c, \mathrm{d};\ \omega)$, then we define $W_T(A)$ as

$$W_T(A) = \frac{1}{2}(\overline{w}_A^L + \overline{w}_A^R).\left[1 + \exp\left(\frac{\omega}{p}\right)\right]. \quad (9)$$

Where p is a non-vanishing ($p \neq 0$) rational number. Since the weighted mean values represent the location of the fuzzy number on the real axis, therefore, the value of p higher than one ($i.e.\ p > 1$) represents lower importance of the altitude (ω) to the location of the fuzzy number on the real axis. Thus, we generally have $p > 1$ for weight score computation to illustrate the proposed approach by numerical examples.
The weight score of fuzziness of the image $A' = (-d, -c, -b, -a;\ \omega)$ is expressed by

$$W_T(A') = \frac{1}{2}\left(\overline{w}_{A'}^L + \overline{w}_{A'}^R\right).\left[1 + \exp\left(\frac{\omega}{p}\right)\right]. \quad (10)$$

Remark 2. Let $E = \{A_1, A_2, A_3, \ldots\ldots.\ A_n\}$ is the set of fuzzy numbers whose membership function is defined by Eq. (1). Then, for any two fuzzy numbers $A_i = (a_i, b_i, c_i, \mathrm{d}_i;\ \omega_i),\ A_j = \left(a_j, b_j, c_j, \mathrm{d}_j;\ \omega_j\right) \in E$, the ranking decision can be made by using the weight score defined of the fuzzy number defined in Eq. (9) in the sense of a ranking function as follows,

$$\left.\begin{array}{lll} (i) & \text{If } W_T(A_i) > W_T(A_j), & \text{then } A_i \succ A_j \\ (ii) & \text{If } W_T(A_i) < W_T(A_j), & \text{then } A_i \prec A_j \\ (iii) & \text{If } W_T(A_i) = W_T(A_j), & \text{then } A_i \sim A_j \end{array}\right\}. \quad (11)$$

3.3. *Attributes of the weight score*

If $A_i = (a_i, b_i,\ c_i, d_i; \omega_i),\ i \in \overline{1,n}$ are generalized fuzzy numbers and $A_i' = (-d_i, -c_i, -b_i, -a_i; \omega_i)$ are their respective images, then,

(i) $\overline{w}_{A_i}^L = -\overline{w}_{A_i'}^R$ and $\overline{w}_{A_i}^R = -\overline{w}_{A_i'}^L$,

(ii) $W_T(A_i) = -W_T(A_i')$,

(iii) $W_T(A_i) \geq W_T(A_j) \Leftrightarrow W_T(A_i') \leq W_T(A_j')$,

(iv) $W_T(A_i) < W_T(A_j) \Leftrightarrow W_T(A_i') > W_T(A_j')$.

Proof. (i) Using Eq. (5) to Eq. (8) we have

$$\overline{w}_{A_i}^L = \frac{\int_{a_i}^{b_i} x\, \mathcal{M}_{A_i}^L(x)\, dx}{\int_{a_i}^{b_i} \mathcal{M}_{A_i}^L(x)\, dx} = -\frac{\int_{-b_i}^{-a_i} x\, \mathcal{M}_{A_i'}^R(x)\, dx}{\int_{-b_i}^{-a_i} \mathcal{M}_{A_i'}^R(x)\, dx} = -\overline{w}_{A_i'}^R,$$

and $$\overline{w}_{A_i}^R = \frac{\int_{c_i}^{d_i} x\, \mathcal{M}_{A_i}^R(x)\, dx}{\int_{c_i}^{d_i} \mathcal{M}_{A_i}^R(x)\, dx} = -\frac{\int_{-d_i}^{-c_i} x\, \mathcal{M}_{A_i'}^L(x)\, dx}{\int_{-d_i}^{-c_i} \mathcal{M}_{A_i'}^L(x)\, dx} = -\overline{w}_{A_i'}^L.$$

(ii) From Eq. (9) and Eq. (10) we have

$$W_T(A_i) = \frac{1}{2}\left(\overline{w}_{A_i}^L + \overline{w}_{A_i}^R\right).\left[1 + \exp\left(\frac{\omega}{p}\right)\right],$$

$$= -\frac{1}{2}\left(\overline{w}_{A_i'}^L + \overline{w}_{A_i'}^R\right).\left[1 + \exp\left(\frac{\omega}{p}\right)\right], \quad \text{by (i)}$$

$$= -W_T(A_i')\ .$$

(iii) Let $W_T(A_i) \geq W_T(A_j)$,

$\Leftrightarrow -W_T(A'_i) \geq -W_T(A'_j)$, by (ii)

$\Leftrightarrow W_T(A'_i) \leq W_T(A'_j)$.

(iv) Let $W_T(A_i) < W_T(A_j)$,

$\Leftrightarrow -W_T(A'_i) < -W_T(A'_j)$, by (ii)

$\Leftrightarrow W_T(A'_i) > W_T(A'_j)$.

