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Higher Dimensional Spherically Symmetric String Cosmological Model with Zero Mass Scalar Field in Lyra Geometry

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Abstract

In this article, A higher-dimensional string cosmological model with a zero-mass scalar field in Lyra's geometry has been investigated by considering a five-dimensional spherically symmetric space-time. To obtain the deterministic solutions for field equations, some physically plausible conditions are taken into account. The relation between the metric coefficients was assumed to be $B = nA$, and due to the highly non-linear nature of the field equations, the case $\lambda = \rho$. was considered. Furthermore, calculations were performed for various physical and kinematical parameters, and their astrophysical implications were analyzed, revealing a strong resemblance to recent observational data.

Keywords: Spherically Symmetric Space-time; Zero mass scalar field; Lyra geometry.

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1. Introduction

As the universe evolves, the extra dimensions are not observable due to dynamical contraction and compactification with the passage of time and it ultimately reduces to four-dimensional continuum [1]. There is an ample literature on higher dimensional cosmological models in general relativity [2-6]. Einstein [7] has geometrized gravitational field in his general theory of relativity. This has inspired several researchers to geometrize the other physical fields. Weyl [8] has formulated a unified theory to geometrize gravitation and electromagnetism. Lyra [9] proposed a modification of Riemannian geometry by introducing an additional gauge function into the structure less manifold as a result of which the cosmological constant arises naturally from the geometry. In this theory both the scalar and tensor fields have geometrical significance. Subsequently Sen [10] and Sen and Dunn [11] suggested a new scalar tensor theory of gravitation. Jeavons et al. [12] pointed out that the field equations proposed by Sen and Dunn are heuristically useful even though they are not derived from the usual variational principle. A brief note on Lyra's geometry is given

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by Singh *et al*. [13]. It has been shown by Halford [14] that the energy conservation law does not hold in the cosmological theory based on Lyra's geometry. Halford [15] has also shown that the scalar tensor theory of gravitation in Lyra manifold gives same effects, within observational limits, as in the Einstein theory. Soleng [16] pointed out that the constant gauge vector φ_i in Lyra's geometry together with a creation field becomes Hoyle's [17] creation field cosmology or contains a special vacuum field, which together with the gauge vector may be considered as a cosmological term. Further, Soleng [18] showed that for matter with zero spin the field equations of his scalar tensor theory reduce to those of Brans-Dicke theory. Bali *et. al.* studied [19,20] Bianchi type-I cosmological model for perfect fluid distribution in Lyra geometry and string dust magnetized in Bianchi type I cosmological models in Lyra geometry. Bhamra [21], Kalyanshetty and Waghmode [22], Reddy and Innaiah [23], Beesham [24], Reddy and Venkateswarlu [25], and Singh and Desikan [26] are some of the authors who have investigated various aspects of the four-dimensional cosmological models in Lyra's manifold. Reddy *et al*. [27] have investigated Bianchi type-I cosmological model with extra dimensions in Lyra manifold while Mohanty *et al.* [28-29] showed the nonexistence of five-dimensional perfect fluid cosmological model in this manifold. and obtained the exact solutions of the field equations for empty universe. Also, Mohanty *et al*. [30] showed that in a five-dimensional space-time the general perfect fluid distribution does not survive but degenerates into stiff fluid distribution in this particular manifold. Kaluza-Klein FRW cosmological models have been constructed by Mohanty *et al.* [31] in Lyra geometry. Higher dimensional cosmological models, in this geometry, have also been discussed by Rahaman [32] and Rahaman *et al*. [33,34].

Samanta and Dhal [35] found a new class of higher dimensional cosmological models of the early universe filled with perfect fluid source in the frame work of (R,T) gravity with the help of five dimensional spherically symmetric metric. Pawar *et. al*. [36] investigated Kaluza-Klein Cosmological Model with strange-quark-matter in Lyra Geometry. Recently Rao and Jayasudha [37,38] obtained five dimensional spherically symmetric perfect fluid models in Saez-Ballester [39] and Brans-Dicke [40] scalar-tensor theories of gravitation. Reddy [41] investigated Five-Dimensional Spherically Symmetric Perfect Fluid Cosmological Model in Lyra Manifold. Five-dimensional spherically symmetric cosmological models based on Lyra geometry are significant because they provide insights into the evolution of the universe during its early stages. Brahma *et al*. [42] have reviewed Bianchi type-V dark energy cosmological model with the electromagnetic field in Lyra based on f (R, T) gravity.

