On Some $R_1$-Properties in Fuzzy Topological Spaces

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Abstract

In this paper, we introduce six $R_1$-axioms for fuzzy topological spaces (in short, fts). We study their interrelations, goodness and initialities. Besides we recall nine $R_0$-axioms for fts. A complete answer is given with regard to all possible $(R_1 \Rightarrow R_0)$-type implications for fts.

Keywords: Fuzzy topological space; Fuzzy $R_1$-axiom; Fuzzy $R_0$-axiom.

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1. Introduction

The concept of fuzzy sets was first introduced, in 1965, by L. A. Zadeh in his new classical paper [1] as an attempt to mathematically handle those phenomena which are inherently vague, imprecise or fuzzy in nature. He interpreted a fuzzy set on a set $X$ as a mapping from $X$ into the closed unit interval $I = [0, 1]$. Various merits and applications as well as some limitations of fuzzy set theory have since been demonstrated by Zadeh and a large number of subsequent workers.

The advent of fuzzy set theory has also led to the development of some new areas of study in mathematics. It has become a concern and a new tool for the mathematicians working in many different areas of mathematics. These have been generally accomplished by replacing subsets, in various existing mathematical structures, by fuzzy sets. In 1968, Chang [2] did ‘fuzzification’ of topology by replacing ‘subsets’ in the definition of fuzzy topology by ‘fuzzy sets’. Since then a large body of concepts and results have been growing in this area which has come to be known as “fuzzy topology”.

A major deviation in the definition of fuzzy topology was made by Lowen [3, 4]. He gave a modified definition of fuzzy topology by including all constant fuzzy sets in a fuzzy topology.

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In this paper, we introduce six new concepts of $R_1$-properties of fts each of which is shown as the good extension of the topological $R_1$-property. We study their interrelations and initialities. In addition, we recall nine concepts of $R_0$-properties of fts from [6]. In analogy with the well known topological property $(R_1 \Rightarrow R_0)$ we study the relations of this type for fts. It is also shown that, the property $(R_0 \Leftrightarrow R_1)$, in general, is also true for fts.

2. Preliminaries

In this section, we recall some definitions and basic results (which we label as facts) on fuzzy sets and fts. This section is considered as the base and background for the study of subsequent sections.

Definition 2.1 [1]: Let $X$ be a non-empty set and $I$ the unit closed interval $[0, 1]$. A fuzzy set on $X$ is a function $u : X \rightarrow I$. $\forall x \in X$, $u(x)$ denotes a degree or the grade of membership of $x$. The set of all fuzzy sets in $X$ is denoted by $I^X$. Ordinary subsets of $X$ (crisp sets) are also considered as the members of $I^X$ which take the values 0 and 1 only. A crisp set which always takes the value 0 is denoted by 0, similarly a crisp set which always takes the value 1 is denoted by 1.

Definition 2.2 [9]: Let $u \in I^X$. The set $\{x \in X : u(x) > 0\}$ is called the support of $u$ and is denoted by $u_o$ or $\text{supp}(u)$. By $u^c$, we denote the complement of $u$ which is defined as $u^c(x) = 1 - u(x) \ \forall x \in X$.

Definition 2.3 [6]: If $A \subseteq X$, by $1_A$ we denote the characteristics function $A$. The characteristics function of a singleton set $\{x\}$ is denoted by $1_x$.

Definition 2.4 [9, 11]: A fuzzy point $x_\alpha$ in $X$ is a special type of fuzzy set in $X$ with the membership function $x_\alpha(x) = \alpha$ and $x_\alpha(y) = 0$ if $x \neq y$, where $0 < \alpha < 1$ and $x, y \in X$. The fuzzy point $x_\alpha$ is said to have support $x$ and value $\alpha$. We also write this as $\alpha 1_x$.

Definition 2.5 [2]: Let $f : X \rightarrow Y$ be a mapping and $u$ a fuzzy set in $X$. Then the image $f(u)$ is a fuzzy set in $Y$ which is defined as
Definition 2.6 [2]: Let $f : X \to Y$ be a mapping and $u$ be a fuzzy set in $Y$. Then the inverse image $f^{-1}(u)$ is the fuzzy set in $X$ which is defined as $f^{-1}(u)(x) = u(f(x)) \quad \forall x \in X$.