Remark 3. Let $A_k = (a_k, b_k, c_k, d_k; \omega_k)$, $k = \overline{1, n}$ are the trapezoidal fuzzy numbers and $A'_k = (-d_k, -c_k, -b_k, -a_k; \omega_k)$ are their respective partnered images. Then, using the ordering Remark 2, and attributes of the weight score, the following statements can be made for pair-wise comparison of fuzzy numbers A_i, A_j and their respective images A'_i , A'_j for $i, j \in k$.

(i) $A_i \succ A_j$ if and only if $A'_i \prec A'_j$,

(ii) $A_i \prec A_j$ if and only if $A'_i \succ A'_j$,

(iii) $A_i \sim A_j$ if and only if $A'_i \sim A'_j$.

4. Comparative Numerical Examples

This section compares the ordering results of the proposed approach in contrast to existing ordering methods using several model fuzzy numbers which are available in the literature and very common for a wide range of comparative studies. Based on Remark 3, the weight score of fuzziness for the images does not need to be presented in the comparative Tables for their ordering. In examples 4.1 to 4.7, the detailed explanations of the proposed approach in contrast to the existing approaches are subsequently described. For weight score computations, the value of p is taken as $p = 10$ throughout the numerical study.

Example 4.1. Consider the following two sets of triangular and trapezoidal fuzzy numbers taken from Nasseri *et al.* [10].

Set A: $A_1 = (0.4, 0.5, 0.5, 1; 1)$, $A_2 = (0.4, 0.7, 0.7, 1; 1)$, $A_3 = (0.4, 0.9, 0.9, 1; 1)$,

Set B: $A_1 = (0.3, 0.4, 0.7, 0.9; 1)$, $A_2 = (0.3, 0.7, 0.7, 0.9; 1)$, $A_3 = (0.5, 0.7, 0.7, 0.9; 1)$

Figs. 2 and 3 are the visual representations of the membership functions of the fuzzy numbers in the above two sets of Ex. 4.1. Using formulae in Eq. (9), the weight score of fuzziness for the fuzzy numbers is obtained and displayed in Table 1.

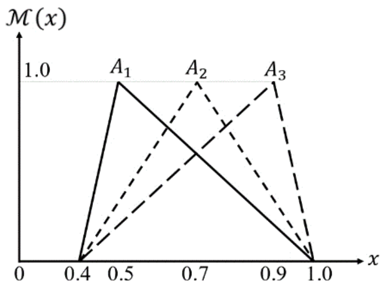

Fig. 2. Fuzzy numbers in set-A of Ex. 4.1.

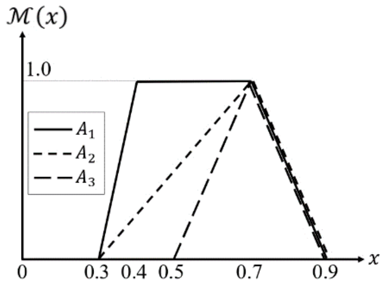

Fig. 3. Fuzzy numbers in set-B of Ex. 4.1.

Table 1. Comparative ordering results of the fuzzy numbers in Ex. 4.1.

Author	Fuzzy Numbers	Weight score	
		Set-A	Set-B
Cheng [4]	A_1	0.7901	0.7594
(distance)	A_2	0.8602	0.8150
	A_3	0.9269	0.8602
Ordering results		$A_1 \prec A_2 \prec A_3$	$A_1 \prec A_2 \prec A_3$
Abbasbandy	A_1	1.20	1.15
and Asady [6]	A_2	1.40	1.30
$(P = 1)$	A_3	1.60	1.40
Ordering results		$A_1 \prec A_2 \prec A_3$	$A_1 \prec A_2 \prec A_3$
	A_1	0.8869	0.8756
$(P = 2)$	A_2	1.0198	0.9522
	A_3	1.1605	1.0033
Ordering results		$A_1 \prec A_2 \prec A_3$	$A_1 \prec A_2 \prec A_3$
Asady and	A_1	0.60	0.575
Zendehnam [7]	A_2	0.70	0.65
	A_3	0.80	0.70
Ordering results		$A_1 \prec A_2 \prec A_3$	$A_1 \prec A_2 \prec A_3$
Abbasbandy	A_1	0.5333	0.5583
and Hajjari [9]	A_2	0.70	0.6834
	A_3	0.8667	0.70
Ordering results		$A_1 \prec A_2 \prec A_3$	$A_1 \prec A_2 \prec A_3$
Nasseri *et al.* [10]	A_1	1.6228	1.6281
	A_2	1.8174	1.7189
	A_3	2.0228	1.8615
Ordering results		$A_1 \prec A_2 \prec A_3$	$A_1 \prec A_2 \prec A_3$
Patra [30]	A_1	0.60	0.575
	A_2	0.70	0.40
	A_3	0.80	0.26
Ordering results		$A_1 \prec A_2 \prec A_3$	$A_3 \prec A_2 \prec A_1$
Proposed	A_1	1.19	1.19
Approach	A_2	1.47	1.40
	A_3	1.75	1.47
Ordering results		$A_1 \prec A_2 \prec A_3$	$A_1 \prec A_2 \prec A_3$

The detailed discussions of the ordering results are as follows:

Set A: The fuzzy numbers A_1, A_2, and A_3 in set-A are the approximation of 0.5, 0.7, and 0.9, respectively. The left and right spreads of all the fuzzy numbers are the same. Therefore, the intuitive order perception will be $A_1 \prec A_2 \prec A_3$. From Table 1, the ordering outcome of our proposed method is the same as the intuitive order perception and the ranking results of the other methods in Cheng [4], Abbasbandy and Asady [6], Asady and Zendehnam [7], Abbasbandy and Hajjari [9], Nasseri *et al*. [10] and Patra [30]. As a result, the proposed method can be used reliably to rank the fuzzy numbers.

