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On a Pair of (σ, τ)-derivations of Semiprime Γ-rings

K. K. Dey[*] and A. C. Paul

Department of Mathematics, Rajshahi University, Rajshahi-6205, Bangladesh

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Abstract

Let M be a 2-torsion free Γ-ring satisfying an assumption and let σ,τ be centralizing epimorphisms on M. Let f and g be (σ, τ)-derivations on M such that $f(x)\alpha x + x\alpha g(x) = 0$ for all $x\in M$, $\alpha\in\Gamma$. Then we prove that $f(u)\beta[x, y]_\alpha = g(u)\beta[x, y]_\alpha = 0$ for all $x, y, u\in M$, $\alpha,\beta\in\Gamma$ and f, g map M into its center.

Keywords. Epimorphism; Commuting; Map; Centralizing map; α-derivation; (α, β)-derivation; Prime Γ-ring; Semiprime Γ-ring.

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1. Introduction

Let M and Γ be additive abelian groups. M is called a Γ-ring if for all $x,y,z\in M$, $\alpha,\beta\in\Gamma$ the following conditions are satisfied:

(i) $x\beta y\in M$,

(ii) $(x + y)\alpha z = x\alpha z + y\alpha z$, $x(\alpha + \beta)y = x\alpha y + x\beta y$,
$x\alpha(y + z) = x\alpha y + x\alpha z$,

(iii) $(x\alpha y)\beta z = x\alpha(y\beta z)$.

For any $x, y\in M$, the notation $[x, y]_\alpha$ and $(x, y)_\alpha$ will denote $x\alpha y - y\alpha x$ and $x\alpha y + y\alpha x$ respectively. We know that $[x\beta y, z]_\alpha = x\beta[y, z]_\alpha + [x, z]_\alpha\beta y + x[\beta,\alpha]_z y$ and $[x, y\beta z]_\alpha = y\beta[x, z]_\alpha + [x,y]_\alpha\beta z + y[\beta,\alpha]_x z$, for all $x,y,z\in M$ and for all $\alpha,\beta\in\Gamma$. We shall take an assumption (*) $x\alpha y\beta z = x\beta y\alpha z$ for all $x,y,z\in M$, $\alpha,\beta\in\Gamma$. Using this assumption the identities $[x\beta y, z]_\alpha = x\beta[y,z]_\alpha + [x,z]_\alpha\beta y$ and $[x,y\beta z]_\alpha = y\beta[x,z]_\alpha + [x,y]_\alpha\beta z$, for all $x,y,z\in M$ and for all $\alpha,\beta\in\Gamma$ are used extensively in our results. An additive mapping d from M into itself is called a derivation if $d(x\alpha y) = x\alpha d(y) + d(x)\alpha y$ for all $x, y\in M$, $\alpha\in\Gamma$. A mapping f from M into itself is commuting if $[f(x), x]_\alpha = 0$, and centralizing if $[f(x), x]_\alpha \in Z(M)$ for all $x\in M$, $\alpha\in\Gamma$. We call a mapping $f : M \to M$ central if $f(x)\in Z(M)$ for all $x\in M$. Recall that if f is an additive commuting mapping from M into itself, then a linearization of $[f(x), x]_\alpha = 0$ yields $[f(x), y]_\alpha = [x, f(y)]_\alpha$ for all $x, y\in M$, $\alpha\in\Gamma$.

[*] *Corresponding author*: kkdmath@yahoo.com

Let σ, τ be mappings of M into itself. An additive mapping d of M into itself is called a (σ, τ)-derivation if $d(x\alpha y) = \sigma(x)\alpha d(y) + d(x)\alpha\tau(y)$ for all $x, y\in M$, $\alpha\in\Gamma$. If $\tau = 1$, where 1 is the identity mapping of M, then d is called a σ-derivation or a (σ, 1)-derivation or a skew-derivation. Of course, a (1, 1)-derivation or a 1-derivation is a derivation.

In classical ring theories, Chaudhry and Thaheem [1] worked on (α, β)-derivations in semiprime rings. Quite a few Mathematicians studied (α, β) or (σ, τ)-derivations in prime and semiprime rings and they obtained some fruitful results in these fields.

In this paper we work on semiprime Γ-rings with a pair of (σ, τ)-derivations. Some characterizations are obtained relating to (σ, τ)-derivations.

2. The Results

First we prove the following lemma.

Lemma 2.1 Let T be an endomorphism of the prime Γ-ring M, and let I be a nonzero left ideal of M. Then

(i) if $T(r) = r$ for all $r\in I$, T is the identity map on M,

(ii) if T is one-to-one on I, it is one-to-one on M.

Proof

(i) For arbitrary $x\in M$ and $r\in I$, $x\alpha r = T(x\alpha r) = T(x)\alpha T(r) = T(x)\alpha r$, $\alpha\in\Gamma$, hence $(x - T(x))\alpha r = 0$. Thus we have $(x - T(x))\alpha y\beta r = 0$, $x,y\in M$, $\alpha,\beta\in\Gamma$, and therefore by the primeness of M we get, $x = T(x)$ for all $x\in M$.

(ii) Observe that $\ker(T)\Gamma I \subseteq \ker(T) \cap I = \{0\}$, and since $I \neq \{0\}$, $\ker(T) = \{0\}$.