Definition 2.7 [2]: Chang C. L. defined a fuzzy topological space as follows:

Let $X$ be a set. A class $t$ of fuzzy sets in $X$ is called a fuzzy topology on $X$ if $t$ satisfies the following conditions:

(i) $0, 1 \in t$,

(ii) if $u, v \in t$ then $u \land v \in t$

and (iii) if $\{u_i : i \in K\}$ is a family of fuzzy sets in $t$, then $\bigvee_{i \in K} u_i \in t$.

The pair $(X, t)$ is then called a fuzzy topological space (in short, fts). The members of $t$ are called $t$-open sets (or open sets) and their complements are called $t$-closed set (or closed sets).

Definition 2.8: Lowen [3] modified the definition of an fts defined by Chang [2] by adding another condition. In the sense of Lowen R. the definition of an fts is as follows:

Let $X$ be a set and $t$ is a family of fuzzy sets in $X$. Then $t$ is called a fuzzy topology on $X$ if the following conditions hold:

(i) $0, 1 \in t$,

(ii) if $u, v \in t$ then $u \land v \in t$

(iii) if $\{u_i : i \in K\}$ is a family of fuzzy sets in $t$, then $\bigvee_{i \in K} u_i \in t$

and (iv) $t$ contains all constant fuzzy sets in $X$.

The pair $(X, t)$ is called an fts.

We shall use the concept of fts due to Lowen R. unless otherwise stated.

Definition 2.9 [6, 9]: Let $u$ be a fuzzy set in an fts $(X, t)$. Then the fuzzy closure $\overline{u}$ and the fuzzy interior $u^\circ$ of $u$ are defined as follows:

$\overline{u} = \inf \{ \lambda : u \leq \lambda \text{ and } \lambda \in t^c \}$.

$u^\circ = \sup \{ \lambda : \lambda \leq u \text{ and } \lambda \in t \}$.

Fact 2.10 [6, 9]: For a fuzzy topological space $(X, t)$ and for $u \in I^X$, the following hold:

$$f(u)(y) = \begin{cases} \sup \{ u(x) : f(x) = y \} & \text{if } f^{-1}(y) \neq \emptyset \\ 0 & \text{if } f^{-1}(y) = \emptyset \end{cases}$$
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(i) $\bar{u} = 1 - u^o$

(ii) For any fuzzy set $u$ in $X$, $u^o \leq u \leq \bar{u}$.

(iii) If $u \leq v$, then $\bar{u} \leq \bar{v}$ and $u^o \leq v^o$.

**Definition 2.11** [9]: Let $(X, t)$ and $(Y, s)$ be two fts. A function $f : (X, t) \rightarrow (Y, s)$ is called

(i) continuous if and only if $f^{-1}(u) \in t$ for each $u \in s$.

(ii) open if and only if $f(u) \in s$ for each $u \in t$.

(iii) closed if and only if $f(u) \in s^c$ for each $u \in t^c$.

**Definition 2.12** [9]: Let $\{ (X_i, t_i) : i \in K \}$ be a collection of fts. Let $X = \prod_{i \in K} X_i$ be their Cartesian product and $p_i : X \rightarrow X_i$ be the projection map. Then the fuzzy topology $t$ on $X$ generated by $\{ p_i^{-1}(u_i) : i \in K, u_i \in t_i \}$ is called the product fuzzy topology on $X$ and the pair $(X, t)$ is called the product fts. It can be verified that $p_i^{-1}(u_i), i \in K$, as defined above, can be expressed as $\prod_{k \in K} \lambda_k$ where $\lambda_k = u_i$ if $k = i$ and $\lambda_k = X_k$ if $k \neq i$.

The product fuzzy topology $t$ is also called the coarsest fuzzy topology on $X$.

**Fact 2.13** [9]: For a family $\{ (X_i, t_i) : i \in K \}$ of fts and a fuzzy topology $t$ on $X = \prod_{i \in K} X_i$, the following are equivalent:

(i) $t$ is the product of the fuzzy topologies $t_i$’s.

(ii) $t$ is the smallest fuzzy topology on $X$ which makes each projection $p_i : X \rightarrow X_i$, $i \in K$ continuous.

(iii) For each fts $(Y, s)$ the function $f : (X, t) \rightarrow (Y, s)$ is continuous if and only if for all $i \in K$, $p_i(f)$ is continuous.