Set B: Based on the left spreads of the fuzzy numbers A_1, A_2, and A_3 in set B, the logical order outcome will be $A_1 \prec A_2 \prec A_3$. From Table 1, the ordering results of our proposed technique are the same as the logical order outcome. Other approaches Cheng [4], Abbasbandy and Asady [6], Asady and Zendehnam [7], Abbasbandy and Hajjari [9], Nasseri *et al*. [10] also demonstrate the same ordering results, whereas Patra [30] infers counterintuitive ranking results as $A_3 \prec A_2 \prec A_1$. Hence, the proposed approach has intuitive ordering strength.

Example 4.2. Consider the following two sets of triangular and trapezoidal fuzzy numbers taken from Nasseri *et al*. [10].

Set A: $A_1 = (0.3, 0.5, 0.5, 0.7; 1)$, $A_2 = (0.3. 0.5, 0.8, 0.9; 1)$, $A_3 = (0.3, 0.5, 0.5, 0.9; 1)$,

Set B: $A_1 = (0, 0.4, 0.7, 0.8; 1)$, $A_2 = (0.2, 0.5, 0.5, 0.9; 1)$, $A_3 = (0.1, 0.6, 0.6, 0.8; 1)$.

Figs. 4 and 5 are the visual representations of the membership functions of the fuzzy numbers in the above two sets of Ex. 4.2.

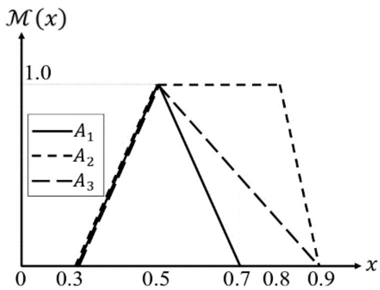

Fig. 4. Fuzzy numbers in set-A of Ex. 4.2.

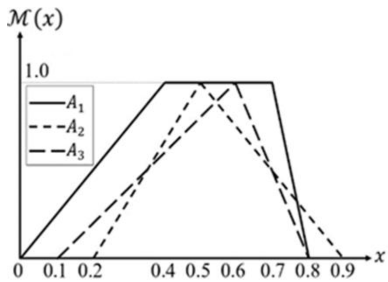

Fig. 5. Fuzzy numbers in set-B of Ex. 4.2.

The detailed discussions of the ordering results are as follows:

Set A: The intuitive and logical order perception will be $A_1 \prec A_3 \prec A_2$ on account of the left-right spreads of the fuzzy numbers A_1, A_2, and A_3 in set-A and their approximate values. From Table 2, the ordering results of the proposed approach are found to be the same as those of logical perception. Other approaches, Cheng [4], Abbasbandy and Asady [6], Asady and Zendehnam [7], Abbasbandy and Hajjari [9], Nasseri *et al*. [10], and Patra [30] also infer the same intuitive ordering outcomes. Hence, the proposed technique has a strong discrimination power.

Set B: Intuitive ordering perception among the fuzzy numbers in set B is not clear as in the last examples due to the overlapping of the fuzzy numbers. From Table 2, the proposed method yields the ordering result $A_1 \prec A_2 \prec A_3$, consistent with those of Nasseri *et al*. [10] and Abbasbandy and Asady [6] for $p = 2$. Other methods, Cheng [4], Abbasbandy and Asady [6] for $p = 1$, Asady and Zendehnam [7], Abbasbandy and Hajjari [9], and Patra [30] demonstrate ranking orders differently.

Table 2. Comparative ordering results of the fuzzy numbers in Ex. 4.2.