Lemma 2.2 Let $I \neq \{0\}$ be a left ideal of the semiprime Γ-ring M satisfying the condition (*). If T is an endomorphism of M which is centralizing on I, then T is commuting on I.

Proof

Linearizing the condition that $[x, T(x)]_\alpha \in Z$ for all $x\in I$, $\alpha\in\Gamma$, we obtain

$$[x, T(y)]_\alpha + [y, T(x)]_\alpha \in Z \text{ for all } x,y\in I,\ \alpha\in\Gamma. \quad (1)$$

Replacing y by $x\beta x$ in (1) we then get $[x, T(x\beta x)]_\alpha + [x\beta x, T(x)]_\alpha$

$= x\beta[x, T(x)]_\alpha + [x, T(x)]_\alpha\beta x + [x, T(x)\beta T(x)]_\alpha$

$= x\beta[x, T(x)]_\alpha + [x, T(x)]_\alpha\beta x + T(x)\beta[x, T(x)]_\alpha + [x, T(x)]_\alpha\beta T(x)$

$= x\beta[x, T(x)]_\alpha + x\beta[x, T(x)]_\alpha + T(x)\beta[x, T(x)]_\alpha + T(x)\beta[x, T(x)]_\alpha$

$= 2x\beta[x, T(x)]_\alpha + 2T(x)\beta[x, T(x)]_\alpha \in Z$ for all $x\in I$, $\alpha,\beta\in\Gamma$,

and since the first summand commutes with x, we have

$2[T(x)\beta[x, T(x)]_\alpha, x]_\alpha = 0$, from which it follows that

$2[T(x), x]_\alpha\beta[x, T(x)]_\alpha + 2T(x)\beta[[x, T(x)]_\alpha, x]_\alpha$

$= 2[x, T(x)]_\alpha\beta[x, T(x)]_\alpha = 0$ for all $x\in I$, $\alpha,\beta\in\Gamma$. Since the center of a semiprime Γ-ring contains no nonzero nilpotent elements, we conclude that

$$2[x, T(x)]_\alpha = 0 \text{ for all } x\in I, \alpha\in\Gamma, \quad (2)$$

and hence

$$2([x, T(y)]_\alpha + [y, T(x)]_\alpha) = 0 \text{ for all } x,y\in I, \alpha\in\Gamma. \quad (3)$$

Now, we have,

$$\begin{aligned}
&[x\beta y + y\beta x, T(x)]_\alpha + [x\beta x, T(y)]_\alpha \\
&= [x\beta y, T(x)]_\alpha + [y\beta x, T(x)]_\alpha + [x\beta x, T(y)]_\alpha \\
&= x\beta[y, T(x)]_\alpha + [x, T(x)]_\alpha\beta y + y\beta[x, T(x)]_\alpha + [y, T(x)]_\alpha\beta x + x\beta[x, T(y)]_\alpha + [x, T(y)]_\alpha\beta x \\
&= x\beta[y, T(x)]_\alpha + y\beta[x, T(x)]_\alpha + y\beta[x, T(x)]_\alpha + x\beta[y, T(x)]_\alpha + x\beta[x, T(y)]_\alpha + x\beta[x, T(y)]_\alpha \\
&= x\beta[y, T(x)]_\alpha + 2y\beta[x, T(x)]_\alpha + x\beta[y, T(x)]_\alpha + x\beta[x, T(y)]_\alpha + x\beta[x, T(y)]_\alpha \\
&= 2x\beta([y, T(x)]_\alpha + [x, T(y)]_\alpha) + 2y\beta[x, T(x)]_\alpha
\end{aligned}$$

Applying (2) and (3), we get the identity

$$[x\beta y + y\beta x, T(x)]_\alpha + [x\beta x, T(y)]_\alpha = 0 \text{ for all } x,y\in I, \alpha,\beta\in\Gamma. \quad (4)$$

For $x\in I$, take $y = T(x)\delta x\beta x$ in (4), thereby obtaining

$$\begin{aligned}
&[x\beta T(x)\delta x\beta x + T(x)\delta x\beta x\beta x, T(x)]_\alpha + [x\beta x, T(T(x)\delta x\beta x)]_\alpha \\
&= x\beta T(x)\beta[x\beta x, T(x)]_\alpha + [x\beta T(x), T(x)]_\alpha\beta x\beta x + T(x)\delta x\beta[x\beta x \\
&= T(x)]_\alpha + [T(x)\delta x, T(x)]_\alpha\beta x\beta x + T(T(x))\beta[x\beta x, T(x)\beta T(x)]_\alpha + [x\beta x, T(T(x))]_\alpha\beta T(x)\beta T(x) \\
&= x\beta T(x)\beta[x\beta x, T(x)]_\alpha + [x\beta T(x), T(x)]_\alpha\beta x\beta x + T(x)\delta x\beta[x\beta x, T(x)]_\alpha + [T(x)\delta x, T(x)]_\alpha\beta x\beta x \\
&\quad + T(T(x))\beta[x\beta x, T(x)\beta T(x)]_\alpha + [x\beta x, T(T(x))]_\alpha\beta T(x)\beta T(x) \\
&= 0, \qquad \text{for all } x,y\in I, \alpha,\beta\in\Gamma.
\end{aligned}$$

