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A Note on Semiprime Hyperideal in Ternary Hypersemirings

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Abstract

This research aims to define and investigate the properties of semiprime hyperideals in ternary hypersemirings. The researchers introduce the notion of semiprime hyperideal in a ternary hypersemiring and characterize it. The study has expanded by introducing the ideas of weak n-system and strong n-system in a ternary hypersemiring, and using these; the researchers characterize semiprime hyperideals. Again, the researchers introduce the prime radical $\boldsymbol{\beta}(\mathbf{I})$ of a hyperideal **I** in a ternary hypersemiring and obtain the important result that for a proper hyperideal **I** of a ternary hypersemiring **R**, $\boldsymbol{\beta}(\mathbf{I}) = \{\mathbf{r} \in \mathbf{R}$: every weak **m**-system in **R** which contains **r** has a nonempty intersection with $\mathbf{I}\}$. Finally, the work has been concluded by introducing the concept of fully idempotent ternary hypersemirng, and using this concept. It has been proved that if **S** is a commutative ternary hypersemiring with hyperidentity, then **S** is a regular commutative ternary hypersemiring with hyperidentity.

Keywords: Hyperideal, Semiprime hyperideal; Ternary hypersemiring; Weak n-system; Strong n-system; Fully idempotent.

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1. Introduction

Now a highly well-established area of mathematics, algebraic hyperstructure is founded on the idea of hyperoperation, first proposed by Marty [1] in 1934. Hyperstructure theory has grown remarkably in the modern era. The idea of hyperstructure, which has several implications in the pure and applied sciences, has piqued the curiosity of many scholars. Corsini *et al.* [2] and Leoreanu-Fotea *et al.* [3] list their applications in a variety of disciplines, including computer science and theoretical physics. Algebraic hyperstructure is an extension of classical algebra, where the composition of elements of a set is a subset of that set, in contrast to classical algebra, where the combination of two elements is again an element. The idea of multiplicative hyperring was developed by Rota [4], in which addition is a binary operation and multiplication is a binary hyperoperation. In 2011, Dasgupta [5] proposed the idea of multiplicative hypersemiring, which is a generalization of multiplicative hyperring. In multiplicative hypersemirings, Salim [6] investigated the quasi-hyperideals and bi-hyperideals. The concept of a ternary ring was first proposed,

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and several of its key characteristics were examined by Lister [7]. After that, in 2003, Dutta and Kar [8] generalized the notion of a ternary ring and introduced ternary semiring.

A study on regular and very regular equivalency relations on multiplicative ternary hyperring was conducted by Salim *et al*. [9]. An extension of multiplicative ternary hyperrings, ternary semirings, and ternary hypersemirings was first proposed by Tamang *et al*. [10]. In [11], Tamang *et al*. studied the properties of hyperideals in ternary hypersemirings. Mandal *et al*. [12] studied the radical properties, prime hyperideals, and primary hyperideals of ternary hypersemirings. Salim and Sinha [13] discussed the relations on ternary hypersemirings and obtained three isomorphism theorems on ternary hypersemirings. Again, Hilla *et al.* [14] studied the quasi-hyperideals and bi-hyperideals in ternary hypersemirings. Up until now, different kinds of hyperideals have been discussed in ternary hypersemirings, like primary hyperideal, prime hyperideal, quasi-hyperideal, and bi-hyperideal. But in this exploration, a special type of hyperideal has been initiated that is not a prime hyperideal, called semirprime hyperideal of ternary hypersemirings. The researchers obtain some important results on semiprime hyperideals and characterize ternary hypersemiring by semiprime hyperideals. Also, weak n-system and strong n-system have been introduced in ternary hypersemirings and prove that if a hyperideal is semiprime, then its complement is a strong n-system, and conversely.

2. Preliminaries

Definition 2.1 [11] An additive commutative semigroup (S,+) equipped with a ternary hyperoperation '∘' that satisfies the subsequent requirements is called a ternary hypersemiring (S,+,∘).

(i) $(\mathbf{a} \circ \mathbf{b} \circ \mathbf{c}) \circ \mathbf{d} \circ \mathbf{e} = \mathbf{a} \circ (\mathbf{b} \circ \mathbf{c} \circ \mathbf{d}) \circ \mathbf{e} = \mathbf{a} \circ \mathbf{b} \circ (\mathbf{c} \circ \mathbf{d} \circ \mathbf{e})$;

(ii) $(a + b) \circ c \circ d \subseteq a \circ c \circ d + b \circ c \circ d$;

(iii) $a \circ (b + c) \circ d \subseteq a \circ b \circ d + a \circ c \circ d$;

(iv) $a \circ b \circ (c + d) \subseteq a \circ b \circ c + a \circ b \circ d$.

for all $a, b, c, d, e \in S$. Hence, the ternary hypersemiring is referred to as a strongly distributive ternary hypersemiring if the inclusions in (ii) are substituted with equalities.

Definition 2.2 [11] A nonempty finite subset $\varepsilon = \{(e_i, f_i)\}_{i=1}^{n}$ of $R \times R$ where R is a ternary hypersemiring is called a left (resp. lateral, right) identity set of R if for any $a \in R$,

$$a \in \sum_{i=1}^{n} e_i \circ f_i \circ a \quad (\text{resp. } a \in \sum_{i=1}^{n} e_i \circ a \circ f_i \text{ , } a \in \sum_{i=1}^{n} a \circ e_i \circ f_i \text{ })$$

An element e of a ternary hypersemiring $(R, +, \circ)$ is called an unital element of R if $a \in (e \circ e \circ a) \cap (e \circ a \circ e) \cap (a \circ e \circ e)$.

Example 2.3 Let us consider $\mathbb{Z}^{-}$, ring of non-negative integers and $A = \{2,5\}$, then $R = \mathbb{Z}_{\mathbb{A}}^{-}$ is an additive commutative semigroup with respect to usual addition of integers. We now define a ternary hyperoperation on R as follows $a \circ b \circ c = \{a \cdot n_1 \cdot b \cdot n_2 \cdot c : n_1, n_2 \in A\}$. Then $(R, +, \circ)$ is a ternary hypersemiring.