Author	Fuzzy Numbers	Weight Score	
		Set-A	Set-B
Cheng [4]	A_1	0.7071	0.7015
(distance)	A_2	0.8025	0.7257
	A_3	0.7458	0.7242
ordering results		$A_1 \prec A_3 \prec A_2$	$A_1 \prec A_3 \prec A_2$
Abbasbandy	A_1	1.00	0.95
and Asady [6]	A_2	1.25	1.05
$(P = 1)$	A_3	1.10	1.05
ordering results		$A_1 \prec A_3 \prec A_2$	$A_1 \prec A_2 \sim A_3$
	A_1	0.7257	0.7853
$(P = 2)$	A_2	0.9416	0.7958
	A_3	0.8165	0.7979
ordering results		$A_1 \prec A_3 \prec A_2$	$A_1 \prec A_2 \prec A_3$
Asady and	A_1	0.50	0.475
Zendehnam [7]	A_2	0.625	0.525
	A_3	0.55	0.525
ordering results		$A_1 \prec A_3 \prec A_2$	$A_1 \prec A_2 \sim A_3$
Abbasbandy	A_1	0.5001	0.5250
and Hajjari [9]	A_2	0.6417	0.5083
	A_3	0.5167	0.5750
ordering results		$A_1 \prec A_3 \prec A_2$	$A_2 \prec A_1 \prec A_3$
Nasseri *et al*. [10]	A_1	1.4617	1.3935
	A_2	1.7281	1.4414
	A_3	1.5189	1.4447
ordering results		$A_1 \prec A_3 \prec A_2$	$A_1 \prec A_2 \prec A_3$
Patra [30]	A_1	0.185	0.475
	A_2	0.625	0.296
	A_3	0.338	0.298
ordering results		$A_1 \prec A_3 \prec A_2$	$A_2 \prec A_3 \prec A_1$
Proposed	A_1	1.05	1.05
Approach	A_2	1.33	1.09
	A_3	1.12	1.16
ordering results		$A_1 \prec A_3 \prec A_2$	$A_1 \prec A_2 \prec A_3$

Example 4.3. Consider the following two symmetrical triangular fuzzy numbers, given in Liou and Wang [3], Wang and Lee [8], and Nasseri *et al*. [10].

$$A_1 = (3, 5, 5, 7; 1), \qquad A_2 = (3, 5, 5, 7; 0.8),$$

Fig. 6 represents the visual representation of the membership functions of these two fuzzy numbers.

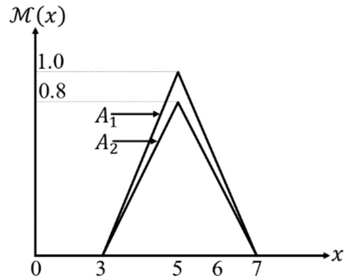

Fig. 6. Visual representation of the fuzzy numbers of Ex. 4.3.

Using formulae in Eq. (9), the weight score of fuzziness for the fuzzy numbers is obtained and displayed in Table 3.

Table 3. Comparative ordering results of the fuzzy numbers in Ex. 4.3.

Author	Weight score		Ordering results
	A_1	A_2	
Abbasbandy and Asady [6]			
$P = 1$	10.00	10.00	$A_1 \sim A_2$
$P = 2$	7.257	7.257	$A_1 \sim A_2$
Asady and Zendehnam [7]	5.00	5.00	$A_1 \sim A_2$
Abbasbandy and Hajjari [9]	5.00	5.00	$A_1 \sim A_2$
Wang and Lee [8]	0.50	0.40	$A_2 \prec A_1$
Nasseri *et al.* [10]	9.70	9.66	$A_2 \prec A_1$
Patra [30]	5.00	4.90	$A_2 \prec A_1$
Proposed method	10.53	10.42	$A_2 \prec A_1$

From Fig. 6, we can see that the two triangular fuzzy numbers A_1 and A_2 are symmetrical about the line $x = 5$ and have the same support but different weights. Therefore, based on weights, the intuitive perception will be $A_2 \prec A_1$. From Table 3, the ordering results of the proposed method coincide with intuitive perception. In other methods, Wang and Lee [8], Nasseri *et al.* [10], and Patra [30] also demonstrate intuitive ordering results. However, methods in Abbasbandy and Asady [6], Asady and Zendehnam [7], and Abbasbandy and Hajjari [9] are inconsistent to discriminate the fuzzy numbers A_1 and A_2 and yield $A_1 \sim A_2$.

Example 4.4. Consider the following three trapezoidal fuzzy numbers, given in Liou and Wang [3], Wang and Lee [8], and Nasseri *et al.* [10].

$B_1 = (5, 7, 9, 10; 1)$, $B_2 = (6, 7, 9, 10; 0.6)$ and $B_3 = (7, 8, 9, 10; 0.4)$

Fig. 7 presents the visual representation of the membership functions of these trapezoidal fuzzy numbers.

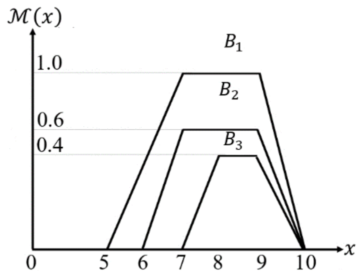

Fig. 7. Visual representation of the fuzzy numbers of Ex. 4.4.

Using formulae in Eq. (9), the weight score of fuzziness for the fuzzy numbers is obtained and displayed in Table 4.

Table 4. Comparative ordering results of the fuzzy numbers in Ex. 4.4.