**Definition 2.14** [9]: Let $\{ f_j : X \rightarrow (X_j, t_j) : j \in J \}$ be a family of functions from a set $X$ to fts $(X_j, t_j), j \in J$. Then the initial fuzzy topology on $X$ induced by the family $\{ f_j : j \in J \}$, say $t$, is the smallest fuzzy topology on $X$, making each $f_j, j \in J$, continuous. It can be verified that $t$ is generated by the family of fuzzy sets $f_j^{-1}(u_j) : u_j \in t_j$ and $j \in J$. For example, the product fuzzy topology is the initial fuzzy topology induced by the family of projections. Similarly, the subspace topology is also the initial fuzzy topology induced by the inclusion map.
Definition 2.15 [9]: A fuzzy topological property FP is said to be an initial property if for each family of functions \( \{ f_j : X \to (X_j, t_j) ; j \in J \} \), whenever each fts \( (X_j, t_j) ; j \in J \), has FP, \( (X, t) \) also has FP, \( t \) being the initial fuzzy topology on \( X \) induced by the family \( \{ f_j : j \in J \} \).

Definition 2.16 [9]: A function \( f : X \to R \) is called lower semicontinuous (l.s.c.) if and only if for every \( \alpha \in R \), \( \{ x \in X : f(x) > \alpha \} \) is an open set. For a topological space \( (X, T) \), the l.s.c. fuzzy topology on \( X \) associated with \( T \) is denoted by \( \omega(T) \) and is defined as \( \omega(T) = \{ u \in I^X : u \text{ is l.s.c.} \} \).

Fact 2.17 [9]: Let \( (X, T) \) be a topological space. Then
(i) \( u \in I^X \) is \( \omega(T) \) closed if and only if for all \( \alpha \in I \), \( u^{-1}[\alpha, 1] \) is \( T \)-closed.
(ii) \( A \subseteq X \) is \( T \)-open if and only if \( 1_A \) is \( \omega(T) \)-open.
(iii) \( A \subseteq X \) is \( T \)-closed if and only if \( 1_A \) is \( \omega(T) \)-closed.
(iv) \( u^{-1}(\alpha, 1] \subseteq (u^{-1}[\alpha, 1] \).
(v) \( \alpha 1_A = \alpha 1_A \).
(vi) \( \{ 1_U : U \in T \} \) is a subbase for \( \omega(T) \).
(vii) \( \{ \alpha 1_U : \alpha \in I_0 \text{ and } U \in T \} \) is a base for \( \omega(T) \).

Definition 2.18 [2]: Let \( P \) be a property of a topological space and FP its fuzzy topological analogue. Then FP is called a good extension of \( P \) if and only if the statement “\((X, T) \) has \( P \) if and only if \((X, \omega(T)) \) has FP” holds good for every topological space \((X, T)\).

3. \( R_1 \)-properties

In this section we introduce six \( R_1 \)-axioms for fts.

Definitions 3.1: We define, for fts \((X, t)\), \( R_1 \)-properties as follows:

\( FR_1(1) \): If \( \forall x, y \in X, x \neq y \) and \( \forall \alpha \in I_{0,1} \) \( \exists w \in t \) such that either \( w(x) > \alpha \) and \( w(y) = 0 \) or \( w(y) > \alpha \) and \( w(x) = 0 \), then \( \exists \mu, \nu \in t \) such that \( \overline{1_x} \leq \mu, \overline{1_y} \leq \nu \) and \( \mu \land \nu = 0 \).

\( FR_1(2) \): If \( \forall x, y \in X, x \neq y \) and \( \forall \alpha \in I_{0,1} \) \( \exists w \in t \) such that either \( w(x) > \alpha \) and \( w(y) = 0 \) or \( w(y) > \alpha \) and \( w(x) = 0 \), then \( \exists \mu, \nu \in t \) such that \( \overline{1_x} \leq \mu, \overline{1_y} \leq \nu \) and \( \mu \leq 1 - \nu \).
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FR₁ (3): If ∀ x, y ∈ X, x ≠ y and ∀ α ∈ I₀,₁ ∃ w ∈ t such that either w(x) > α and w(y) = 0 or w(y) > α and w(x) = 0, then ∃ µ, ν ∈ t such that µ(x) = 1 = ν(y) and µ ∨ ν = 0.

FR₁ (4): If ∀ x, y ∈ X, x ≠ y and ∀ α ∈ I₀,₁ ∃ w ∈ t such that either w(x) > α and w(y) = 0 or w(y) > α and w(x) = 0, then ∃ µ, ν ∈ t such that µ(x) = 1 = ν(y) and µ ≤ 1 − ν.

