Some Remarks on Fuzzy $R_0$, $R_1$ and Regular Topological Spaces

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Abstract

In this paper, five regular-axioms, eighteen $R_1$-axioms and nine $R_0$-axioms for fuzzy topological spaces are recalled. A complete answer is given with regard to all possible $(R_1 \Rightarrow R_0)$-type implications for fuzzy topological spaces. It is also shown that, though the regular-axiom implies $R_1$-axiom in ‘general topological spaces’, this is not true for ‘fuzzy topological spaces’, in general.

Keywords: Fuzzy Topological Space; Fuzzy $R_1$-axiom; Fuzzy $R_0$-axiom; Fuzzy regular axiom.

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1. Introduction

In 1965, Zadeh [1] defined fuzzy sets with a view to study and formulate mathematically those situations which are imprecise and vaguely defined. Since then, fuzzy set theory has been developed in many directions by many scholars. Chang [2] gave the concept of ‘fuzzy topology’. He did the ‘fuzzification’ of topology by replacing ‘subsets’ in the definition of topology by ‘fuzzy sets’. In 1976, Lowen [3] gave a modified definition of ‘fuzzy topology’. Hutton and Reilly [4] introduced the concept of fuzzy $R_0$ and $R_1$ axioms. These studies were further carried out by many researchers [5-13]. In this paper we recall nine $R_0$-axioms from [9], eighteen $R_1$-axioms from [11] and five regular axioms from [7, 8] for fuzzy topological spaces (fts, in short). In analogy with the well known topological properties like $(regular \Rightarrow R_1)$ and $(R_1 \Rightarrow R_0)$, we study these types of properties for fts. We give a complete answer with regard to all possible $(R_1 \Rightarrow R_0)$-type implications for fts. It is also shown that, the property $(R_1 \Rightarrow R_0)$ is also true for fts; however, the property $(regular \Rightarrow R_1)$ is not true for fts, in general.

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1.1 Preliminaries

In this section, we recall some definitions on fuzzy sets and fts which will be needed in the sequel.

Definition-1.1.1. [1]: Let \( X \) be a non-empty set and \( I \) the unit closed interval \([0, 1]\). A fuzzy set is a function \( u: X \to I, \forall \ x \in X \); \( u(x) \) denotes a degree or the grade of membership of \( x \). The set of all fuzzy sets in \( X \) is denoted by \( I^X \). Ordinary subsets of \( X \) (crisp sets) are also considered as the members of \( I^X \) which take the values 0 and 1 only. A crisp set which always takes the value 0 is denoted by 0; similarly a crisp set which always takes the value 1 is denoted by 1.

Definition-1.1.2. [10]: Let \( u: X \to I \). Then the set \( \{ x \in X: u(x) > 0 \} \) is called the support of \( u \) and is denoted by \( u_0 \) or \( \text{supp}(u) \). Let \( A \subseteq X \), then by \( 1_A \) we denote the characteristic function \( A \). The characteristic function of a singleton set \( \{ x \} \) is denoted by \( 1_x \).

Definition-1.1.3. [10]: Let \( u \) be a fuzzy set in \( X \). Then by \( u^c \), we denote the complement of \( u \) which is defined as \( (u^c)(x) = 1 - u(x), \forall \ x \in X \).

Definition-1.1.4. [1]: Let \( u \) and \( v \) be two fuzzy sets in \( X \). We define

(i) \( u = v \) if and only if \( u(x) = v(x), \forall \ x \in X \).

(ii) \( u \subseteq v \) if and only if \( u(x) \leq v(x), \forall \ x \in X \).

(iii) \( (u \cup v)(x) = \max\{u(x), v(x)\}, \forall \ x \in X \).

(iv) \( (u \cap v)(x) = \min\{u(x), v(x)\}, \forall \ x \in X \).

Definition-1.1.5. [1]: For a family of fuzzy sets \( \{ u_i : i \in J \} \) in \( X \). We define

(i) \( \bigcup_{i \in J} u_i(x) = \sup\{u_i(x)\}, \forall \ x \in X \).