Definition 2.4 [11] Let $(S, +, \circ)$ be ternary hypersemiring. An additive subsemigroup I of S is called

(i) a left hyperideal of **S** if $\mathbf{s_1} \circ \mathbf{s_2} \circ \mathbf{x} \subseteq \mathbf{I}$, for all $\mathbf{x} \in \mathbf{I}$ and for all $\mathbf{s_1}, \mathbf{s_2} \in \mathbf{S}$;

(ii) a right hyperideal of **S** if $\mathbf{x} \circ \mathbf{s_1} \circ \mathbf{s_2} \subseteq \mathbf{I}$, for all $\mathbf{x} \in \mathbf{I}$ and for all $\mathbf{s_1}, \mathbf{s_2} \in \mathbf{S}$;
(iii) a lateral hyperideal of **S** if $\mathbf{s_1} \circ \mathbf{x} \circ \mathbf{s_2} \subseteq \mathbf{I}$, for all $\mathbf{x} \in \mathbf{I}$ and for all $\mathbf{s_1}, \mathbf{s_2} \in \mathbf{S}$;
(iv) if I is a left and a right hyperideal of S, then I is a two-sided hyperideal of S;
(v) a hyperideal of S if I correspond to S's left, right, and lateral hyperideals.

Definition 2.5 [5] If for any three hyperideals I, J, and K of S, a proper hyperideal P of a ternary hypersemiring S is a prime hyperideal of S if $I \circ J \circ K \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$ or $K \subseteq P$.

Proposition 2.6 If $(S, +, \circ)$ is a ternary hypersemiring and $a \in S$, then $\langle a\rangle_r = a \circ S \circ S + \{na: n \in \mathbb{Z}\}$ is the right hyperideal of S generated by a. Similarly, if $(S, +, \circ)$ is a ternary hypersemiring and $a \in S$, then $\langle a\rangle_m = S \circ a \circ S + S \circ S \circ a \circ S \circ S + \{na: n \in \mathbb{Z}\}$ is the lateral hyperideal of S generated by a, and $\langle a\rangle_l = S \circ S \circ a + \{na: n \in \mathbb{Z}\}$ is the left hyperideal of S generated by a. Finally, $\langle a\rangle = S \circ S \circ a + a \circ S \circ S + S \circ a \circ S + S \circ S \circ a \circ S \circ S + \{na: n \in \mathbb{Z}\}$ is the hyperideal of S generated by a.

Proposition 2.7 If $(S, +, \circ)$ is a ternary hypersemiring with a unital element e or with an identity set, and $a \in S$ then $\langle a\rangle_r = a \circ S \circ S$ is the right hyperideal of S generated by a. Similarly, $\langle a\rangle_m = S \circ S \circ a \circ S \circ S$ is the lateral hyperideal of S generated by a and $\langle a\rangle_l = S \circ S \circ a$ is the left hyperideal of S generated by a. Finally, $\langle a\rangle = S \circ S \circ a \circ S \circ S$ is the hyperideal of S generated by a.

3. Semiprime Hyperideal

Definition 3.1 An appropriate hyperideal Q is referred to be a semiprime hyperideal of a ternary hypersemiring (S,+,∘) if $I \circ I \circ I \subseteq Q$ implies $I \subseteq Q$ for any hyperideal I of S.

Example 3.2 Consider the ring of non-negative integers $\mathbb{Z}^-$. Let $A = \{-2, -5\}$. Now we define a ternary hyperoperation '∘' on $\mathbb{Z}^-$ as follows: $a \circ b \circ c = \{a \cdot x \cdot b \cdot y \cdot c: x, y \in A\}$. Then $(\mathbb{Z}^-, +, \circ)$ is a ternary hypersemiring. We denote it by $(\mathbb{Z}_A^-, +, \circ)$. Now $6\mathbb{Z}^-$ is a hyperideal of $(\mathbb{Z}_A, +, \circ)$. Let U be any hyperideal of $(\mathbb{Z}_A^-, +, \circ)$ and $U \circ U \circ U \subseteq 6\mathbb{Z}^-$. Let $U \nsubseteq 6\mathbb{Z}^-$, then there exists an element $a \in U$ such that $a \notin 6\mathbb{Z}^-$. Now $a \circ a \circ a \subseteq 6\mathbb{Z}^-$. This implies that $6|a \circ a \circ a = a \cdot x \cdot a \cdot y \cdot a$ for $x, y \in A$. This implies that $6|a^3 \cdot x \cdot y \Rightarrow 6|a^3$, since $6 \nmid x, y$. Then $3|a^3 \Rightarrow 3|a$ and $2|a^3 \Rightarrow 2|a$, since 2 and 3 are prime. This implies $6|a$. So $a \in 6\mathbb{Z}^-$, a contradiction. Therefore $U \subseteq 6\mathbb{Z}^-$ and hence $6\mathbb{Z}^-$ is a semiprime hyperideal of $(\mathbb{Z}_A^-, +, \circ)$.

Proposition 3.3 Every prime hyperideal in a ternary hypersemiring is a semiprime hyperideal, but the converse is not true.
We can show this by the following example:

Example 3.4 Let us consider the ternary hypersemiring $(\mathbb{Z}_{\mathbb{A}}^-, +, \circ)$, where $A = \{3,4\}$, $a \circ b \circ c = \{a \cdot x \cdot b \cdot y \cdot c: x, y \in A\}$ and consider $15\mathbb{Z}^-$. Let $I \circ I \circ I \subseteq 15\mathbb{Z}^-$, where I be any hyperideal of $\mathbb{Z}_{\mathbb{A}}^-$. If possible, let I is not a subset of $15\mathbb{Z}^-$, so there exist $x \in I$ such that $x \notin 15\mathbb{Z}^- \Longrightarrow 15 \nmid x \Longrightarrow$ either $3 \nmid x$ or $5 \nmid x$. Again $x \circ x \circ x \in I^3 \subseteq 15\mathbb{Z}^-$. Then $x \cdot a \cdot x \cdot b \cdot x \in 15\mathbb{Z}^-$ for some $a, b \in A$. This implies that $15|x^3 \cdot a \cdot b \Longrightarrow 15|ab$, which is a contradiction. So, $I \subseteq 15\mathbb{Z}^-$ i.e. $15\mathbb{Z}^-$ is a semiprime hyperideal.