Author	Weight score			Ordering results
	B_1	B_2	B_3	
Abbasbandy and Asady [6] $P = 1$	15.50	16.00	17.00	$B_1 \prec B_2 \prec B_3$
$P = 2$	11.255	11.518	12.110	$B_1 \prec B_2 \prec B_3$
Asady and Zendehnam [7]	7.75	8.00	8.50	$B_1 \prec B_2 \prec B_3$
Abbasbandy and Hajjari [9]	7.917	8.00	8.50	$B_1 \prec B_2 \prec B_3$
Wang and Lee [8]	7.714	8.00	8.5	$B_1 \prec B_2 \prec B_3$
Nasseri *et al*. [10]	15.34	15.84	16.80	$B_1 \prec B_2 \prec B_3$
Patra [30]	7.75	5.36	2.81	$B_1 \succ B_2 \succ B_3$
Proposed method	16.491	16.495	17.35	$B_1 \prec B_2 \prec B_3$

From Fig. 7, we see that the fuzzy numbers B_1, B_2, and B_3 have different supports and weights. In this regard, Wang and Lee [8] presented a modification to Chu and Tsao's method [5] and suggested that the importance of the degree of representative location on the real axis is higher than the average height of the fuzzy number and obtained the ordering results as $B_1 \prec B_2 \prec B_3$. From Table 4, the ordering outcome of the proposed method coincides with Wang and Lee [8]. Other methods, Abbasbandy and Asady [6], Asady and Zendehnam [7], Abbasbandy and Hajjari [9], and Nasseri *et al*. [10] also agree with Wang and Lee and conclude $B_1 \prec B_2 \prec B_3$. However, Patra [30] is incongruent with the others and yields $B_1 \succ B_2 \succ B_3$.

Example 4.5. Consider a pair of fuzzy triangular numbers $A_1 = (1, 4, 4, 5; 1)$ and $A_2 = (2, 3, 3, 6; 1)$ which are congruent and overlapped, as visualized in Fig. 8.

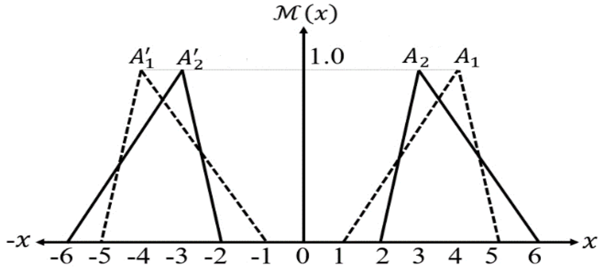

Fig. 8. Visual representation of the fuzzy numbers and their partnered images of Ex. 4.5.

Using formulae in Eq. (9), the weight scores $W_T(A_1)$ and $W_T(A_2)$ of fuzziness for the fuzzy numbers are obtained and displayed in Table 5.

Table 5. Comparative ordering results of the fuzzy numbers in Ex. 4.5.

Author	Weight score				Ordering results
	A_1	A_2	A'_1	A'_2	
Abbasbandy and Asady [6]					
$P = 1$	7.00	7.00	-7.00	-7.00	$A'_1 \sim A'_2 \prec A_2 \sim A_1$
$P = 2$	5.23	5.23	-5.23	-5.23	$A'_1 \sim A'_2 \prec A_2 \sim A_1$
Asady and Zendehnam [7]	3.50	3.50	-3.50	-3.50	$A'_1 \sim A'_2 \prec A_2 \sim A_1$
Abbasbandy and Hajjari [9]	3.83	3.17	-3.83	-3.17	$A'_1 \prec A'_2 \prec A_2 \prec A_1$
Nasseri *et al.* [10]	6.77	6.77	-7.22	-7.22	$A'_1 \sim A'_2 \prec A_2 \sim A_1$
Yu and Dat [11] (*Me*)	3.45	3.55	-3.45	-3.55	$A'_2 \prec A'_1 \prec A_1 \prec A_2$
Nguyen [17] $\lambda = 0.5$	11.67	12.83	-11.67	-12.83	$A'_2 \prec A'_1 \prec A_1 \prec A_2$
K. Patra [30]	3.50	3.50	-3.50	-3.50	$A'_1 \sim A'_2 \prec A_2 \sim A_1$
Proposed Method	7.72	7.02	-7.72	-7.02	$A'_1 \prec A'_2 \prec A_2 \prec A_1$

Fuzzy numbers are taken from Nguyen [14]. The partnered images $A'_1 = (-5, -4, -4, -1; 1)$ and $A'_2 = (-6, -3, -3, -2; 1)$ are on the left of the membership axis. There is an unclear situation for intuition to distinguish these fuzzy numbers due to overlapping after flipping and sliding. From Table 5, The ordering outcome of the proposed method is $A'_1 \prec A'_2 \prec A_2 \prec A_1$, consistent with those of Abbasbandy and Hajjari [9]. The ordering results of Yu and Dat [11], and Nguyen [17] are different and concluded $A'_2 \prec A'_1 \prec A_1 \prec A_2$. However, Abbasbandy and Asady [6], Asady and Zendehnam [7], Nasseri *et al*. [10], and Patra [30] are inconsistent in inferring any preference. Hence, the proposed approach is capable of ranking the fuzzy numbers and their images in an unclear situation for intuition.

Example 4.6. Consider the two sets of crisp numbers visualized in Fig. 9, which are considered by Nguyen [17].