(ii) \( \bigcap_{i \in J} u_i(x) = \inf\{u_i(x)\}, \forall \ x \in X \).

Definition-1.1.6. [14]: A fuzzy point \( x_\alpha \) in \( X \) is a special type of fuzzy set in \( X \) with the membership function \( x_\alpha(x) = \alpha \) and \( x_\alpha(y) = 0 \) if \( x \neq y \), where \( 0 < \alpha < 1 \) and \( x, y \in X \). The fuzzy point \( x_\alpha \) is said to have support \( x \) and value \( \alpha \). We also write this as \( \alpha 1_x \).

Definition-1.1.7. [14]: Let \( \alpha 1_x \) be a fuzzy point in \( X \) and \( u \in I^X \). Then \( \alpha 1_x \in u \) if and only if \( \alpha \leq u(x) \).

Definition-1.1.8. [10]: Let \( f: X \to Y \) be a mapping and \( u \in I^X \). Then the image \( f(u) \) is a fuzzy set in \( Y \) which is defined as
\[
f(y) = \begin{cases} 
\sup \{u(x) : f(x) = y\} & \text{if } f^{-1}(y) \neq \emptyset \\
0 & \text{if } f^{-1}(y) = \emptyset 
\end{cases}
\]

**Definition-1.1.9.** [10]: Let \( f: X \to Y \) be a mapping and \( u \) be a fuzzy set in \( Y \). Then the inverse image \( f^{-1}(u) \) is a fuzzy set in \( X \) which is defined by \( f^{-1}(u)(x) = u(f(x)) \quad \forall \ x \in X \).

**Definition-1.1.10.** [2]: Chang [2] defined an fts as follows:

Let \( X \) be a set. A class \( t \) of fuzzy sets in \( X \) is called a fuzzy topology on \( X \) if \( t \) satisfies the following conditions:

(i) \( 0, 1 \in t \),

(ii) if \( u, v \in t \) then \( u \land v \in t \) and

(iii) if \( \{u_i : i \in K\} \) is a family of fuzzy sets in \( t \), then \( \bigvee_{i \in K} (u_i) \in t \).

The pair \((X, t)\) is then called an fts. The members of \( t \) are called \( t \)-open sets (or open sets) and their complements are called \( t \)-closed set (or closed sets).

**Definition-1.1.11.** [3]: Lowen [3] modified the definition of an fts defined by Chang [2] by adding another condition. In the sense of R. Lowen [3], the definition of an fts is as follows:

Let \( X \) be a set and \( t \) a family of fuzzy sets in \( X \). Then \( t \) is called a fuzzy topology of \( X \) if the following conditions hold:

(i) \( 0, 1 \in t \),

(ii) if \( u, v \in t \) then \( u \land v \in t \),

(iii) if \( \{u_i : i \in K\} \) is a family of fuzzy sets in \( t \), then \( \bigvee_{i \in K} (u_i) \in t \) and

(iv) \( t \) contains all constant fuzzy sets in \( X \).

The pair \((X, t)\) is called an fts. Throughout this work, we use the concept of fts due to Lowen [3].

**Definition-1.1.12.** [10]: Let \( u \) be a fuzzy set in an fts \((X, t)\). Then the fuzzy closure \( \overline{u} \) and the fuzzy interior \( u^0 \) of \( u \) are defined as follows: \( \overline{u} = \inf \{\lambda : u \leq \lambda \text{ and } \lambda \in t^c\} \), \( u^0 = \sup \{\lambda : \lambda \leq u \text{ and } \lambda \in t\} \).

**Definition-1.1.13.** [2]: Let \( f: (X, t) \to (Y, s) \) be a mapping between fts. Then \( f \) is called

(i) fuzzy continuous if and only if \( f^{-1}(u) \in t \) for each \( u \in s \).

(ii) fuzzy open if and only if \( f(u) \in s \) for each \( u \in t \).
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(iii) fuzzy closed if and only if \( f(u) \in s^c \) for each \( u \in t^c \).

2. Fuzzy \( R_0 \) topological spaces

In this section, we recall nine \( R_0 \)-axioms of fts from [9].