Next let L, M, N be hyperideals of $\mathbb{Z}_{\mathbb{A}}^{-}$ such that $L \circ M \circ N \subseteq 15\mathbb{Z}^{-}$, where $L = 3\mathbb{Z}^{-}, M = 5Z^{-}, N = 2\mathbb{Z}^{-}$. Then $L \circ M \circ N = \cup \{\sum a_i \circ b_i \circ c_i : a_i \in L, b_i \in M, c_i \in N\} = \cup \{\sum a_i \cdot x \cdot b_i \cdot y \cdot c_i : a_i \in L, b_i \in M, c_i \in N\}$, but none of L, M, N is a subset of $15\mathbb{Z}^{-}$. So this is not a prime hyperideal.

Theorem 3.5 Let $(R, +, \circ)$ be a ternary hypersemiring. An element x of R belongs to a semiprime hyperideal Q of R if and only if $R \circ x \circ R \subseteq Q$.

Proof. Suppose Q is a semiprime hyperideal of R. If $x \in Q$, then obviously $R \circ x \circ R \subseteq R \circ Q \circ R \subseteq Q$. Conversely, let $R \circ x \circ R \subseteq Q$. Then $R \circ R \circ x \circ R \circ R \subseteq R \circ Q \circ R \subseteq Q$. Now

$$\begin{aligned}&\langle x\rangle \circ \langle x\rangle \circ \langle x\rangle \\ &= (R \circ R \circ x + x \circ R \circ R + R \circ x \circ R + R \circ R \circ x \circ R \circ R + \{nx : n \in \mathbb{Z}^{-}\}) \circ \\ &(R \circ R \circ x + x \circ R \circ R + R \circ x \circ R + R \circ R \circ x \circ R \circ R + \{nx : n \in \mathbb{Z}^{-}\}) \circ \\ &(R \circ R \circ x + x \circ R \circ R + R \circ x \circ R + R \circ R \circ x \circ R \circ R + \{nx : n \in \mathbb{Z}^{-}\}) \\ &\subseteq (R \circ x \circ R + R \circ R \circ x \circ R \circ R) \subseteq Q.\end{aligned}$$

Since Q is semiprime hyperideal, we have $\langle x\rangle \subseteq Q$ and hence $x \in Q$.

Theorem 3.6 Let $(R, +, \circ)$ be a ternary hypersemiring and Q be a hyperideal of R. Then the following conditions are equivalent:

(i) Q is a semiprime hyperideal of R;

(ii) $a \circ R \circ a \circ R \circ a \subseteq Q$, $a \circ R \circ R \circ a \circ R \circ R \circ a \subseteq Q$, $a \circ R \circ R \circ a \circ R \circ a \circ R \subseteq Q$, $R \circ a \circ R \circ a \circ R \circ R \circ a \subseteq Q$ implies $a \in Q$;

(iii) For any $a \in R$, if $\langle a\rangle \circ \langle a\rangle \circ \langle a\rangle \subseteq Q$ then $a \in Q$;

(iv) If U is a right as well as a lateral hyperideal of R such that $U \circ U \circ U \subseteq Q$, then $U \subseteq Q$;

(v) If V is a left as well as a lateral hyperideal of R such that $V \circ V \circ V \subseteq Q$ then $V \subseteq Q$.

Proof. The proof is similar to the proof of the Theorem 3.4 of [15], and hence we omit it.

Corollary 3.7 Let $(R, +, \circ)$ be a ternary hypersemiring with a unital e or with an identity set. Then a hyperideal Q of R is semiprime if and only if $a \circ R \circ R \circ a \circ R \circ R \circ a \subseteq Q$ implies $a \in Q$.

Proof. Let Q be a semiprime hyperideal of R. Suppose $a \circ R \circ R \circ a \circ R \circ R \circ a \subseteq Q$. Then, $a \circ R \circ a \circ R \circ a \subseteq a \circ R \circ (e \circ a \circ e) \circ R \circ a \subseteq a \circ R \circ R \circ a \circ R \circ R \circ a \subseteq Q$. Similarly $a \circ R \circ R \circ a \circ R \circ a \circ R \subseteq Q$ and $R \circ a \circ R \circ a \circ R \circ R \circ a \subseteq Q$. Now, by Theorem 3.6, $a \in Q$. Similar is the proof when R contains identity set.

Conversely, suppose that $a \circ R \circ R \circ a \circ R \circ R \circ a \subseteq Q$ implies that $a \in Q$ wherea $\in R$. Let U be a hyperideal of R such that $U \circ U \circ U \subseteq Q$. Let $x \in U$. Now $x \circ R \circ R \circ x \circ R \circ R \circ x \subseteq U \circ U \circ U \subseteq Q$. This implies that $x \in Q$. Thus $U \subseteq Q$ and hence Q is a semiprime hyperideal of R.

Corollary 3.8 A proper hyperideal Q of a commutative ternary hypersemiring $(R, +, \circ)$ is semiprime if and only if $a \circ a \circ a \subseteq Q$ implies that $a \in Q$ for any element a of R.