Now

$$\begin{aligned}
&[x\beta x, T(x)]_\alpha = x\beta[x, T(x)]_\alpha + [x, T(x)]_\alpha\beta x \\
&= x\beta[x, T(x)]_\alpha + x\beta[x, T(x)]_\alpha = 2x\beta[x, T(x)]_\alpha = 0, \text{ for all } x,y\in I, \alpha,\beta\in\Gamma \quad (5)
\end{aligned}$$

Replacing $y = T(x)$ in above relation, we get for all $x\in I$, $\alpha,\beta\in\Gamma$,

$$[x\beta T(x) + T(x)\beta x, T(x)]_\alpha\beta T(x)\beta T(x) + [x\beta x, T(T(x)]_\alpha\beta T(x)\beta T(x) = 0 \quad (6)$$

Replacing y by $T(x)$ in (4), we get,

$$\begin{aligned}
&[x\beta T(x) + T(x)\beta x, T(x)]_\alpha = x\beta[T(x), T(x)]_\alpha + [x, T(x)]_\alpha\beta T(x) + T(x)\beta[x, T(x)]_\alpha \\
&+ [T(x), T(x)]_\alpha\beta x \\
&= [x, T(x)]_\alpha\beta T(x) + T(x)\beta[x, T(x)]_\alpha \\
&= T(x)\beta[x, T(x)]_\alpha + T(x)\beta[x, T(x)]_\alpha, \\
&= 2T(x)\beta[x, T(x)]_\alpha = 0, \quad \text{for all } x,y\in I, \alpha,\beta\in\Gamma.
\end{aligned}$$

So we get from (6) for all $x\in I$, $\alpha,\beta\in\Gamma$,

$$[x\beta x, T(T(x))]_\alpha\beta T(x)\beta T(x) = 0 \quad (7)$$

On the other hand, taking $y = T(x)\delta x$ in (4) yields

$[x\beta T(x)\delta x + T(x)\delta x\beta x, T(x)]_\alpha + [x\beta x, T(T(x)\delta x)]_\alpha$

$= [x\beta T(x)\delta x + T(x)\delta x\beta x, T(x)]_\alpha + [x\beta x, T(T(x))\delta T(x)]_\alpha$

$= [x\beta T(x)\delta x + T(x)\delta x\beta x, T(x)]_\alpha + [x\beta x, T(T(x)\delta x)]_\alpha$,

Hence

$[([x, T(x)]_\alpha + 2T(x)\beta x), T(x)]_\alpha\beta T(T(x)) + [x\beta x, T(x)]_\alpha\beta T(x) + [x\beta x, T(x)]_\alpha = 0$

Or, $[x, T(x)]_\alpha\beta[x, T(x)]_\alpha + [x\beta x, T(T(x))]_\alpha\beta T(x) = 0$ for all $x\in I$, $\alpha,\beta\in\Gamma$ (8)

From (8) it follows that $w = [x\beta x, T(T(x))]_\alpha\beta T(x)$ is central, and from (7) that $w\gamma w = 0$. It is now apparent from (8) that $[x, T(x)]_\alpha\beta[x, T(x)]_\alpha\gamma[x, T(x)]_\alpha\beta[x, T(x)]_\alpha = 0$, and the absence of nonzero central nilpotent elements implies that $[x, T(x)]_\alpha = 0$ for all $x\in I$, $\alpha\in\Gamma$.

Lemma 2.3

Let M be a semiprime Γ-ring satisfying the condition (*). Let $a\beta[x, y]_\alpha = 0$, for $a,x,y\in M$, $\alpha,\beta\in\Gamma$, then $a\in Z(M)$.

Proof

Since $a\beta[x, y]_\alpha = 0$, for $a,x,y\in M$, $\alpha,\beta\in\Gamma$, then replace y by a, we get $a\beta[x, a]_\alpha = 0$, for $a,x\in M$, $\alpha,\beta\in\Gamma$. Thus we get $a\beta x\alpha a = a\beta a\alpha x$, for all $a,x\in M$, $\alpha,\beta\in\Gamma$.

Now $[a, x]_\alpha\beta[a, y]_\alpha = (a\alpha x - x\alpha a)\beta(a\alpha y - y\alpha a)$

$= a\alpha x\beta a\alpha y - a\alpha x\beta y\alpha a - x\alpha a\beta a\alpha y + x\alpha a\beta y\alpha a$

$= a\alpha(x\beta a)\alpha y - a\alpha(x\beta y)\alpha a - x\alpha a\beta a\alpha y + x\alpha a\beta(y\alpha a)$

$= a\alpha a\beta x\alpha y - a\alpha a\alpha x\beta y - x\alpha a\beta a\alpha y + x\alpha a\beta a\alpha y$

$= a\alpha a\beta x\alpha y - a\alpha a\alpha x\beta y = a\alpha a\beta x\alpha y - a\alpha a\beta x\alpha y = 0$, for all $a,x,y\in M$, $\alpha,\beta\in\Gamma$.

Hence $[a, x]_\alpha\beta[a, y]_\alpha = 0$, for all $a,x,y\in M$, $\alpha,\beta\in\Gamma$.