Definitions-2.1. [9]: We define, for fts \((X, t)\), \( R_0 \)-axioms as follows:

\( R_0^1 \) : For every pair \( x, y \in X, x \neq y, \bar{I}_y(x) = 0 \Rightarrow \bar{I}_x(y) = 0 \)

\( R_0^2 \) : For every pair \( x, y \in X, x \neq y, \left( \forall \alpha \in I_0, \bar{\alpha}I_x(y) = \alpha \right) \Leftrightarrow \left( \beta \bar{I}_y(x) = \beta, \forall \beta \in I_0 \right) \)

\( R_0^3 \) : \( \forall \lambda \in t, \forall x \in X \) and \( \forall \alpha < \lambda(x) \), \( \bar{\alpha}I_x \leq \lambda \)

\( R_0^4 \) : \( \forall \lambda \in t, \forall x \in X \) and \( \forall \alpha \leq \lambda(x) \), \( \bar{\alpha}I_x \leq \lambda \)

\( R_0^5 \) : For every pair \( x, y \in X, x \neq y \), \( \bar{I}_x(y) = 1 \Rightarrow \bar{I}_y(x) = 1 \)

\( R_0^6 \) : For every pair \( x, y \in X, x \neq y \), \( \bar{I}_x(y) = \bar{I}_y(x) \)

\( R_0^7 \) : For every pair \( x, y \in X, x \neq y \), \( \bar{I}_x(y) = \bar{I}_y(x) \in \{0, 1\} \)

\( R_0^8 \) : For every pair \( x, y \in X, x \neq y \) and \( \forall \alpha \in I_0, \bar{\alpha}I_x(y) = \alpha \Rightarrow \bar{\alpha}I_y(x) = \alpha \)

\( R_0^9 \) : For every pair \( x, y \in X, x \neq y \) and \( \forall \alpha \in I_0, \bar{\alpha}I_x(y) = \bar{\alpha}I_y(x) \)

Theorem-2.1 [9]: The accompanying diagram (Fig. 1) illustrates the interrelations among the \( R_0 \)-properties mentioned in the section 2:

![Fig. 1. Interrelations among the \( R_0 \)-properties [9].](image)

For proof see [9]. □
3. Fuzzy $R_1$-topological spaces

In this section, we recall eighteen definitions of fuzzy $R_1$-topological spaces from [11].

Definitions-3.1 [11]: An fts $(X, t)$ is said to have the property

1. **P1**, if $\forall x, y \in X, x \neq y$, $\exists w \in t$ such that $w(x) \neq w(y)$.

2. **P2**, if $\forall x, y \in X, x \neq y$, $\exists w \in t$ such that $w(x) > 0 = w(y)$.

3. **P3**, if $\forall x, y \in X, x \neq y$, $\exists w \in t$ such that either $w(x) = 1$, $w(y) = 0$ or $w(x) = 0$, $w(y) = 1$.

4. **Q1**, if $\forall x, y \in X, x \neq y$, $\exists u, v \in t$ such that $\bar{x} \leq u, \bar{y} \leq v$ and $u \wedge v = 0$.

5. **Q2**, if $\forall x, y \in X, x \neq y$, $\exists u, v \in t$ such that $\bar{x} \leq u, \bar{y} \leq v$ and $u \leq 1 - v$.

6. **Q3**, if $\forall x, y \in X, x \neq y$, $\exists u, v \in t$ such that $u(x) = 1 = v(y)$ and $u \wedge v = 0$.

7. **Q4**, if $\forall x, y \in X, x \neq y$, $\exists u, v \in t$ such that $u(x) = 1 = v(y)$ and $u \leq 1 - v$.

8. **Q5**, if $\forall x, y \in X, x \neq y$ and $\forall \alpha, \beta \in I_{0,1}$, $\exists u, v \in t$ such that $u(x) > \alpha$ and $v(y) > \beta$ and $u \wedge v = 0$.

9. **Q6**, if $\forall x, y \in X, x \neq y$, $\exists u, v \in t$ such that $u(x) > 0$, $v(y) > 0$ and $u \wedge v = 0$.