Proof. Assume that a commutative ternary hypersemiring R has a semiprime hyperideal Q and $a \circ a \circ a \subseteq Q$. Now $a \circ R \circ (a \circ R \circ a) = a \circ R \circ (a \circ a \circ R) = a \circ (R \circ a \circ a) \circ R =$

$(a \circ a \circ a) \circ R \circ R = Q \circ R \circ R \subseteq Q$ i.e., $a \circ R \circ a \circ R \circ a \subseteq Q$. Similarly $a \circ R \circ R \circ a \circ R \circ R \circ a \subseteq Q$, $a \circ R \circ R \circ a \circ R \circ a \circ R \subseteq Q$ and $R \circ a \circ R \circ a \circ R \circ R \circ a \subseteq Q$. Since Q is a semiprime hyperideal of R. Now by (ii) of the Theorem 3.6, we get $a \in Q$.
Converse is similar to the converse part of Corollary

Definition 3.9 A nonempty subset A of a ternary hypersemiring $(R, +, \circ)$ is called a weak m-system if for any $a, b, c \in A$, $(a \circ R \circ b \circ R \circ c) \cap A \neq \phi$ or $(a \circ R \circ R \circ b \circ R \circ R \circ c) \cap A \neq \phi$ or $(a \circ R \circ R \circ b \circ R \circ c \circ R) \cap A \neq \phi$ or $R(\circ a \circ R \circ b \circ R \circ R \circ c) \cap A \neq \phi$.

Definition 3.10 Let $(R, +, \circ)$ be a ternary hypersemiring. Then a nonempty subset A of R is called a strong m-system if for any $a, b, c \in A$ and there exist $x, y, s, t \in R$ such that $(a \circ x \circ b \circ y \circ c) \cap A \neq \phi$ or $(a \circ x \circ y \circ b \circ s \circ t \circ c) \cap A \neq \phi$ or $(a \circ x \circ y \circ b \circ s \circ c \circ t) \cap A \neq \phi$ or $(x \circ a \circ y \circ b \circ s \circ t \circ c) \cap A \neq \phi$.

Example 3.11 Consider the ring of non-negative integers $\mathbb{Z}^-$. Let $A = \{1,2\}$. Now, we define a ternary hyperoperation '$\circ$' on $\mathbb{Z}^-$ as follows: $a \circ b \circ c = \{a \cdot x \cdot b \cdot y \cdot c : x, y \in A\}$. Then $(\mathbb{Z}^-, +, \circ)$ is a commutative ternary hypersemiring. We denote it by $({\mathbb{Z}^-}_A, +, \circ)$. Now $\mathbb{Z}^- \setminus 3\mathbb{Z}^-$ is a strong m-system as well as a weak m-system.

Definition 3.12 A nonempty subset N of a ternary hypersemiring $(R, +, \circ)$ is called a weak n-system if for any $a \in N$, $(a \circ R \circ a \circ R \circ a) \cap N \neq \phi$ or $(a \circ R \circ R \circ a \circ R \circ R \circ a) \cap N \neq \phi$ or $(a \circ R \circ R \circ a \circ R \circ a \circ R) \cap N \neq \phi$ or $(R \circ a \circ R \circ a \circ R \circ R \circ a) \cap N \neq \phi$.

Definition 3.13 A nonempty subset N of a ternary hypersemiring R is called a strong n-system if for each $a \in N$, and there exist elements x_1, x_2, x_3, x_4 of R such that $(a \circ x_1 \circ a \circ x_2 \circ a) \cap N \neq \phi$ or $(a \circ x_1 \circ x_2 \circ a \circ x_3 \circ x_4 \circ a) \cap N \neq \phi$ or $(a \circ x_1 \circ x_2 \circ a \circ x_3 \circ a \circ x_4) \cap N \neq \phi$ or $(x_1 \circ a \circ x_2 \circ a \circ x_3 \circ x_4 \circ a) \cap N \neq \phi$.

Example 3.14 Consider the ring of negative integers $\mathbb{Z}^-$. Let $A = \{-1, -2\}$. Now, we define a ternary hyperoperation '$\circ$' on $\mathbb{Z}^-$ as follows: $a \circ b \circ c = \{a \cdot x \cdot b \cdot y \cdot c : x, y \in A\}$. Then $(\mathbb{Z}^-, +, \circ)$ is a commutative ternary hypersemiring. We denote it by $(\mathbb{Z}^-_A, +, \circ)$. Now $6\mathbb{Z}^-$ is a hyperideal of $(\mathbb{Z}^-_A, +, \circ)$. Now $\mathbb{Z}^- \setminus 6\mathbb{Z}^-$ is a strong n-system as well as a weak n-system.

Theorem 3.15 Let $(R, +, \circ)$ be a ternary hypersemiring and Q be a hyperideal of R. Then the following conditions are equivalent :

(i) **Q** is a semiprime hyperideal of **R**;

(ii) $\mathbf{Q^c}$(complement of **Q** in **R**) is a strong **n**-system;

(iii) $\mathbf{Q^c}$ is a weak **n**-system.

Proof. (i) ⇒ (ii) Let $a \in Q^c$. Suppose that $(a \circ x \circ a \circ y \circ a) \cap Q^c = \phi$, for all $x, y \in R$. Then $a \circ x \circ a \circ y \circ a \subseteq Q$ for all $x, y \in R$. This implies that $a \circ R \circ a \circ R \circ a \subseteq Q$. Next, suppose that $(a \circ x \circ y \circ a \circ s \circ t \circ a) \cap Q^c = \phi$ for all $x, y, s, t \in R$. Then $a \circ x \circ y \circ a \circ s \circ t \circ a \subseteq Q$ for all $x, y, s, t \in R$. This implies that $a \circ R \circ R \circ a \circ R \circ R \circ a \subseteq Q$. Similarly, if $(a \circ x \circ y \circ a \circ s \circ a \circ t) \cap Q^c = \phi$ and $(x \circ a \circ y \circ a \circ s \circ t \circ a) \cap Q^c = \phi$ for all $x, y, s, t \in R$, then we can show that $a \circ R \circ R \circ a \circ R \circ a \circ R \subseteq Q$ and $R \circ a \circ R \circ a \circ R \circ R \circ a \subseteq Q$ respectively. Since Q is semiprime, $a \in Q$, a contradiction. Thus Q^c is a strong n-system.
(ii) ⇒ (iii) It is obvious.
(iii) ⇒ (i) Let $a \circ R \circ a \circ R \circ a \subseteq Q$, $a \circ R \circ R \circ a \circ R \circ R \circ a \subseteq Q$, $a \circ R \circ R \circ a \circ R \circ a \circ R \subseteq Q$ and $R \circ a \circ R \circ a \circ R \circ R \circ a \subseteq Q \cdots (A)$. If possible, let $a \in Q^c$. Since Q^c is a weak