Replace y by $y\delta x$, we get,

$[a, x]_\alpha\beta[a, y\delta x]_\alpha = [a, x]_\alpha\beta y\delta[a, x]_\alpha + [a, x]_\alpha\beta[a, y]_\alpha\delta x = [a, x]_\alpha\beta y\delta[a, x]_\alpha = 0$, for all $a,x,y\in M$, $\alpha,\beta,\delta\in\Gamma$. By the semiprimeness of M we get, $[a, x]_\alpha = 0$, for all $a,x\in M$, $\alpha\in\Gamma$.

Hence $a\in Z(M)$, for all $a\in M$.

Lemma 2.4 Let σ,τ be epimorphisms of a semiprime Γ-ring M satisfying the assumption (*) and such that τ is centralizing. If d is a commuting (σ, τ)-derivation of M, then $[x, y]_\alpha\beta d(u) = 0 = d(u)\beta[x, y]_\alpha$ for all $x, y, u\in M$, $\alpha,\beta\in\Gamma$, in particular, d maps M into its center.

Proof

Since τ is a centralizing epimorphism, by Lemma 2.2 τ is commuting. Then we have $[\tau(x), x]_\alpha = 0$ and $[d(x), x]_\alpha = 0$, for all $x\in M$, $\alpha\in\Gamma$.

Thus $[\tau(x), y]_\alpha = [x, \tau(y)]_\alpha$. Also, $[d(x), y]_\alpha = [x, d(y)]_\alpha$ for all $x, y\in M$, $\alpha\in\Gamma$.

We consider

$$[d(y\beta x), x]_\alpha = [y\beta x, d(x)]_\alpha = y\beta[x, d(x)]_\alpha + [y, d(x)]_\alpha\beta x = [y, d(x)]_\alpha\beta x \quad (9)$$

and

$$[d(y\beta x), x]_\alpha = [\sigma(y)\beta d(x) + d(y)\beta\tau(x), x]_\alpha$$

$$= \sigma(y)\beta[d(x), x]_\alpha + [\sigma(y), x]_\alpha\beta d(x) + d(y)\beta[\tau(x), x]_\alpha + [d(y), x]_\alpha\beta\tau(x)$$

$$= [\sigma(y), x]_\alpha\beta d(x) + [d(y), x]_\alpha\beta\tau(x), \text{ for } x, y\in M, \alpha,\beta\in\Gamma \quad (10)$$

From (9) and (10), we get $[y, d(x)]_\alpha\beta x = [\sigma(y), x]_\alpha\beta d(x) + [d(y), x]_\alpha\beta\tau(x)$

Thus $[y, d(x)]_\alpha\beta x - [x, d(y)]_\alpha\beta\tau(x) = [\sigma(y), x]_\alpha\beta d(x)$, for all $x, y\in M, \alpha,\beta\in\Gamma$.

$$[y, d(x)]_\alpha\beta x - [y, d(x)]_\alpha\beta\tau(x) = [\sigma(y), x]_\alpha\beta d(x), \text{ for all } x, y\in M, \alpha,\beta\in\Gamma.$$

$$[y, d(x)]_\alpha\beta(x - \tau(x)) = [y, \sigma(x)]_\alpha\beta d(x), \quad \text{for all } x, y\in M, \alpha,\beta\in\Gamma \quad (11)$$

We further consider

$$[x, \tau(y\beta x)]_\alpha = [x, \tau(y)]_\alpha\beta\tau(x), \quad (12)$$

Again,

$$[x, \tau(y\beta x)]_\alpha = [\tau(x), y\beta x]_\alpha = [x, \tau(y)]_\alpha\beta x + \tau(y)\beta[x, \tau(x)]_\alpha. = [x, \tau(y)]_\alpha\beta x \quad (13)$$

From (12) and (13), we get $[x, \tau(y)]_\alpha\beta\tau(x) = [x, \tau(y)]_\alpha\beta x$. Since τ is onto, we get

$$[x, y]_\alpha\beta\tau(x) = [x, y]_\alpha\beta x \quad \text{for all } x, y\in M, \alpha,\beta\in\Gamma. \quad (14)$$

Replacing y by $d(y)$ in (14), we have

$$[x, d(y)]_\alpha\beta\tau(x) = [x, d(y)]_\alpha\beta x \text{ for all } x, y\in M, \alpha,\beta\in\Gamma$$

$$[x, d(y)]_\alpha\beta x - [x, d(y)]_\alpha\beta\tau(x) = 0$$

$$[x, d(y)]_\alpha\beta(x - \tau(x)) = [d(x), y]_\alpha\beta(x - \tau(x)) = 0 \quad (15)$$

Using (15), from (11) we get $[\sigma(y), x]_\alpha\beta d(x) = 0$. Since σ is onto, we get

$$[y, x]_\alpha\beta d(x) = 0 \text{ for all } x, y\in M, \alpha,\beta\in\Gamma \quad (16)$$

Replacing y by $y\delta z$ in (16), we get $y\delta[z, x]_\alpha\beta d(x) + [y, x]_\alpha\delta z\beta d(x) = 0$, which along with (16) yields

$$[y, x]_\alpha\delta z\beta d(x) = 0 \text{ for all } x, y, z\in M, \alpha,\beta,\delta\in\Gamma \quad (17)$$