FR₁ (5): If ∀ x, y ∈ X, x ≠ y and ∀ α ∈ I₀,₁ ∃ w ∈ t such that either w(x) > α and w(y) = 0 or w(y) > α and w(x) = 0, then ∀ β, δ ∈ I₀,₁ ∃ µ, ν ∈ t such that µ(x) > β, ν(y) > δ and µ ∧ ν = 0.

FR₁ (6): If ∀ x, y ∈ X, x ≠ y and ∀ α ∈ I₀,₁ ∃ w ∈ t such that either w(x) > α and w(y) = 0 or w(y) > α and w(x) = 0, then ∃ µ, ν ∈ t such that µ(x) > 0, ν(y) > 0 and µ ∨ ν = 0.

Theorem 3.1: The following implications hold among the R₁-properties mentioned above:

\[
\begin{array}{c}
FR₁ (1) \iff FR₁ (3) \implies FR₁ (5) \\
FR₁ (2) \iff FR₁ (4) \implies FR₁ (6)
\end{array}
\]

Proof:
FR₁ (1) ⇒ FR₁ (3): Let (X, t) be an fts which has the property FR₁ (1). Let x, y ∈ X, x ≠ y, and w ∈ t such that w(x) > α ∈ I₀,₁ and w(y) = 0. Then, by the FR₁ (1) - property of (X, t), there exist u, v ∈ t such that \( \uparrow x \leq u \), \( \uparrow y \leq v \) and \( u \land v = 0 \). Clearly, u(x) = 1 = v(y) and \( u \lor v = 0 \). Hence, (X, t) has the property FR₁ (3). Thus FR₁ (1) ⇒ FR₁ (3).

FR₁ (4) ⇒ FR₁ (2): Consider a FR₁ (4) -fts (X, t). Let x, y ∈ X, x ≠ y, α ∈ I₀,₁ and w ∈ t such that w(x) > α and w(y) = 0. Then by FR₁ (4), ∃ u, v ∈ t such that u(x) = 1 = v(y) and \( u \leq 1 \land v \). Let \( z ∈ X \) and β ∈ I₀,₁ such that β \( \not \models z \). This implies that β > u(z). Then u(z) = δ ∈ I₀,₁. Then u(z) = δ ∈ I₀,₁ and u(y) = 0 together imply that \( \exists \eta, \lambda ∈ t \) such that \( \eta(y) = 1 = \lambda(z) \) and \( \lambda \leq 1 \land \eta \). Now \( 1 \land \lambda(y) = 1 \). Therefore, \( \uparrow y \leq 1 \land \lambda \). Now, \( \uparrow y(z) \leq 1 \land \lambda(z) = 0 \) and so \( \beta \not \models \uparrow y \). Therefore, \( \uparrow y \leq u \), which is a contradiction as \( u(y) \neq 1 \). Therefore, \( u(z) = 0 \). Now, \( u(x) = 1 > \alpha \), ∀ α ∈ I₀,₁ and u(z) = 0, together imply that \( \exists \eta, \lambda ∈ t \) such that \( \eta(x) = 1 = \lambda(z) \) and \( \lambda \leq 1 \land \eta \). Now,
Therefore, \(\beta l_z \leq 1 - \lambda\). Thus we see that, \(\beta l_z \leq u\) implies \(\beta l_z \leq \beta l_x\). Hence, \(\beta l_x \leq u\). Similarly we can show that \(\beta l_z \leq v\). Therefore, \((X, t)\) is \(R_1^2\). Thus \(FR_1(4) \Rightarrow FR_1(2)\).

All other proofs are similar. □

Now we give some counter examples to show the non-implications among the fuzzy \(R_1\) properties mentioned above.

**Example 3.1** [9]: Let \(X\) be an infinite set and for any \(x, y \in X\), we define \(u_{xy}\), a fuzzy set in \(X\), as follows:

\[
u_{xy}(x) = 1, \quad \nu_{xy}(y) = 0, \quad \nu_{xy}(z) = 0.5 \quad \forall z \in X, \quad z \neq x, y.
\]

Now consider the fuzzy topology \(t\) on \(X\) generated by \(\{u_{xy}: x, y \in X, \quad x \neq y\} \cup \{\text{constants}\}\). It is clear that, \(\nu_{xy} \leq \nu_{y} \land \nu_{x}\) and \(\nu_{xy} \leq 1 - \nu_{yx}\). Thus, \((X, t)\) is \(FR_1(2)\). But \((X, t)\) is not \(FR_1(6)\) as \(\nu_{xy} \land \nu_{yx}\) can never be zero. Thus, we see that, \(FR_1(2) \nRightarrow FR_1(6)\) and therefore, \(FR_1(p) \nRightarrow FR_1(q)\); \((p = 2, 4\) and \(q = 1, 3, 5, 6)\).