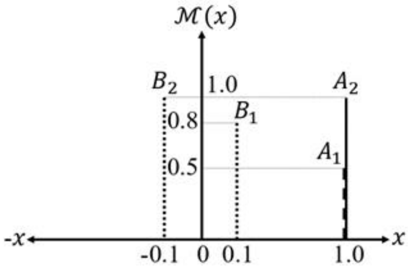

Fig. 9. Visual representation of crisp numbers of Ex. 4.6.

The first set consists $A_1 = (1, 1, 1, 1; 0.5)$ and $A_2 = (1, 1, 1, 1; 1.0)$ and the second consists $B_1 = (0.1, 0.1, 0.1, 0.1; 0.8)$, $B_2 = (-0.1, -0.1, -0.1, -0.1; 1)$. Using Eq. (9), the weight scores of fuzziness for these crisp numbers are obtained and found as $W_T(A_1) = 2.051$, $W_T(A_2) = 2.105$, $W_T(B_1) = 0.208$ and $W_T(B_2) = -0.211$, they scored as $W_T(A_1) < W_T(A_2)$ and $W_T(B_1) > W_T(B_2)$, therefore, A_1, A_2 and B_1, B_2 are ranked as $A_1 < A_2$ and $B_1 \succ B_2$. Rezvani [12], Chutia and Chutia [16], Nguyen [17], and Prasad and Sinha [31] all come up with the same ordering results, indicating that the proposed method is relevant with crisp numbers as well.

Example 4.7. Considering a triangular fuzzy number $A_1 = (1, 2, 2, 5; 1)$ and a general fuzzy number $A_2 = (1, 2, 2, 4; 1)$ with non-linear membership function $\mathcal{M}_{A_2}(x)$, given by

$$\mathcal{M}_{A_2}(x) = \begin{cases} \mathcal{M}^L_{A_2}(x) = \sqrt{1-(x-2)^2} \ ; & 1 \le x \le 2 \\ \mathcal{M}^R_{A_2}(x) = \sqrt{1-\frac{1}{4}(x-2)^2} \ ; & 2 \le x \le 4 \\ 0; & otherwise \end{cases},$$

The fuzzy numbers are taken from Liou and Wang [3]. The visual representation of their membership functions is shown in Fig. 10.

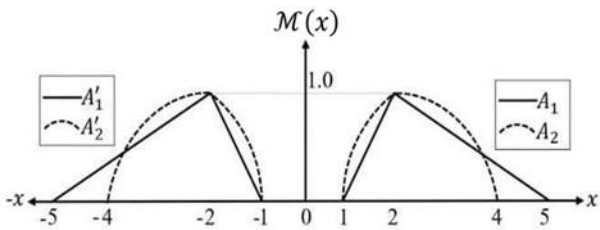

Fig. 10. Visual representation of the fuzzy numbers and their images of Ex. 4.7.

The intuitive perception can realize $A_1 \succ A_2$ $(A'_1 \prec A'_2)$ based on the right spreads. Using Eq. (5) and Eq. (6), we obtain the weighted mean values of the left-right fuzziness regions for the fuzzy number $A_2 = (1, 2, 2, 4; 1)$, as follows:

$$\overline{w}^L_{A_2} = \frac{\int_1^2 x\sqrt{1-(x-2)^2}\,dx}{\int_1^2 \sqrt{1-(x-2)^2}\,dx} = 1.5756,$$

and $$\overline{w}^R_{A_2} = \frac{\int_2^4 x\sqrt{1-\frac{1}{4}(x-2)^2}\,dx}{\int_2^4 \sqrt{1-\frac{1}{4}(x-2)^2}\,dx} = 2.8488.$$

Substituting these values of $\overline{w}^L_{A_2}$ and $\overline{w}^R_{A_2}$ in Eq. (9), the weight score $W_T(A_2)$ for the fuzzy number A_2 is obtained and displayed in Table 6. Using the formulae in Eq. (9), the weight score $W_T(A_1)$ for the fuzzy number A_1 is obtained and displayed in Table 6. Based on ranking Remark 2 and Remark 3, the ordering outcome of our proposed approach is found as $A_1 \succ A_2$ $(A'_1 \prec A'_2)$ which is in support of intuitive perception. From Table 6, we find that the ordering results of the proposed approach coincide with the neutral decision $(\alpha = 0.5)$ of Liou and Wang [3] and Nguyen [17]. The ranking results of Patra [30] also coincide with the proposed approach. Chutia and Chutia [16] are inconsistent with the proposed approach and with the other methods cited in Table 6. As a result, the suggested method has the potential to discriminate the fuzzy numbers with non-linear membership functions in addition to triangular and trapezoidal fuzzy numbers.

Table 6. Comparative ordering results of the fuzzy numbers in Ex. 4.7.