n-system, $(a \circ R \circ a \circ R \circ a) \cap Q^c \neq \phi$ or $(a \circ R \circ R \circ a \circ R \circ R \circ a) \cap Q^c \neq \phi$ or $(a \circ R \circ R \circ a \circ R \circ a \circ R) \cap Q^c \neq \phi$ or $(R \circ a \circ R \circ a \circ R \circ R \circ a) \cap Q^c \neq \phi$, which contradicts (A). So $a \in Q$. Hence, Q is a semiprime hyperideal of R.

Lemma 3.16 Let $(R, +, \circ)$ be a ternary hypersemiring with an unital element e and Q be a hyperideal of R and $a \in R$. Then Q^c is weak n-system if and only if $a \in Q^c$ implies that $(a \circ R \circ R \circ a \circ R \circ R \circ a) \cap Q^c \neq \phi$.

Proof. Let Q be a hyperideal of R and Q^c be a weak n-system $\Leftrightarrow Q$ is a semiprime hyperideal of R [by Theorem 3.15] $\Leftrightarrow (a \circ R \circ R \circ a \circ R \circ R \circ a) \subseteq Q \Rightarrow a \in Q$ [by (ii) of the Corollary 3.7] $\Leftrightarrow a \notin Q \Rightarrow (a \circ R \circ R \circ a \circ R \circ R \circ a) \nsubseteq Q$
$\Leftrightarrow a \in Q^c \Rightarrow (a \circ R \circ R \circ a \circ R \circ R \circ a) \cap Q^c \neq \phi$

Lemma 3.17 Let Q be a semiprime hyperideal of a ternary hypersemiring $(R, +, \circ)$ with an unital element e and $a \in Q^c$, then there exists a weak m-system M in R such that $a \in M \subseteq Q^c$.

Proof. We take a subset $M = \{a_1, a_2, a_3, \cdots\}$ of R, in which the elements $a_1, a_2, \cdots$ are defined as follows:

Let $a_1 = a$. Since $a \in Q^c$ and Q^c is a weak n-system (as Q is semiprime hyperideal of R with unital element e), $(a_1 \circ R \circ R \circ a_1 \circ R \circ R \circ a_1) \cap Q^c \neq \phi$ (by Lemma 3.16). Take an element of $(a_1 \circ R \circ R \circ a_1 \circ R \circ R \circ a_1) \cap Q^c$ as a_2. Take $a_3 \in (a_2 \circ R \circ R \circ a_2 \circ R \circ R \circ a_2) \cap Q^c$. Repeating this process, we get elements $a_1, a_2, \cdots$ of M. Then $a \in M \subseteq Q^c$. It remains to show that M is a weak m-system in R. Let $a_i, a_j, a_k \in M$. Without loss of generality, we may assume that $i \leq j \leq k$

Now

$$\begin{aligned}
a_{k+1} &\in a_k \circ R \circ R \circ a_k \circ R \circ R \circ a_k \\
\Rightarrow a_{k+1} &\in a_k \circ R \circ R \circ (a_{k-1} \circ R \circ R \circ a_{k-1} \circ R \circ R \circ a_{k-1}) \circ R \circ R \circ a_k \\
&\subseteq a_k \circ R \circ R \circ a_{k-1} \circ R \circ R \circ a_k \\
&\subseteq (a_{k-1} \circ R \circ R \circ a_{k-1} \circ R \circ R \circ a_{k-1}) \circ R \circ R \circ a_{k-1} \circ R \circ R \circ a_k \\
&\subseteq a_{k-1} \circ R \circ R \circ a_{k-1} \circ R \circ R \circ a_k \\
&\subseteq (a_{k-2} \circ R \circ R \circ a_{k-2} \circ R \circ R \circ a_{k-2}) \circ R \circ R \circ a_{k-1} \circ R \circ R \circ a_k \\
&\subseteq a_{k-2} \circ R \circ R \circ a_{k-1} \circ R \circ R \circ a_k \\
&\dots\dots\dots\dots \\
&\subseteq a_i \circ R \circ R \circ a_j \circ R \circ R \circ a_k
\end{aligned}$$

i.e., $a_{k+1} \in a_i \circ R \circ R \circ a_j \circ R \circ R \circ a_k$. This implies that $(a_k \circ R \circ R \circ a_k \circ R \circ R \circ a_k) \cap M \neq \phi$. So M is a weak m-system.

Definition 3.18 The prime radical $\beta(I)$ of a hyperideal I in a ternary hypersemiring R is defined to be the intersection of all prime hyperideals of R which contain I.

Theorem 3.19 For a proper hyperideal I of a ternary hypersemiring R, $\beta(I) = \{r \in R$: every weak m-system in R which contains r has a nonempty intersection with I$\}$.