**Example 3.2** [9]: Let \(X = I = \{x, y\}\) and \(t\) be the fuzzy topology on \(X\) generated by \(B = B_1 \cup B_2 \cup B_3 \cup B_4\). Where, \(B_1 = \{1, x : x \in I_{0,1}\}, \quad B_2 = \{u_m : m \in \mathbb{N}\}; \ u_m\) is a fuzzy set in \(X\) defined by \(u_m = \left[0, \frac{1}{m+1}\right]\), \(B_3 = \{v_n, F : n \in \mathbb{N}\} \land \ F\) is a finite crisp subset of \(X\), \(\{\text{constants}\}\). Where \(v_{n,F}\) is a fuzzy set in \(X\) defined by \(v_{n,F} = \left[0, \frac{1}{n+1}\right] - F\) and \(B_4 = \{\text{constants}\}\).

It can be checked that \((X, t)\) is \(FR_1(5)\). But \((X, t)\) is not \(FR_1(4)\). For, if we take \(x = 1, \quad y = 0\) and \(u_1 \in B_2\), we see that \(u_1(x) = 0, \quad u_1(y) = 1 > \alpha \quad \forall \alpha \in I_{0,1}\), but there exist no \(u, v \in t\) such that \(u(x) = 1 = v(y)\) and \(u \leq 1 - v\). Thus we see that, \(FR_1(5) \nRightarrow FR_1(4)\) and therefore \(FR_1(p) \nRightarrow FR_1(q)\); \((p = 5, 6\) and \(q = 1, 2, 3, 4)\).

**Example 3.3** [10]: Let \(X = \{x, y\}\) and \(t = \left\{\frac{1}{2} x, \frac{1}{2} y\right\} \cup \{\text{constants}\}\). Then \((X, t)\) is an fts and it is \(FR_1(6)\). But it is not \(FR_1(5)\). For, if we take \(\beta, \delta \in I_{0,1}\) such that \(\beta > 0.5\) and \(\delta > 0.5\), we see that there exist no \(u, v \in t\) such that \(u(x) > \beta, \quad v(y) > \delta\) and \(u \land v = 0\). Thus we see that, \(FR_1(6) \nRightarrow FR_1(5)\).
Theorem 3.2: All \( FR_k \) \( (1 \leq k \leq 6) \) are good extensions of the topological \( R_1 \)-property. That is, \((X, T)\) is an \( R_1 \)-space, if and only if \((X, \omega(T))\) satisfies \( FR_k \) \( (1 \leq k \leq 12) \).

Note: By theorem 3.1, we have only to prove the following:

(a) If \((X, T)\) is an \( R_1 \)-space, then \((X, \omega(T))\) satisfies \( FR_1 \).

(b) If \((X, \omega(T))\) satisfies \( FR_k \) \( (k \in \{4, 6\}) \), then \((X, T)\) is an \( R_1 \)-space.

Proof:

(a) Suppose \((X, T)\) is an \( R_1 \)-topological space. Let \( x, y \in X \), \( x \neq y \), and \( \alpha \in I_{0,1} \), and \( w \in t \) such that \( w(x) > \alpha \) and \( w(y) = 0 \). Now \( w^{-1}(\alpha, 1] \in \omega(T) \) such that \( x \in w^{-1}(\alpha, 1] \) and \( y \notin w^{-1}(\alpha, 1] \). This implies that \( x \notin \{y\} \) in \( T \). Hence there exist \( U, V \in T \) such that \( x \in U \), \( y \in V \) and \( U \cup V = \emptyset \). Since an \( R_1 \)-topological space is also an \( R_0 \)-topological space, \( \{x\} \subseteq U \) and \( \{y\} \subseteq V \). Also we know that, \( 1_{\{x\}} = 1_x \) and \( 1_{\{y\}} = 1_y \). Therefore, \( 1_x \leq 1_U \) and \( 1_y \leq 1_V \). Moreover, \( 1_U \land 1_V = 0 \). Hence \((X, \omega(T))\) satisfies \( FR_1 \).