The investigation of yet unsolved interacting fields in reference to modified gravitation theories assuming one of the fields is a massless scalar field is a basic attempt to study the unification of the quantum and gravitational theories. In recent years, there has been a lot of interest in the set of field equations that represent a zero-mass scalar field coupled with gravitational theories. Venkateswarlu *et al.* [43-45] studied various cosmological models with zero mass scalar field, Godani and Samanta [46] studied Wormhole modeling in f (R, T) gravity with minimally-coupled massless scalar field, Singh *et al*. [47] discussed Causal viscous universe coupled with zero-mass scalar field in higher derivative theory, Singh[48] studied Rotational perturbations of radiating Universes coupled with zero-mass scalar field, Patra [49] studied spherically symmetric space-time with magnetic field and zero mass scalar field, Adhav *et al.* [50] examined Zero mass scalar field with bulk viscous cosmological solutions in Lyra geometry, Dixit *et al*. [51] discussed Transit cosmological models coupled with zero-mass scalar field with high redshift in higher derivative theory, Katore *et al.* [52] studied FRW Cosmological Solutions with Zero-Mass Scalar Field Attached to Bulk Viscous Fluid in Saez-Ballester Theory of Gravitation, Pawar *et. al*. [53] investigated Plane Symmetric String Cosmological Model with Zero Mass Scalar Field in f (R) Gravity, Mete *et al.* [54] studied a five-dimensional Bianchi type-III string cosmological model with a one-dimensional cosmic string in the presence of zero mass scalar field in the context of the Lyra manifold and Cadoni and Franzin [55], Pawar *et al.* [56,57] are some of the authors who have vigorously studied interacting fluid with one matter content as a zero mass scalar field. In recent years strings are widely receiving significant interest from researchers as they play an important role in explaining the early phase of cosmic evolution. In modern cosmology, the substantial theoretical development of string theory has been done using different types of gravitation theories. The primary objective of the study was to explore five-dimensional spherically symmetric cosmological models with zero-mass scalar fields in Lyra's geometry. Studying spherically symmetric space time in Lyra geometry offers a rich area of research evolving physical aspects of gravitation and cosmology. According to literature [58-60], the extra dimension produces a vast amount of entropy, which may provide solutions to the flatness and horizon problems. Since humans inhabit a four-dimensional space-time, it has been suggested that the hidden extra dimension in five-dimensional space is likely connected to the elusive dark matter and dark energy. Singh *et al.* [61-64] studied higher dimensional spherically symmetric space times in various gravitational theories.

This paper is organized as follows: In Section-2, The field equation has been provided in the normal gauge for Lyra geometry. In Section 3, the metric was introduced, the field equations were solved, and the spherically symmetric model was presented. Section-4 contains some physical and kinematical properties. Section 5: Discussion and conclusions.

2. Field Equation of Lyra Geometry

The modified Einstein's field equation in normal gauge for Lyra geometry proposed by Sen [10] and Sen and Dunn [11] are given by

$$R_j^i - \frac{1}{2}\delta_j^i R + \frac{3}{2}\varphi_j\varphi^i - \frac{3}{4}\varphi_k\varphi^k\delta_j^i = -T_j^i \quad (1)$$

Where ϕ_i is the time-dependent displacement field vector, defined by

$$\varphi_i = (0,0,0,0.\beta) \quad (2)$$

3. Metric and Field Equation

The five dimensional spherically symmetric space-time in the form

$$ds^2 = dt^2 - e^A(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\varphi^2) - e^B d\zeta^2 \tag{3}$$

Where A and B are functions of cosmic time t only.

The energy-momentum tensor for one-dimensional cosmic string coupled with zero mass scalar fields is given by

$$T_j^i = \rho u^i u_j - \lambda x^i x_j + \left(\psi'^i \psi_{,j} - \frac{1}{2}\partial_j^i \psi'^i \psi_{,j}\right) \tag{4}$$

Where λ and ρ are the tension density and rest energy density of cloud string fluid respectively. u^i denotes the four time-like velocity vectors and x^i denotes a unit space-like vector which represents the anisotropic direction of cloud string and satisfies the conditions

$$u^i = (1,0,0,0,0) \text{ and s}-x^i = (0,0,0,0,1). \tag{5}$$

The scalar field ψ satisfies the condition

$$\psi_{;i}^{i} = 0. \tag{6}$$

Here the scalar field ψ indirectly coupled to matter. It indirectly interacts with matter through gravity. Equation (2) gives the components of energy momentum tensor as follows

$$T_1^1 = \frac{1}{2}\dot{\psi}^2 = T_2^2 = T_3^3\ ,\ T_4^4 = \lambda + \frac{1}{2}\dot{\psi}^2,\ T_0^0 = \rho + \frac{1}{2}\dot{\psi}^2 \tag{7}$$