Author	Weight score				Ordering results
	A_1	A_2	A'_1	A'_2	
Liou and Wang [3], $\alpha = 0.5$	2.50	2.40	-2.50	-2.40	$A'_1 \prec A'_2 \prec A_2 \prec A_1$
Nguyen [17], $\alpha = 0.35$	6.67	5.80	-6.67	-5.80	$A'_1 \prec A'_2 \prec A_2 \prec A_1$
Chutia and Chutia [16], $\alpha = 0.5$	1.67	1.72	-1.67	-1.72	$A'_2 \prec A'_1 \prec A_1 \prec A_2$
K. Patra [30]	2.12	1.83	-2.12	-1.83	$A'_1 \prec A'_2 \prec A_2 \prec A_1$
Proposed method	4.91	4.66	-4.91	-4.66	$A'_1 \prec A'_2 \prec A_2 \prec A_1$

5. Application of the Proposed Method to Multi-Criteria Decision-Making (MCDM)

In this section, the method of solving multi-criteria decision-making problems based on the proposed ordering approach is presented and illustrated by an example. The procedure of solving the Fuzzy Multi-Criteria Decision-Making problem involves mainly two following steps,

(1) Finding of collective performance of each alternative concerning all criteria.

(2) Ordering of collective performances to obtain the preference of alternatives.

In the Fuzzy MCDM problem, the linguistic criterion values are represented by fuzzy numbers. Therefore, the collective performance of alternatives is fuzzy numbers. Thus, it is necessary for us to define an ordering procedure on the set of fuzzy numbers.

5.1. *Method for solving fuzzy multi-criteria decision-making problem*

Let $A = \{A_1, A_2, A_3, ... A_i, ... A_m\}$ be the set of m alternatives, and for each alternative, there are n different criteria $C_1, C_2, C_3, ... C_j ... C_n$. Let

$$D = [C_j(A_i)]_{m\times n} = \begin{array}{c|cccccc} & C_1 & C_2 & - & - & C_n \\ \hline A_1 & C_1(A_1) & C_2(A_1) & - & - & C_n(A_1) \\ A_2 & C_1(A_2) & C_2(A_2) & - & - & C_n(A_2) \\ | & | & | & | & | & | \\ | & | & | & | & | & | \\ A_m & C_1(A_m) & C_2(A_m) & - & - & C_n(A_m) \end{array}$$

be the multi-criteria decision matrix, where $C_j(A_i)$ represents the criterion value of the alternative A_i concerning the criteria C_j. Let w_j $(j = \overline{1,n})$ be the weight of the criterion C_j, respectively which indicates the relative importance of the criteria C_1, C_2, C_3, $...\ C_j\ ...C_n$ where $\sum_{j=1}^{n} w_j = 1$. The criterion values $C_j(A_i)$ of the alternatives in the multi-criteria decision matrix are not initially precise but rather considered logical linguistic values such as low, medium, high, etc., which are further expressed in terms of trapezoidal fuzzy numbers. The weights w_j of the criteria C_j are taken positive rational numbers such that $\sum_{j=1}^{n} w_j = 1$. The collective performance P_i $(i = \overline{1,m})$ of each alternative A_i concerning the criteria C_j $(j = 1, 2\ ..\ n)$ and their weight score $W_T(P_i)$ can be obtained step by step as follows:

Step 1a: Prepare the multi-criteria decision matrix $D = [C_j(A_i)]_{m\times n}$ with logical linguistic criterion values and then represent the linguistic criterion values in terms of generalized trapezoidal fuzzy numbers.

Step 1b: Assign the positive rational value to the weights w_j $(j = \overline{1,n})$ of the criteria C_j such that $\sum_{j=1}^{n} w_j = 1$.

Step 1c: The collective performance P_i of each alternative A_i concerning the criteria C_j $(j = 1, 2\ ..\ n)$ is obtained by the formulae as follows:

$$P_i = \sum_{j=1}^{n} w_j \otimes C_j(A_i) = (p_{i1}, p_{i2}, p_{i3}, p_{i4};\ \omega_i) \text{ (say)} \tag{12}$$

Since the criteria C_j and are expressed in terms of generalized trapezoidal fuzzy numbers, therefore, the performance score $P_i = (p_{i1}, p_{i2}, p_{i3}, p_{i4}; \omega_i)$ will also be a generalized trapezoidal fuzzy number.

Step 2: Using formulae in Eq. (9), determine the weight score $W_T(P_i)$ of each P_i and obtain the order. Performance score P_i with a higher weight score corresponding to a better alternative A_i.

5.2. *An illustrative example of multi-criteria decision-making (MCDM)*

In this section, a numerical example is presented to illustrate the application of the proposed method. Consider intuitively that a fund manager of a financial institution has to invest money in the stock market. After getting fundamental inquiry and investigation of different sectors of stocks, the fund manager chooses five sectors of stocks as alternatives for further evaluation to invest the money. They are

(1) Banking and services sector (A_1),
(2) Petroleum sector (A_2),
(3) Metal sector (A_3),
(4) Auto sector (A_4), and

(5) the Textile sector (A_5).

For future growth of the stock sectors, the fund manager fixes three beneficial criteria for the alternatives, which are

(1) Public demand (C_1),

(2) Political stability (C_2), and (3) Global peace (C_3).

Before investing money, the fund manager must prepare an order of preference among the five alternatives under the above three criteria. The main objective of this problem is to select the best choice among the five alternatives or to obtain an order of preference for investing. The decision procedure for the problem is in the following steps as follows.