(b) Suppose \((X, \omega(T))\) satisfies \( FR_k \) \( (k \in \{4, 6\}) \). Let \( x, y \in X \) such that \( x \notin \{y\} \) in \( T \). Then \( \exists w \in T \) such that \( x \in w \) and \( y \notin w \). Now \( 1_w \in \omega(T) \) such that \( 1_w(y) = 0 \) and \( 1_w(x) = 1 > \alpha \forall \alpha \in I_{0,1} \). Therefore \( \exists \mu, \nu \in \omega(T) \) such that \( \mu(x) = 1 = \nu(y) \) and \( \mu \leq 1 - \nu \). Take \( U = \mu^{-1}(\frac{1}{2}, 1] \) and \( V = \nu^{-1}(\frac{1}{2}, 1] \). Clearly, \( U, V \in T \) such that \( x \in U \), \( y \in V \) and \( U \cap V = \emptyset \). Therefore, \((X, T)\) is an \( R_1 \)-topological space.

Again, suppose \((X, \omega(T))\) satisfies \( FR_k \) \( (k \in \{4, 6\}) \). Let \( x, y \in X \) such that \( x \notin \{y\} \) in \( T \). Then \( \exists w \in T \) such that \( x \in w \) and \( y \notin w \). Now \( 1_w \in \omega(T) \) such that \( 1_w(y) = 0 \) and \( 1_w(x) = 1 > \alpha \forall \alpha \in I_{0,1} \). Therefore \( \exists \mu, \nu \in \omega(T) \) such that \( \mu(x) > 0 \), \( \nu(y) > 0 \) and \( \mu \land \nu = 0 \). Now, \( x \in \mu^{-1}(0, 1] \in T \), \( y \in \nu^{-1}(0, 1] \in T \) such that \( \mu^{-1}(0, 1] \cap \nu^{-1}(0, 1] = \emptyset \). Therefore, \((X, T)\) is an \( R_1 \)-topological space. □

Theorem 3.3: The properties \( FR_k \) \( (1 \leq k \leq 6) \) are initial, i.e., if \((f_j: X \rightarrow (X_j, t_j))\) is a source in fts where all \((X_j, t_j)\) are \( FR_k \), then the initial fuzzy topology \( t \) on \( X \) is also \( FR_k \).
Proof:
Let \( \{ (X_j, t_j): j \in J \} \) be a family of \( FR_1(1) \)-fts, \( \{ f_j: X \to (X_j, t_j): j \in J \} \) a family of functions and \( t \) the initial fuzzy topology on \( X \) induced by the family \( \{ f_j: j \in J \} \). Let \( x, y \in X, x \neq y, \alpha \in I_{0, 1} \) and \( w \in t \) such that \( w(x) > \alpha \) and \( w(y) = 0 \). Since \( w \in t \), there exist basic \( t \)-open sets, \( w_p \) such that \( w = \sup \{ w_p: p \in P \} \). Also each \( w_p \) must be expressible as \( w_p = \inf \{ f_{p_k}^{-1}w_{p_k}: 1 \leq k \leq n \} \). As \( w(x) > \alpha \) and \( w(y) = 0 \), we can find some \( k \) such that \( 1 \leq k \leq n \), say \( k' \) such that \( f_{p_{k'}}^{-1}w_{p_{k'}}(x) > \alpha \) and \( f_{p_{k'}}^{-1}w_{p_{k'}}(y) = 0 \). This implies that \( w_{p_{k'}}(x) > \alpha \) and \( w_{p_{k'}}(y) = 0 \). Since \( f_{p_{k'}}(\overline{x}) \leq \mu_{p_{k'}} \), \( f_{p_{k'}}(\overline{y}) \leq \nu_{p_{k'}} \) and \( \mu_{p_{k'}} \wedge \nu_{p_{k'}} = 0 \). Also since \( f_{p_{k'}} \) is continuous, we have \( f_{p_{k'}}(\overline{x}) \leq \overline{f_{p_{k'}}(x)} \). Now put \( \mu = f_{p_{k'}}^{-1}(\mu_{p_{k'}}) \) and \( \nu = f_{p_{k'}}^{-1}(\nu_{p_{k'}}) \). Then \( \mu, \nu \in t \) such that \( \overline{x} \leq \mu, \overline{y} \leq \nu \) and \( \mu \wedge \nu = 0 \). Hence \( (X, t) \) is \( FR_1(1) \). Thus we see that \( FR_1(1) \) is an initial property.

All other proofs are similar.