Linearizing (16) (in x), we get

$$[y, x + u]_\alpha\beta d(x + u) = 0 \text{ for all } x, y\in M, \alpha,\beta\in\Gamma$$

$$[y, x]_\alpha\beta d(x) + [y, x]_\alpha\beta d(u) + [y, u]_\alpha\beta d(x) + [y, u]_\alpha\beta d(u) = 0 \text{ for all } x,y,u\in M, \alpha,\beta\in\Gamma.$$

$$[y, x]_\alpha\beta d(u) = [u, y]_\alpha\beta d(x) \text{ for all } x, y, u\in M, \alpha,\beta\in\Gamma \quad (18)$$

Replacing z by $d(u)\lambda z\delta[u, y]_\alpha$ in (17) and using (18), we have

$$0 = [y, x]_\alpha\beta d(u)\lambda z\delta[u, y]_\alpha\beta d(x) = [y, x]_\alpha\beta d(u)\lambda z\delta[y, x]_\alpha\beta d(u).$$

The semiprimeness of M implies

$$[y, x]_\alpha\beta d(u) = 0 \text{ for all } x, y, u\in M, \alpha,\beta\in\Gamma \quad (19)$$

Substituting $y\delta z$ for y in (19), we have $[y, x]_\alpha \delta z \beta d(u) = 0$, and so $d(u)\beta[y, x]_\alpha \delta z \beta d(u)\beta[y, x]_\alpha = 0$. Since M is semiprime, we get $d(u)\beta[y, x]_\alpha = 0$ for all $x, y, u \in M$, $\alpha, \beta \in \Gamma$. Thus $[x, y]_\alpha \beta d(u) = 0 = d(u)\beta[x, y]_\alpha$ for all $x, y, u \in M$, $\alpha, \beta \in \Gamma$, and further $d(u) \in Z(M)$.

Now we prove our main result.

Theorem 2.5. Let M be a 2-torsion free semiprime Γ-ring satisfying the assumption (*) and σ, τ be centralizing epimorphisms of M. Let f, g be (σ, τ)-derivations of M such that

$$f(x)\alpha x + x\alpha g(x) = 0 \text{ for all } x \in M, \alpha \in \Gamma. \tag{20}$$

Then $g(u)\beta[x, y]_\alpha = f(u)\beta[x, y]_\alpha = 0$ for all $x, y, u \in M$, $\alpha, \beta \in \Gamma$ and f, g map M into its center.

Proof

Since σ, τ are centralizing epimorphisms, they are commuting by Lemma 2.2 and hence $\sigma - 1$ is a commuting σ-derivation and $\tau - 1$ is a commuting τ-derivation. Thus by Lemma 2.3 we get

$$\sigma(u) - u \in Z(M),\ \sigma(u)\beta[x, y]_\alpha = u\beta[x, y]_\alpha \text{ and}$$
$$[x, y]_\alpha \beta \sigma(u) = [x, y]_\alpha \beta u \text{ for all } x, y, u \in M, \alpha, \beta \in \Gamma \tag{21}$$

and for all $x, y, u \in M$, $\alpha, \beta \in \Gamma$,

$$\tau(u) - u \in Z(M),\ \tau(u)\beta[x, y]_\alpha = u\beta[x, y]_\alpha \text{ and } [x, y]_\alpha \beta \tau(u) = [x, y]_\alpha \beta u \tag{22}$$

Linearizing (20), we get

$$f(x)\alpha y + f(y)\alpha x + x\alpha g(y) + y\alpha g(x) = 0 \text{ for all } x, y \in M, \alpha \in \Gamma \tag{23}$$

Replacing y by $y\beta x$ in (23) and using (21), we get

$$\begin{aligned}
0 &= f(x)\alpha y\beta x + \sigma(y)\beta f(x)\alpha x + f(y)\beta\tau(x)\alpha x + x\alpha\sigma(y)\beta g(x) + x\alpha g(y)\beta\tau(x) + y\beta x\alpha g(x)\\
&= f(x)\alpha y\beta x + \sigma(y)\beta f(x)\alpha x + f(y)\beta(\tau(x) - x)\alpha x + f(y)\beta x\alpha x + x\alpha(\sigma(y) - y)\beta g(x)\\
&\quad + x\alpha y\beta g(x) + x\alpha g(y)\beta\tau(x) + y\alpha x\beta g(x)\\
&= f(x)\alpha y\beta x + \sigma(y)\beta f(x)\alpha x + (\tau(x) - x)\beta f(y)\alpha x + f(y)\beta x\alpha x + (\sigma(y) - y)\alpha x\beta g(x)\\
&\quad + x\alpha y\beta g(x) + x\alpha g(y)\beta\tau(x) + y\alpha x\beta g(x)\\
&= f(x)\alpha y\beta x + \sigma(y)\beta(f(x)\alpha x + x\alpha g(x)) + (\tau(x) - x)\alpha f(y)\alpha x + f(y)\beta x\alpha x - y\alpha x\beta g(x)\\
&\quad + x\alpha y\beta g(x) + x\alpha g(y)\alpha(\tau(x) - x) + x\alpha g(y)\beta x + y\alpha x\beta g(x)\\
&= f(x)\alpha y\beta x + f(y)\alpha x\beta x + x\alpha y\beta g(x) + x\alpha g(y)\beta x + (\tau(x) - x)\beta(f(y)\alpha x + x\alpha g(y))\\
&= (f(x)\alpha y + f(y)\alpha x + x\alpha g(y))\beta x + x\alpha y\beta g(x) + (\tau(x) - x)\beta(f(y)\alpha x + x\alpha g(y)).
\end{aligned}$$