Proof. Let $S = \{r \in R$: every weak m-system in R which contains r has a nonempty intersection with I$\}$. Let $r \notin \beta(I)$. Then, there exists a prime hyperideal P of R containing I such that $r \notin P$. Then $r \in P^c$. Since P is a prime hyperideal of R, P^c is a weak m-system of R (by Proposition 3.10). Thus, we get a weak m-system P^c containing r, which has an empty intersection with I. So $r \notin S$. Thus $S \subseteq \beta(I) \cdots (1)$

Conversely, suppose that $r \notin S$, then there exists a weak m-system M containing r such that M has an empty intersection with I, i.e., $M \cap I = \phi$. Let $\mathcal{F} = \{J: J$ is a hyperideal of R containing I such that $M \cap J = \phi\}$. Obviously $\mathcal{F} \neq \phi$ and $\mathcal{F}$ is a poset with respect to the set inclusion. Using Zorn's lemma, we can find a maximal element P in $\mathcal{F}$. Then P is a hyperideal of R containing I and $P \cap M = \phi \cdots (2)$. We claim that P is a prime hyperideal of R. If possible let P be not prime. Then there exist three elements $a \notin P, b \notin P$ and $c \notin P$ but $\langle a \rangle \circ \langle b \rangle \circ \langle c \rangle \subseteq P$. Now by maximality of P in $\mathcal{F}$, $M \cap (P + \langle a \rangle) \neq \phi, M \cap (P + \langle b \rangle) \neq \phi$ and $M \cap (P + \langle c \rangle) \neq \phi$. Let $m_1 \in M \cap (P + \langle a \rangle)$, $m_2 \in M \cap (P + \langle b \rangle)$ and $m_3 \in M \cap (P + \langle c \rangle)$. Then $m_1, m_2, m_3 \in M$. Since M is a weak m-system, $(m_1 \circ R \circ m_2 \circ R \circ m_3) \cap M \neq \phi$ or $(m_1 \circ R \circ R \circ m_2 \circ R \circ R \circ m_3) \cap M \neq \phi$ or $(m_1 \circ R \circ R \circ m_2 \circ R \circ m_3 \circ R) \cap M \neq \phi$ or $(R \circ m_1 \circ R \circ m_2 \circ R \circ R \circ m_3) \cap M \neq \phi$. Let $(m_1 \circ R \circ m_2 \circ R \circ m_3) \cap M \neq \phi$. Now $m_1 \circ R \circ m_2 \circ R \circ m_3 \subseteq (P + \langle a \rangle) \circ (P + \langle b \rangle) \circ (P + \langle c \rangle) \subseteq P$. So $P \cap M \neq \phi$, a contradiction. Similarly, we can show that in each of the above cases we arrive at a contradiction. Thus P is a prime hyperideal of R. Since $P \cap M = \phi$ and $r \in M, r \notin P$. This implies that $r \notin \beta(I)$. Therefore $\beta(I) \subseteq S \cdots (3)$. Combining (1) and (3) we get $\beta(I) = S$.

Theorem 3.20 A proper hyperideal Q of a ternary hypersemiring $(R, +, \circ)$ with an unital element 'e' is semiprime if and only if $\beta(Q) = Q$.

Proof. Let $\beta(Q) = Q$. Now $I \circ I \circ I \subseteq Q$ where I is a hyperideal of R. $\Rightarrow I \circ I \circ I \subseteq \beta(Q)$

$\Rightarrow I \circ I \circ I \subseteq \cap \{P : P$ is a prime hyperideal of R containing $Q\}$ $\Rightarrow I \circ I \circ I \subseteq P$; P is a prime hyperideal of R containing Q

$\Rightarrow I \subseteq P$, since P is a prime hyperideal of R

$\Rightarrow I \subseteq \cap\{P : P$ is a prime hyperideal of R containing $Q\}$

$\Rightarrow I \subseteq \beta(Q) = Q$

Therefore Q is a semiprime hyperideal.

Conversely, we assume that Q is a semiprime hyperideal of R. We shall show that $\beta(Q) = Q$. Obviously $Q \subseteq \beta(Q)$. Now let $r \notin Q$. Then $r \in Q^c$. Since Q is a semiprime hyperideal of R, by the Lemma 3.17, there exists a weak m-system M in R such that $r \in M \subseteq Q^c$. Then $M \cap Q = \phi$. Thus, there exists a weak m-system M in R containing r which has an empty intersection with Q. So, $r \notin \beta(Q)$. Hence $\beta(Q) \subseteq Q$. So $\beta(Q) = Q$.

Corollary 3.21 A proper hyperideal Q of a ternary hypersemiring $(R, +, \circ)$ with an unital element 'e' is semiprime if and only if it is the intersection of some prime hyperideals.

Proof. We first suppose that Q is a semiprime hyperideal of R. Then $Q = \beta(Q)$. This implies that Q is the intersection of some prime hyperideals of R.

Converse is obvious.

Corollary 3.22 Let I be a hyperideal of a ternary hypersemiring $(R, +, \circ)$ with an unital element 'e', then $\beta(I)$ is the smallest semiprime hyperideal of R containing I.

Proof. Obviously $I \subseteq \beta(I)$. Since $\beta(I)$ is the intersection of some prime hyperideals of R, $\beta(I)$ is a semiprime hyperideal of R containing I. Let $T = \{P: P$ is a prime hyperideal of R such that $I \subseteq P\}$ and $S = \{P: P$ is a prime hyperideal of R such that $Q \subseteq P$ where Q is a semiprime hyperideal of R containing I}. Then $S \subseteq T \Rightarrow \cap T \subseteq \cap S \Rightarrow \beta(I) \subseteq \beta(Q) \Rightarrow \beta(I) \subseteq Q$ [$\because$ Q is semiprime]. Hence $\beta(I)$ is the smallest semiprime hyperideal of R containing I.

Proposition 3.23 Let I be a hyperideal of a ternary hypersemiring (R, +,∘). Then the following conditions are equivalent :

(i) I is a semiprime hyperideal of R;

(ii) $\cup\{a \circ x \circ a: x \in R\} \subseteq I \Leftrightarrow a \in I$.

Proof. (i) ⇒ (ii) : Let $a \in R$ and $I' = \cup\{a \circ x \circ a: x \in R\}$. If $a \in I$, then $I' \subseteq I$ (since I is a hyperideal of R).