Corollary-3.4: Since initiality implies productivity and heredity all the \( FR_1(k) \) properties; \( k = 1, 2, ..., 6 \) are productive and hereditary.

4. \( R_0 \)-properties

We recall from [6], nine definitions of the \( R_0 \)-axioms of an fts used in the sequel:

Definitions-4.1 [6]: We define, for fts \((X, t)\), \( R_0 \)-properties as follows:

\[ R_0^1: \text{For every pair } x, y \in X, x \neq y, \overline{1_y(x)} = 0 \Rightarrow \overline{1_x(y)} = 0 \]

\[ R_0^2: \text{For every pair } x, y \in X, x \neq y, (\forall \alpha \in I_0: \alpha \overline{1_x(y)} = \alpha \Leftrightarrow (\forall \beta \in I_0: \overline{1_y(x)} = \beta) \]

\[ R_0^3: \forall \lambda \in t, \forall x \in X \text{ and } \forall \alpha < \lambda(x), \overline{1_x} \leq \lambda \]

\[ R_0^4: \forall \lambda \in t, \forall x \in X \text{ and } \forall \alpha < \lambda(x), \overline{1_x} \leq \lambda \]

\[ R_0^5: \text{For every pair } x, y \in X, x \neq y, \overline{1_x(y)} = 1 \Rightarrow \overline{1_y(x)} = 1 \]

\[ R_0^6: \text{For every pair } x, y \in X, x \neq y, \overline{1_x(y)} = \overline{1_y(x)} \]

\[ R_0^7: \text{For every pair } x, y \in X, x \neq y, \overline{1_x(y)} = \overline{1_y(x)} \in \{0, 1\} \]
$R_0^8$: For every pair $x, y \in X, x \neq y$, and $\forall \alpha \in I_0$, $\overline{\alpha 1_x (y)} = \alpha \Rightarrow \overline{\alpha 1_y (x)} = \alpha$

$R_0^9$: For every pair $x, y \in X, x \neq y$, and $\forall \alpha \in I_0$, $\overline{\alpha 1_x (y)} = \overline{\alpha 1_y (x)}$

**Theorem 4.1** [6]: Between the fuzzy $R_0$-properties, mentioned above, there exist the following implications:

![Diagram](image)

**Theorem 4.2**: The following relations hold between the fuzzy $R_1$-axioms and fuzzy $R_0$-axioms discussed above:

(a) $R_1^1 \Rightarrow R_0^2$, and so $R_1^k \Rightarrow R_0^m$, where $k \in \{1, 2, 3, 4, 5, 6\}$ and $m \in \{2, 3, 4, 8, 9\}$.

(b) $R_1^4 \Rightarrow R_1^1$, and so $R_1^k \Rightarrow R_0^1$, where $k \in \{1, 2, 3, 4\}$.

(c) $R_1^6 \Rightarrow R_0^5$, and so $R_1^k \Rightarrow R_0^5$, where $k \in \{1, 3, 5, 6\}$.

(d) $R_1^5 \Rightarrow R_1^1$, and so $R_1^k \Rightarrow R_0^m$, where $k \in \{5, 6\}$ and $m \in \{1, 4, 6, 7, 9\}$.

(e) $R_1^4 \Rightarrow R_0^7$, and so $R_1^k \Rightarrow R_0^m$, where $k \in \{1, 2, 3, 4\}$ and $m \in \{1, 5, 6, 7\}$.

(f) $R_0^m \Rightarrow FR_1(k)$, where $k \in \{i-xviii\}$ and $m \in \{1-9\}$.

**Proof (a):**

**Example 4.1**: Consider an fts $(X, t)$ where $X = \{x, y\}$, $u(x) = 0.1$, $u(y) = 0.2$ and $t = \{u\} \cup \{\text{constants}\}$. It can be verified that $(X, t)$ is $FR_1(1)$ but it is not $R_0^2$.

**Proof (b):** Consider an $FR_1(4)$-fts $(X, t)$. Let $x, y \in X, x \neq y$ such that $\bar{I}_x(y) = 0$. Put $w = 1 - \bar{I}_x$. Now $w \in t$ such that $w(x) = 0$ and $w(y) = 1 > \alpha \in I_{0.1}$. Since $(X, t)$ is
which is a contradiction. Again let \( v \leq \lambda \) and so \( \overline{I_x}(x) = 0 \). Hence \((X, t)\) is \( R_0^1 \).