That is for all $x, y \in M$, $\alpha, \beta \in \Gamma$,

$$(f(x)\alpha y + f(y)\alpha x + x\alpha g(y))\beta x + x\alpha y\beta g(x) + (\tau(x) - x)\beta(f(y)\alpha x + x\alpha g(y)) = 0 \tag{24}$$

By (23) and (24), we get

$$\begin{aligned}
0 &= -y\alpha g(x)\beta x + x\alpha y\beta g(x) + (\tau(x) - x)\beta(f(y)\alpha x + x\alpha g(y))\\
&= -[y\beta g(x), x]_\alpha + (\tau(x) - x)\beta(f(y)\alpha x + x\alpha g(y)).
\end{aligned}$$

That is,

$$-[y\beta g(x), x]_\alpha + (\tau(x) - x)\beta(f(y)\alpha x + x\alpha g(y)) = 0 \text{ for all } x, y\in M,\ \alpha,\beta\in\Gamma \quad (25)$$

Let $z\in M$. Then by (25), we get

$$0 = [[-y\beta g(x), x]_\alpha, z]_\alpha + [(\tau(x) - x)\beta(f(y)\alpha x + x\alpha g(y)), z]_\alpha$$
$$= -[[y\beta g(x), x]_\alpha, z]_\alpha + (\tau(x) - x)\beta[f(y)\alpha x + x\alpha g(y), z]_\alpha + [\tau(x) - x, z]_\alpha\beta(f(y)\alpha x + x\alpha g(y)).$$

Using (22), we get

$$[[y\beta g(x), x]_\alpha, z]_\alpha = 0 \quad \text{for all } x, y, z\in M,\ \alpha,\beta\in\Gamma \quad (26)$$

From (26) we get $[y\beta g(x), x]_\alpha \in Z(M)$ for all $x, y\in M$, $\alpha,\beta\in\Gamma$ and, in particular,

$$[[y\beta g(x), x]_\alpha, x]_\alpha = 0 \quad \text{for all } x, y\in M,\ \alpha,\beta\in\Gamma \quad (27)$$

Replacing y by $z\alpha y$ in (27) we get for all $x, y\in M$, $\alpha,\beta\in\Gamma$,

$$[[z\alpha y\beta g(x), x]_\alpha, x]_\alpha$$
$$= [z\alpha[y\beta g(x), x]_\alpha, x]_\alpha + [z, x]_\alpha\alpha[y\beta g(x), x]_\alpha$$
$$= [z, x]_\alpha\alpha[y\beta g(x), x]_\alpha + z\alpha[y\beta[g(x), x]_\alpha, x]_\alpha + [z, x]_\alpha\alpha[y\beta g(x), x]_\alpha$$
$$= 2[z, x]_\alpha\alpha[y\beta g(x), x]_\alpha + z\alpha[[y\beta g(x), x]_\alpha, x]_\alpha = 0 \quad (28)$$

Replacing z by $y\beta g(x)$ in (28) and using (27), we get $2[y\beta g(x), x]_\alpha\alpha[y\beta g(x), x]_\alpha = 0$. Since M is 2-torsion free and, being semiprime, has no nonzero central nilpotents, we have,

$$[y\beta g(x), x]_\alpha = 0 \quad \text{for all } x, y\in M,\ \alpha,\beta\in\Gamma \quad (29)$$

Replacing y by $z\alpha y$ in (29), we get

$$[z, x]_\alpha\alpha y\beta g(x) = 0 \quad \text{for all } x, y, z\in M,\ \alpha,\beta\in\Gamma \quad (30)$$

Replacing y by $g(x)\beta y\gamma[z, x]_\alpha$ in (30), we get

$$[z, x]_\alpha\alpha g(x)\beta y\gamma[z, x]_\alpha\beta g(x) = 0 \text{ for all } x, y, z\in M,\ \alpha,\beta,\gamma\in\Gamma.$$

Since M is semiprime, we get

$$[z, x]_\alpha\beta g(x) = 0 \quad \text{for all } x, z\in M,\ \alpha,\beta\in\Gamma \quad (31)$$

Using (29) and (31), we get $y\beta[g(x), x]_\alpha = 0$ for all $x, y\in M$, $\alpha,\beta\in\Gamma$ and hence by the semiprimeness of M, we have $[g(x), x]_\alpha = 0$ for all $x\in M$. Thus g is a commuting (σ, τ)-derivation of M. Hence, by Lemma 2.3, $g(x)\in Z(M)$ and $g(u)\beta[x, y]_\alpha = 0$ for all $u, x, y\in M$, $\alpha,\beta\in\Gamma$. Also, $f(x)\in Z(M)$ and $f(u)\beta[x, y]_\alpha = 0$ for all $u, x, y\in M$, $\alpha,\beta\in\Gamma$ follows analogously.