Conversely, let $I' \subseteq I$, $J = \langle a \rangle = R \circ R \circ a + a \circ R \circ R + R \circ a \circ R + R \circ R \circ a \circ R \circ R + \{na: a \in \mathbb{N}\}$. Let, $x \in J \circ J \circ J = \langle a \rangle \circ \langle a \rangle \circ \langle a \rangle$. $x \in (R \circ R \circ a + a \circ R \circ R + R \circ a \circ R + R \circ R \circ a \circ R \circ R + \{na: a \in \mathbb{N}\}) \circ (R \circ R \circ a + a \circ R \circ R + R \circ a \circ R + R \circ R \circ a \circ R \circ R + \{na: a \in \mathbb{N}\}) \circ (R \circ R \circ a + a \circ R \circ R + R \circ a \circ R + R \circ R \circ a \circ R \circ R + \{na: a \in \mathbb{N}\}) \subseteq I$ (as $I' \subseteq I$). Then $J \circ J \circ J \subseteq I \Rightarrow J \subseteq I$. Hence $a \in I$.

(ii) ⇒ (i) is obvious.

Proposition 3.24 Let I be a semiprime hyperideal of a commutative ternary hypersemiring $(R, +, \circ)$. Then for any $a \in R, a^3 \subseteq I \Rightarrow a \in I$.

Proof. $a^3 \subseteq I \Rightarrow \cup\{a \circ x \circ a: x \in R\} \subseteq I \Rightarrow a \in I$, by Proposition 3.23

Definition 3.25 A nonempty subset A of a ternary hypersemiring $(S, +, \circ)$ is called a weak p-system if for any $a \in A$, $(a \circ S \circ a) \cap A$ is nonempty.

Proposition 3.26 A hyperideal I of a ternary hypersemiring R is semiprime iff I^c is a weak p-system of R.

Note 3.27 Every weak m-system is a weak p-system and union of weak p-systems is also a weak p system.

Proposition 3.28 A nonempty set A is a weak p-system of a ternary hypersemiring R if and only if it is a union of weak m-systems.

Proof. Necessary part is followed by Note 3.27.

Conversely, let A be a weak p-system. Let us construct a sequence $\{a_1, a_2, a_3, \ldots\ldots\}$ from elements of A such that,

(i) $\mathbf{a_1}$ is a fixed element of A, chosen arbitrarily.

(ii) for $\in \mathbb{N}, \mathbf{m > 1}$, $\mathbf{a_m \in (a_{m-1} \circ R \circ a_{m-1}) \cap A (\neq \emptyset)}$.

So for all $m > 1$, $a_m \in A$, then $a_m \in (a_{m-1} \circ R \circ a_{m-1})$. Thus, there exists $s_{m-1} \in R$ such that $a_m \in (a_{m-1} \circ s_{m-1} \circ a_{m-1})$. Let $\mathbb{B}(a_1) = \{a_1, a_2, a_3, \ldots\ldots\}$. We shall prove $\mathbb{B}(a_1)$ is a weak m-system. For any subset $A_{p-k} (k = 0, 1, 2, \ldots, n, n \in 2\mathbb{N} \cup \{0\})$ of R, we write $\prod_{k=0}^{n} A_{p-k} = A_{p-0} \circ A_{p-1} \circ A_{p-2} \circ \ldots \circ A_{p-n}$ and $\prod_{k=0}^{n} A_{p-k} = A_{p-n} \circ A_{p-n+1} \circ A_{p-n+2} \circ \ldots \circ A_p$. Then, for any $m \in \mathbb{N}$, the following two cases can be proved by mathematical induction for $0 \leq k < m$.

(i) $\mathbf{a_{m+1} \in \left(\prod_{i=0}^{2k} a_{m-i}\right) \circ \left(\prod_{i=0}^{2k} s_{m-i}\right) \circ a_{m-k}}$,

(ii) $\mathbf{a_{m+1} \in a_{m-k} \circ \left(\prod_{i=2k}^{0} s_{m-i}\right) \circ \left(\prod_{i=2k}^{0} a_{m-i}\right)}$.

Now for any $a_p, a_q \in \mathbb{B}(a_1)$ with $p > q$, then $a_{p+1} \in \left(\prod_{i=0}^{2(p-q)} a_{p-i}\right) \circ \left(\prod_{i=0}^{2} s_{p-i}\right) \circ a_q = a_p \circ \left(\prod_{i=1}^{2(p-q)} a_{p-i} \circ s_p \circ \prod_{i=1}^{2(p-q)} s_{p-i}\right) \circ a_q \subseteq a_p \circ R \circ a_q$. But, $a_{p+1} \in \mathbb{B}(a_1) \Rightarrow (a_p \circ R \circ$

$a_{q)} \cap \mathbb{B}(a_1) \neq \phi$. Similarly, we can prove when $p < q$ by using case (ii). It is already proved for $p = q$. So $\mathbb{B}(a_1)$ is a weak m-system. Then, by the formation of $\mathbb{B}(a_1)$, obviously $A = \cup\, \mathbb{B}(a_1)$, where $a_1 \in A$.

Proposition 3.29 Intersection of any class of prime hyperideals of a ternary hypersemiring is a semiprime hyperideal.

Proof. It follows from Proposition 3.26, Note 3.27 and Proposition 3.28.

Definition 3.30 Let $V(I)$ be the set of all Prime hyperideals of ternary hypersemirings S, containing a hyperideal I. Then $\cap V(I)$ is a semiprime hyperideal of S and denoted by $\sqrt{I}$.

Proposition 3.31 A hyperideal I of a ternary hypersemiring S is a semiprime hyperideal if and only if $I = \sqrt{I}$.