**Proof (c):** Consider an \( FR_1(6) \)-fts \((X, t)\). Let \( x, y \in X, x \neq y \) such that \( \overline{I_x}(y) = 1 \).

Suppose \( \overline{I_y}(x) < 1 \).

Put \( w = 1 - \overline{I_y} \). Now \( w \in t \) such that \( w(x) > 0 \) and \( w(y) = 1 \). Then for some \( \alpha \in I_{0,1} \), \( w(x) > \alpha \). Since \((X, t)\) is \( FR_1(6) \), \( \exists u, v \in t \) such that \( u(x) > 0, v(y) > 0 \) and \( u \wedge v = 0 \). \( \lambda = 1 - v \in t^c \). Clearly, \( \overline{I_x} \leq \lambda \). Now, \( \overline{I_x}(y) \leq \lambda(y) < 1 \), which is a contradiction. Therefore, \( \overline{I_y}(x) = 1 \) and so \((X, t)\) is \( R_0^5 \).

**Proof (d):** In example 3.2, it can be verified that \((X, t)\) is \( FR_1(5) \), but it is not \( R_0^1 \) [9].

**Proof (e):** Suppose \((X, t)\) is an \( FR_1(4) \)-fts. Let \( x, y \in X, x \neq y \) such that \( \overline{I_y}(x) \notin \{0,1\} \).

This implies that \( \exists m \in t^c \) such that \( m(y) = 1 \) and \( 0 < m(x) < 1 \). Put \( w = 1 - m \in t \).

Now, \( w(x) > 0 \) and \( w(y) = 0 \). Then, \( w(x) > \alpha \) for some \( \alpha \in I_{0,1} \). Since \((X, t)\) is an \( FR_1(4) \)-fts, \( \exists u, v \in t \) such that \( u(x) = 1 = v(y) \) and \( u \wedge v = 0 \). Put \( n = 1 - u \in t^c \).

Now, \( n(x) = 0 \) and \( n(y) = 1 \). Therefore, \( \overline{I_y} \leq n \) and so \( \overline{I_y}(x) = 0 \), which is a contradiction. Again let \( \overline{I_y}(x) \neq \overline{I_x}(y) \). Without any loss of generality, let \( 0 = \overline{I_x}(y) < \overline{I_y}(x) = 1 \). This implies that \( \exists \lambda_1, \lambda_2 \in t^c \) such that \( \lambda_1(x) = \lambda_2(x) = \lambda_2(y) = 1 \) and \( \lambda_1(y) = 0 \). Take \( w = 1 - \lambda_1 \).

Now \( w \in t \) such that \( w(x) = 0 \) and \( w(y) = 1 > \alpha \). \( \forall \alpha \in I_{0,1} \). Since \((X, t)\) is an \( FR_1(4) \)-fts, \( \exists p, q \in t \) such that \( p(x) = 1 = q(y) \) and \( p \wedge q = 0 \). Put \( n_1 = 1 - p \) and \( n_2 = 1 - q \). Now, \( n_1, n_2 \in t^c \) such that \( n_1(x) = 0 = n_2(y) \) and \( n_1(y) = 1 = n_2(x) \). Clearly, \( \overline{I_y} \leq n_1 \) and so \( \overline{I_y}(x) = 0 \), which is also a contradiction. Therefore, \( \overline{I_x}(y) = \overline{I_y}(x) \in \{0,1\} \). Therefore, \((X, t)\) is an \( R_0^7 \)-fts.

**Proof (f):**

**Example-4.2** [12]: Let \( X \) be an infinite set. For \( x, y \in X \), we define \( U_{xy} \in I^{-X} \) as follows:

\[
U_{xy}(z) = \begin{cases} 
0 & \text{if } z \in \{x, y\} \\
1 & \text{if } z \notin \{x, y\}
\end{cases}
\]

Let \( t \) be the fuzzy topology generated by \( \{U_{xy} : x, y \in X\} \cup \{\text{constants}\} \). It can be checked that if \( x \neq y \), then \( \overline{I_x}(y) = 0 \). Therefore, \((X, t)\) is \( R_0^4, R_0^7 \) and \( R_0^9 \). But \((X, t)\) is
neither $FR_1(4)$ nor $FR_1(6)$ as there exists no $u, v \in t$ such that $u \leq 1 - v$. Therefore,

$R_0^m \nRightarrow FR_1(k)$, where $k \in \{i - xvi\}$ and $m \in \{1 - 9\}$.

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