Theorem 2.6 Let M be a 2-torsion free semiprime Γ-ring satisfying the assumption (*). If f, g are derivations on M such that $f(x)\alpha x + x\alpha g(x) = 0$ for all $x\in M$, $\alpha\in\Gamma$, then $f(u)\beta[x, y]_\alpha = g(u)\beta[x, y]_\alpha = 0$ for all $x, y, u\in M$, $\alpha,\beta\in\Gamma$, in particular, f, g map M into its center.

Proof

Since derivations are (1, 1)-derivations, it follows immediately from Theorem 2.5.

Corollary 2.7 Let M be a 2-torsion free prime Γ-ring satisfying the assumption (*) and σ, τ-centralizing epimorphisms of M. Let f, g be (σ, τ)-derivations of M such that $f(x)\alpha x + x\alpha g(x) = 0$ for all $x\in M$, $\alpha\in\Gamma$. Then either M is commutative or $f = g = 0$.

Proof

Since the center of a prime Γ-ring contains no nonzero divisors of zero, this corollary is immediate from Theorem 2.5.

Theorem 2.8 Let M be a 2-torsion free semiprime Γ-ring satisfying the assumption (*) and σ, τ centralizing epimorphisms of M. Let f, g be (σ, τ)-derivations of M such that

$$f(x)\alpha x + x\alpha g(x)\in Z(M) \quad \text{for all } x\in M, \alpha\in\Gamma \tag{32}$$

Then (i) if $Z(M) = 0$, then $f = g = 0$, and

(ii) if $Z(M) \neq 0$, then $c\delta f(u)\beta[x, y]_\alpha = c\delta g(u)\beta[x, y]_\alpha = 0$ and $c\delta f(x)$, $c\delta g(x)\in Z(M)$ for all $x, y, u\in M$, $\alpha,\beta,\delta\in\Gamma$ and nonzero $c\in Z(M)$.

Proof

(i) Assume that $Z(M) = 0$. Then, by hypothesis, $f(x)\alpha x + x\alpha g(x) = 0$ for all $x\in M$, $\alpha\in\Gamma$ and hence by Theorem 2.5, $f(x)$, $g(x)\in Z(M)$. Since $Z(M) = 0$, we have

$f(x) = g(x) = 0$ for all $x\in M$. Thus $f = g = 0$.

(ii) Let $Z(M) \neq 0$ and c be a nonzero element of $Z(M)$. Since σ, τ are centralizing epimorphisms, therefore, as in Theorem 2.5,

$$\sigma(u) - u\in Z(M),\ \sigma(u)\beta[x, y]_\alpha = u\beta[x, y]_\alpha \text{ and } [x, y]_\alpha\beta\sigma(u) = [x, y]_\alpha\beta u \tag{33}$$

And for all $u, x, y\in M$, $\alpha,\beta\in\Gamma$,

$$\tau(u) - u\in Z(M),\ \tau(u)\beta[x, y]_\alpha = u\beta[x, y]_\alpha \text{ and } [x, y]_\alpha\beta\tau(u) = [x, y]_\alpha\beta u \tag{34}$$

Moreover, since σ and τ are onto, therefore $\sigma(c)$ and $\tau(c)\in Z(M)$.

Linearizing (32), we get

$$f(x)\alpha y + f(y)\alpha x + x\alpha g(y) + y\alpha g(x)\in Z(M) \text{ for all } x, y\in M, \alpha\in\Gamma \tag{35}$$

Replacing y by c in (35), we get for all $x\in M$, $\alpha\in\Gamma$,

$$f(x)\alpha c + f(c)\alpha x + x\alpha g(c) + c\alpha g(x)\in Z(M) \tag{36}$$

Replacing y by $c\delta c$ in (35), we get

$$\begin{aligned}&f(x)\alpha c\delta c + f(c\delta c)\alpha x + x\alpha g(c\delta c) + c\delta c\alpha g(x)\\ &= c\delta(f(x)\alpha c + c\alpha g(x)) + (\sigma(c) + \tau(c))\delta(f(c)\alpha x + x\alpha g(c))\\ &= c\delta(f(x)\alpha c + c\alpha g(x) + f(c)\alpha x + x\alpha g(c)) + (\sigma(c) + \tau(c) - c)\delta(f(c)\alpha x + x\alpha g(c))\end{aligned}$$

$= c\delta(f(x)\alpha c + c\alpha g(x) + f(c)\alpha x + x\alpha g(c)) + (\sigma(c) + \tau(c) - c)\delta(f(c)\alpha x + x\alpha g(c)$
$+ f(x)\alpha c + c\alpha g(x)) - (\sigma(c) + \tau(c) - c)\delta(f(x)\alpha c + c\alpha g(x)) \in Z(M).$

That is for all $x, c \in M$, $\alpha, \delta \in \Gamma$,

$$(\sigma(c) + \tau(c))\delta(f(x)\alpha c + c\alpha g(x) + f(c)\alpha x + x\alpha g(c)) - (\sigma(c) + \tau(c) - c)\delta(f(x)\alpha c + c\alpha g(x)) \in Z(M) \quad (37)$$

As $\sigma(c) + \tau(c) \in Z(M)$ and by (36) the first summand in (37) is in $Z(M)$, (37) implies

$(\sigma(c) + \tau(c) - c)\delta(f(x)\alpha c + c\alpha g(x))$
$= (\sigma(c) + \tau(c) - c)\delta c\alpha(f(x) + g(x)) \in Z(M)$ for all $x \in M$, $\alpha, \delta \in \Gamma$.