Proof. If I is a semiprime hyperideal of the ternary hypersemiring S, then by Proposition 3.26 and Proposition 3.28, there exist weak m-systems $B_i (i \in \Lambda)$ such that $A = S \setminus I = \cup_{i\in\Lambda} B_i$, where $I \cap B_i = \phi$. By applying Zorn's lemma we get a prime hyperideal K_i such that $I \subseteq K_i$. Therefore $I \subseteq \cap_{i\in\Omega} K_i \subseteq (S \setminus B_i) \subseteq I$, whence $I = \sqrt{I}$.
Converse part is proved by Proposition 3.29

Proposition 3.32 Let I be a hyperideal of a commutative ternary hypersemiring S, then $J = \{a \in S : a^{2n+1} \subseteq I \text{ for some } n \in \mathbb{N}\}$ is a hyperideal and $I \subseteq J \subseteq \sqrt{I}$.

Proof. Let $a \in I$, $a^1 = \{a\} \subseteq I \Rightarrow a^1 \in J \Rightarrow I \subset J$. Let $a, b \in J$. Then there exists $m, n \in \mathbb{N}$ such that $a^{2m+1}, b^{2n+1} \subseteq I$. For $m = n = 1$, $a^3, b^3 \in I$, then $(a + b)^3 \subseteq I$ whence $a + b \in J$. Let $m, n > 1 \Rightarrow 2(m + n) + 1 > 1$. Now $(a + b)^{2m+2n+1} \subseteq \sum r = 12m + 2n + 1\binom{2m+2n+1}{r} a^{2m+2n+1-r} \circ b^r + a^{2m+2n+1} + b^{2m+2n+1} = A + B$, where $A = a^{2m+2n+1} + b^{2m+2n+1}$, $B = \sum r = 12m + 2n + 1\binom{2m+2n+1}{r} a^{2m+2n+1-r} \circ b^r$. As $m, n > 1$, $a^{2m+2n+1} = a^{2m+1} \circ a^{2n} \subseteq I$. Now $2m + 2n + 1 - r < 2m + 1 \Rightarrow 2n - r < 0 \Rightarrow r > 2n \Rightarrow a^{2m+2n+1-r} \circ b^r \subseteq I$. Otherwise $2m + 2n + 1 - r >= 2m + 1$. In this case we can similarly prove $a^{2m+2n+1-r} \circ b^r \subseteq I$. This implies $A + B \subseteq I \Rightarrow (a + b)^{2m+2n-1} \subseteq I \Rightarrow a + b \in J$. So $(J, +)$ is a subsemigroup of $(S, +)$. For any $r, r' \in S$, let $p \in r \circ a \circ r' \Rightarrow p^{2m-1} \subseteq (r \circ a \circ r')^{2m+1} = r^{2m+1} \circ a^{2m+1} \circ r'^{(2m+1)} \subseteq I \Rightarrow p \in J \Rightarrow r \circ a \circ r' \subseteq J$. J is a lateral hyperideal of S contains I. Similarly, we can show that J is a left and a right hyperideal of S. If $\sqrt{I} = S$, then $J \subseteq \sqrt{I}$. Let $\sqrt{I} \neq S$ and let $a \in J \Rightarrow a^{2n+1} \subseteq I$ for some $n \in \mathbb{N}$. Hence for any prime hyperideal P of S, containing I, $a^{2n+1} \subseteq P \Rightarrow a \in P \Rightarrow a \in \sqrt{I} \Rightarrow J \subseteq \sqrt{I}$.

Definition 3.33 A prime hyperideal I of a ternary hypersemiring $(S, +, \circ)$ is said to be idempotent if $I \circ I \circ I = I$.

A ternary hypersemiring S is called fully idempotent if each of its hyperideals is idempotent.

Definition 3.34 Let $(S, +, \circ)$ be a ternary hypersemiring. An element $x \in S$ is said to be regular if $x \in x \circ S \circ x$, that is, there exist $s_i \in S (i = 1, 2, \cdots, n)$ such that $a \in \sum_{i=1}^{n} a \circ s_i \circ a$.

A ternary hypersemiring $(S, +, \circ)$ is called regular if each of its elements is regular. If S is strongly distributive ternary hypersemiring, an element $x \in S$ is regular if only if there exsists $a \in S$ such that $x \in x \circ a \circ x$.

Proposition 3.35 The following statements are equivalent in a ternary hypersemiring S

(i) **S** is fully idempotent;

(ii) for three hyperideals $\mathbf{I}, \mathbf{J}, \mathbf{K}$ of **S**, $\mathbf{I} \cap \mathbf{J} \cap \mathbf{K} = \mathbf{I} \circ \mathbf{J} \circ \mathbf{K}$;

(iii) If **S** is a commutative ternary hypersemiring with hyperidentity then **S** is a regular commutative ternary hypersemiring with hyperidentity.

Proof. (i) ⇔ (ii) is obvious. (iii) ⇒ (i): S is regular, $I \in R$. For any hyperideal $I \in S$ we have $I \circ I \circ I = I \Rightarrow S$ is fully idempotent. (i) ⇒ (iii): Let S is fully idempotent and $a \in S$ be an arbitrary element. $a \in a \circ S \circ a = (a \circ S \circ a) \circ (a \circ S \circ a) \circ (a \circ S \circ a) = a \circ S \circ a \circ a \circ S \circ a \circ a \circ S \circ a = a \circ a \circ a \circ S \circ S \circ S \circ a \circ a \circ a$, ($S$ is commutative) $\Rightarrow a \in a \circ (a \circ a \circ x^i \circ y^i \circ z^i \circ a \circ a)$. This implies a is regular, since a is arbitrary. Thus S is regular.

4. Conclusion

Several significant findings on semiprime hyperideals in ternary hypersemirings were obtained from this work. Every prime hyperideal has been shown to be a semiprime hyperideal; however, the opposite is not true. That is, in ternary hypersemirings, this is a unique kind of hyperideal. In hyperstructure theory, the n-system, m-system, and p-system are important concepts. In the light of weak and strong p-systems, the researchers characterize the semiprime hyperideal. One may try to prove more standard results on regular ternary hypersemiring. In the future, semiprime hyperideals in ternary hypersemirings can be generalized to quasi-semiprime hyperideals. Lastly, the question remains open: can semiprime hyperideals be used to characterize regular ternary hypersemirings?

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