From equations (3) and (7) the field equation for the metric (3) are obtained as follows.

$$\frac{3}{4}\dot{A}^2 + \frac{3}{4}\dot{A}\dot{B} + \frac{3}{4}\beta^2 = \left(\rho + \frac{\dot{\psi}^2}{2}\right) \tag{8}$$

$$\ddot{A} + \frac{1}{2}\ddot{B} + \frac{3}{4}\dot{A}^2 + \frac{1}{2}\dot{A}\dot{B} + \frac{1}{4}\dot{B}^2 + \frac{3}{4}\beta^2 = \frac{\dot{\psi}^2}{2} \tag{9}$$

$$\frac{3}{2}\ddot{A} + \frac{3}{2}\dot{A}^2 + \frac{3}{4}\beta^2 = \left(\lambda + \frac{\dot{\psi}^2}{2}\right) \tag{10}$$

Where the overhead dot denotes the derivative with respect to cosmic time *t*.

Here, three non-linear differential field equations with six unknowns A, B, λ, ρ, ψ and β were obtained.

Here, the relation between the metric potentials assumed as,

$$B = nA \tag{11}$$

Because the field equations are non-linear.

In order to obtain the exact solutions following case is considered,

$$\lambda = \rho. \tag{12}$$

From equations (8) to (10) and (11), (12) the exact solutions of the field equations are

$$e^A = \left[\frac{(n+3)}{2}(c_1 t + c_2)^{-1}\right]^{\frac{2}{(n+3)}} \tag{13}$$

$$e^B = \left[\frac{(n+3)}{2}(c_1 t + c_2)^{-1}\right]^{\frac{2n}{(n+3)}} \tag{14}$$

The metric in equation (3) becomes

$$ds^2 = dt^2 - \left[\frac{(n+3)}{2}(c_1 t + c_2)^{-1}\right]^{\frac{2}{(n+3)}} (dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\varphi^2) - \left[\frac{(n+3)}{2}(c_1 t + c_2)^{-1}\right]^{\frac{2n}{(n+3)}} d\zeta^2 \tag{15}$$

The metric potentials obtained are constant for any value to t hence the model is singularity-free. The Scalar Field

$$\psi = \frac{2}{3}\left[\frac{2c_1^2(n+2)}{(n+3)} + \frac{6c_1(n+2)}{(n+3)^2}(c_1 t + c_2)^{-2} + \frac{3}{4}\beta^2\right]^{\frac{3}{2}} \tag{16}$$

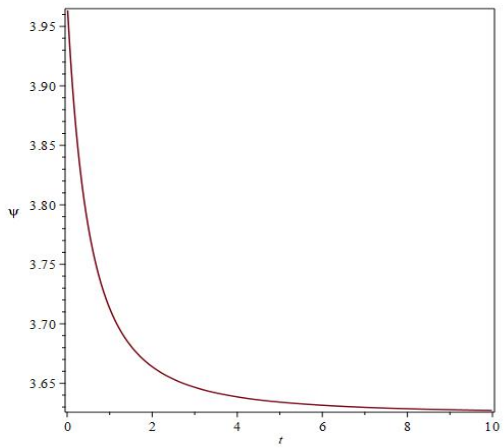

Fig. 1. Behavior of Scalar field vs. *t* for $c_1 = 1.76,\ c_2 = 1.80,\ n = 2,\ \beta = 0.80$.

Energy density

$$\rho = \frac{1}{(n+3)}\left[\frac{c_1^2(n^2+5n+6)+3(c_1t+c_2)^{-2}}{(n+3)}\right] \tag{17}$$

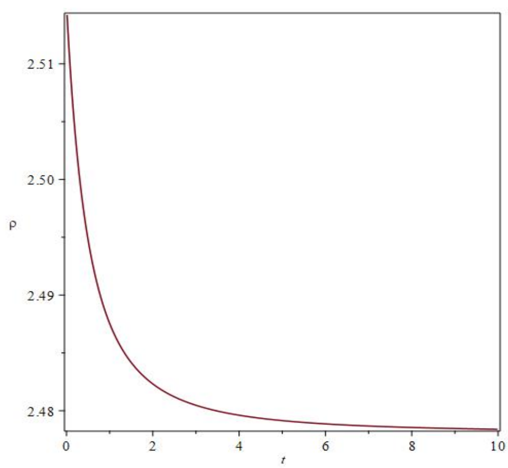

Fig. 2. Behavior of energy density vs. *t* for $c_1 = 1.76,\ c_2 = 1.80,\ n = 2$.