Thus

$$(\sigma(c) + \tau(c) - c)\delta c\alpha(f(x) + g(x)) \in Z(M) \quad \text{for all } x \in M,\ \alpha, \delta \in \Gamma. \quad (38)$$

Since c, $(\sigma(c) + \tau(c) - c)\delta c \in Z(M)$ and f, g are (σ, τ)-derivations, therefore $((\sigma(c) + \tau(c) - c)\delta c)\alpha f$, $((\sigma(c) + \tau(c) - c)\delta c)\alpha g$, $c\delta f$ and $c\delta g$ are (σ, τ)-derivations. Thus $((\sigma(c) + \tau(c) - c)\delta c)\alpha(f + g)$ is an (σ, τ)-derivation and (38) implies that it is central and hence a commuting (σ, τ)-derivation. Thus by Lemma 2.4, we get

$$((\sigma(c) + \tau(c) - c)\delta c)\alpha(f + g)(u)\beta[x, y]_\alpha = 0 \text{ for all } u, x, y \in M,\ \alpha, \beta, \delta \in \Gamma \quad (39)$$

Using (32) and (33), from (31) we get

$0 = (f + g)(u)\beta(\sigma(c) + \tau(c) - c)\delta c\beta[x, y]_\alpha$
$= (f + g)(u)\beta c\delta(\sigma(c) + \tau(c) - c)\beta[x, y]_\alpha$
$= ((f + g)(u)\beta c)\delta(\sigma(c)\beta[x, y]_\alpha + \tau(c)\beta[x, y]_\alpha - c\beta[x, y]_\alpha)$
$= ((f + g)(u)\beta c)\delta(c\beta[x, y]_\alpha + c\beta[x, y]_\alpha - c\beta[x, y]_\alpha = (f + g)(u)\beta c\delta c\beta[x, y]_\alpha$
$= c\beta c\delta(f + g)(u)\beta[x, y]_\alpha$ for all $u, x, y \in M$, $\alpha, \beta \in \Gamma$. That is,

$$c\delta(c\beta f(u) + g(u))\beta[x, y]_\alpha = 0 \text{ for all } u, x, y \in M,\ \alpha, \beta, \delta \in \Gamma \quad (40)$$

As $c \in Z(M)$ and M is semiprime, it follows from (30) that

$$c\delta(f(u) + g(u))\beta[x, y]_\alpha = 0 \text{ for all } u, x, y \in M,\ \alpha, \beta, \delta \in \Gamma \quad (41)$$

Similarly, we have $[x, y]_\alpha\beta c\delta(f(u) + g(u)) = 0$. Thus, by Lemma 2.3 we get $c\delta f(u) + c\delta g(u) \in Z(M)$. Using this and (31), we get $[(c\delta f(u) + c\delta g(u))\beta u, y]_\alpha = (c\delta f(u) + c\delta g(u))\beta[u, y]_\alpha + [c\delta f(u) + c\delta g(u), y]_\alpha\beta u = 0$. That is,

$$[c\delta f(u)\beta u + c\delta g(u)\beta u, y]_\alpha = 0 \text{ for all } u, y \in M,\ \alpha, \beta, \delta \in \Gamma \quad (42)$$

Since $c \in Z(M)$ and $f(u)\beta u + u\beta g(u) \in Z(M)$ (by 32)), we get $c\delta f(u)\beta u + c\delta u\beta g(u) \in Z(M)$. Thus

$$[c\delta f(u)\beta u + c\delta u\beta g(u), y]_\alpha = 0 \text{ for all } u, y \in M,\ \alpha, \beta, \delta \in \Gamma \quad (43)$$

Subtracting (43) from (42), we get $[c\delta g(u)\beta u - c\delta u\beta g(u), y]_\alpha = 0$. That is, $[c\delta(g(u)\beta u - u\beta g(u)), y]_\alpha = [c\delta[g(\underline{u}), u]_\beta, y]_\alpha = [[c\delta g(u), u]_\beta, y]_\alpha = 0$ for all $u, y\in M, \alpha,\beta,\delta\in\Gamma$, which implies $[c\delta g(u), u]_\beta \in Z(M)$. Thus $c\delta g$ is a centralizing (σ, τ)-derivation. We get that $c\delta g$ is a commuting (σ, τ)-derivation. By Lemma 2.3, we get $c\delta g(\underline{u})\in Z(M)$ and $c\delta g(u)\beta[x, y]_\alpha = 0$ for all $u, x, y\in M, \alpha,\beta,\delta\in\Gamma$. Since $c\delta f(u) + c\delta g(u)\in Z(M)$ and $c\delta g(u)\in Z(M)$, therefore $c\delta f(u)\in Z(M)$. Thus $c\delta f$ is central and hence a commuting (σ, τ)-derivation. By Lemma 2.3, we get $c\delta f(u)\in Z(M)$ and $c\delta f(u)\beta[x, y]_\alpha = 0$ for all $u, x, y\in M, \alpha,\beta,\delta\in\Gamma$.

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