Step 1a: After getting logical consideration, the fund manager prepares the following multi-criteria decision matrix with linguistic criterion values as below.

	C_1	C_2	C_3
A_1	Good	Good	Normal
A_2	Fairly very good	Normal	Very good
A_3	Normal	Fairly high	Very good
A_4	High	Good	Fairly good
A_5	Very good	Normal	Normal

Let the linguistic criterion values be represented by generalized trapezoidal fuzzy numbers as given in Table 7.

Table 7. Generalized trapezoidal fuzzy numbers representation of linguistic criterion values.

Linguistic criterion values	Generalized trapezoidal fuzzy numbers
Absolute low	(0, 0, 0, 0; 1)
Fairly Low	(0.02, 0.03, 0.04, 0.05; 1)
Low	(0.02, 0.04, 0.04, 0.05; 1)
Fairly normal	(0.35, 0.40, 0.60, 0.85; 1)
Normal	(0.40, 0.45, 0.65, 0.85; 1)
Fairly good	(0.40, 0.50, 0.70, 0.87; 1)
Good	(0.50, 0.57, 0.72, 0.87; 1)
Fairly very good	(0.60, 0.65, 0.75, 0.90; 1)
Very good	(0.70, 0.70, 0.75, 0.93; 1)
Fairly high	(0.80, 0.80, 0.85, 0.95; 1)
High	(0.90, 0.95, 0.95, 0.98; 1)
Absolute high	(1, 1, 1, 1; 1)

Using Table 7, the multi-criteria decision matrix is converted as follows:

	C1	C2	C3
A1	(0.50, 0.57, 0.72, 0.87; 1)	(0.50, 0.57, 0.72, 0.87; 1)	(0.40, 0.45, 0.65, 0.85; 1)
A2	(0.60, 0.65, 0.75, 0.90; 1)	(0.40, 0.45, 0.65, 0.85; 1)	(0.70, 0.70, 0.75, 0.93; 1)
A3	(0.40, 0.45, 0.65, 0.85; 1)	(0.80, 0.80, 0.85, 0.95; 1)	(0.70, 0.70, 0.75, 0.93; 1)
A4	(0.90, 0.95, 0.95, 0.98; 1)	(0.50, 0.57, 0.72, 0.87; 1)	(0.40, 0.50, 0.70, 0.87; 1)
A5	(0.70, 0.70, 0.75, 0.93; 1)	(0.40, 0.45, 0.65, 0.85; 1)	(0.40, 0.45, 0.65, 0.85; 1)

Step 1b: Let $w_1 = 0.3134,\ w_2 = 0.3364\ and\ w_3 = 0.3502$ are the weights of the criteria C_1, C_2, and C_3, respectively such that $w_1 + w_2 + w_3 = 1$.

Step 1c: Using Eq. (12), the collective performance P_i of each alternative A_i $(i = 1, 2, 3, 4)$ concerning all criteria C_1, C_2 and C_3 are obtained and displayed as follows:

Alternative	Collective performance, $P_i = \sum_{j=1}^{3} w_j \,.\, C_j(A_i)$
A_1	(0.46498, 0.527976, 0.695486, 0.862996; 1)
A_2	(0.56774, 0.60023, 0.71636, 0.893686; 1)
A_3	(0.63962, 0.65529, 0.75230, 0.911656; 1)
A_4	(0.59034, 0.664578, 0.785078, 0.904474; 1)
A_5	(0.49402, 0.52835,0.68134, 0.875072; 1)

Step 2: Using formulae in Eq. (9), the weight score of fuzziness $W_T(P_i)$ of each collective performance P_i is obtained, and the order of P_i $(i = 1, 2, 3, 4)$ is determined, which is given in Table 8.

Table 8. Weight score of each P_i and the ordering results.

Alternative	$W_T(P_i)$	Order
A_1	0.6296	
A_2	0.6825	
A_3	0.7278	$P_4 \succ P_3 \succ P_2 \succ P_5 \succ P_1$
A_4	0.7324	
A_5	0.6314	

From Table 8, the order of preference of alternatives is $A_4 \succ A_3 \succ A_2 \succ A_5 \succ A_1$ and hence A_4 is the best alternative to investing money.

6. Conclusion

The ordering of fuzzy numbers is hindered by inconsistency, counter-intuitiveness, and computational complexity. To lessen this dizziness, this paper defines the weight score of fuzziness regions as an ordering tool. According to comparative studies and investigations, the weight score shows noticeable ordering benefits in terms of consistency, intuitive support, and computational simplicity. The suggested method has four advantages in ordering fuzzy numbers, according to theoretical proofs and comparative reviews. To begin with, the ordering results validate human perception. Secondly, it ensures computational simplicity irrespective of the type of fuzzy numbers. Third, the proposed method has the potential to overcome the limitations of the existing methods that arise due to the compensation of areas. Finally, the suggested method provides a justifiable ordering preference for images. These properties are important in various fields, such as Multi-Criteria Decision-Making, Risk Analysis, Data Analysis, and Optimization Techniques.

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