Recent investigations have confirmed that the universe's associated energy generally falls in tandem with the value of scalar field. From the expressions (16) as well as (17), the scalar field and energy density have been shown to be inverse functions of cosmic time *t*. This has also been confirmed by Figs. 1 and 2, where the graphs of both show a decline as cosmic time *t* increases.

The string tension density

$$\lambda = c_1^2\left[1 - \frac{3n(c_1t+c_2)^{-2}}{(n+3)^2}\right] \tag{18}$$

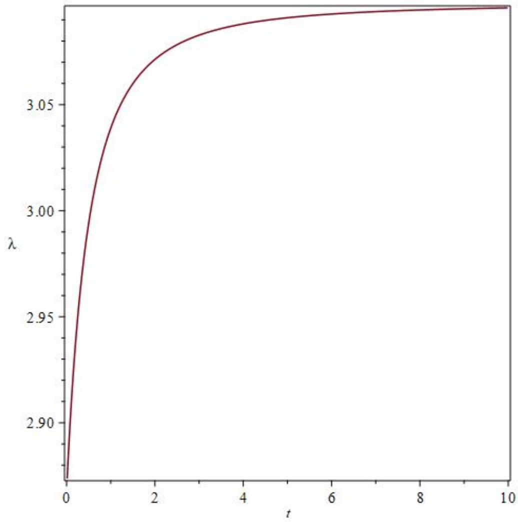

Fig. 3. Behavior of energy density vs. *t* for $c_1 = 1.76,\ c_2 = 1.80,\ n = 2$.

From the Fig. 3, it is observed that as the cosmic time t increases the value of tension density also increases and tend to infinity with infinite time. In the given model, the positive value of λ not only shows the presence of strings in the universe but also the string dominance over particles. This result has been verified with Cadoni and Franzin [55] and Pawar *et al*. [56].

4. The Physical and Kinematical Properties

The directional Hubble factors H_i(i=1,2,3,4) in the directions x, y, z and m obtained as

$$H = \frac{1}{4}\sum_{i=1}^{4} H_i = \frac{1}{4}(H_1 + H_2 + H_3 + H_4)$$

$$H = \frac{c_1 (c_1 t + c_2)^{-1}}{\log\left[\left(\frac{n+3}{2}\right)(c_1 t + c_2)^{-1}\right]} \tag{19}$$

The expansion scalar θ is given by

$$\theta = 4H = \frac{4c_1 (c_1 t + c_2)^{-1}}{\log\left[\left(\frac{n+3}{2}\right)(c_1 t + c_2)^{-1}\right]} \tag{20}$$

The anisotropy parameter A_m

$$A_m = \frac{1}{4}\sum_{i=1}^{4}\left(\frac{H_i}{H} - 1\right)^2$$

$A_m = 1$ (21)

The Spatial Volume

$$V = a^4 = \left(e^{\frac{3A+B}{2}} r^2 \sin^2\theta\right)^{\frac{1}{4}} \quad (22)$$

Shear Scalar

$$\sigma^2 = \frac{3}{2} A_m H^2 = \frac{3c_1(c_1 t + c_2)^{-2}}{4\log\left[\left(\frac{n+3}{2}\right)(c_1 t + c_2)^{-1}\right]} \quad (23)$$

Deceleration Parameter

$$q = -\frac{a\ddot{a}}{\dot{a}^2}$$

$$q = -1 \quad (24)$$

5. Conclusion

The five-dimensional spherically symmetric string cosmological model with zero-mass scalar fields was constructed in the context of Lyra geometry. To obtain the deterministic solutions to the highly non-linear differential field equations, the relation between the metric coefficients as $B = nA$ is assumed and considered the case $\lambda = \rho$. The constructed model is observed to be singularity-free. Furthermore, the scalar field and energy density have been shown to be inverse functions of cosmic time t. This has also been confirmed by behaviour of scalar field and energy density where, the graphs of both show a decline as cosmic time *t* increases. The recent observations validate the comparative behaviour of the scalar field and energy density. Additionally, in the given model, the positive value of λ not only shows the presence of strings in the universe but also string dominance over particles. Also, it is observed that the ratio of the square of shear scalar and expansion scalar is a nonzero constant and the deceleration parameter $q = -1$ shows that the expansion of the universe is accelerated which shows the present state of the universe as per the observational data. Thus, the constructed cosmological model is the standard model of the universe.

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