Definitions-3.2 [11]: An fts $(X, t)$ is called an

1. **FR$_1(i)$**-fts, if $(X, t)$ has **P1** implies $(X, t)$ has **Q1**.

2. **FR$_1(ii)$**-fts, if $(X, t)$ has **P1** implies $(X, t)$ has **Q2**.

3. **FR$_1(iii)$**-fts, if $(X, t)$ has **P1** implies $(X, t)$ has **Q3**.

4. **FR$_1(iv)$**-fts, if $(X, t)$ has **P1** implies $(X, t)$ has **Q4**.

5. **FR$_1(v)$**-fts, if $(X, t)$ has **P1** implies $(X, t)$ has **Q5**.

6. **FR$_1(vi)$**-fts, if $(X, t)$ has **P1** implies $(X, t)$ has **Q6**.

7. **FR$_1(vii)$**-fts, if $(X, t)$ has **P2** implies $(X, t)$ has **Q1**.

8. **FR$_1(viii)$**-fts, if $(X, t)$ has **P2** implies $(X, t)$ has **Q2**.

9. **FR$_1(ix)$**-fts, if $(X, t)$ has **P2** implies $(X, t)$ has **Q3**.

10. **FR$_1(x)$**-fts, if $(X, t)$ has **P2** implies $(X, t)$ has **Q4**.

11. **FR$_1(xi)$**-fts, if $(X, t)$ has **P2** implies $(X, t)$ has **Q5**.

12. **FR$_1(xii)$**-fts, if $(X, t)$ has **P2** implies $(X, t)$ has **Q6**.

13. **FR$_1(xiii)$**-fts, if $(X, t)$ has **P3** implies $(X, t)$ has **Q1**.

14. **FR$_1(xiv)$**-fts, if $(X, t)$ has **P3** implies $(X, t)$ has **Q2**.

15. **FR$_1(xv)$**-fts, if $(X, t)$ has **P3** implies $(X, t)$ has **Q3**.
16. \( \text{FR}_1(xvi) \)-fts, if \((X, t)\) has \(P3 \Rightarrow (X, t)\) has \(Q4\).
17. \( \text{FR}_1(xvii) \)-fts, if \((X, t)\) has \(P3 \Rightarrow (X, t)\) has \(Q5\).
18. \( \text{FR}_1(xviii) \)-fts, if \((X, t)\) has \(P3 \Rightarrow (X, t)\) has \(Q6\).

**Theorem-3.3** [11]: The accompanying diagram (Fig. 2) illustrates the interrelations among the \(\text{FR}_1\)-properties mentioned in Section 3:

Fig. 2. Interrelations among the \(R_1\)-properties [11].

For proof see [11]. □

4. Relations between fuzzy \(R_0\) and \(R_1\)-axioms

In this section, we give a complete answer with regard to all possible \((R_1 \Rightarrow R_0)\)-type implications for fts.

**Theorem-4.1**: The following relations hold between the fuzzy \(R_0\)-axioms and fuzzy \(R_1\)-axioms:

(a) \( \text{FR}_1(xvi) \Rightarrow R_0^1 \), and so \( \text{FR}_1(k) \Rightarrow R_0^1 \), where \( k \in \{i - iv, vii - x, xiii - xvi\} \).

(b) \( \text{FR}_1(xiii) \nRightarrow R_0^5 \), and so \( \text{FR}_1(k) \nRightarrow R_0^m \), where \( k \in \{xiii, xiv, \ldots, xviii\} \) and \( m \in \{5, 6, \ldots, 9\} \).

(c) \( \text{FR}_1(v) \Rightarrow R_0^8 \), and so \( \text{FR}_1(k) \Rightarrow R_0^m \) where \( k \in \{i, iii, v\} \) and \( m \in \{2, 5, 8\} \).

(d) \( \text{FR}_1(vi) \Rightarrow R_0^2 \), and so \( \text{FR}_1(k) \Rightarrow R_0^2 \) where \( k \in \{i, iii, v, vi\} \).

(e) \( \text{FR}_1(vi) \nRightarrow R_0^8 \), and so \( \text{FR}_1(k) \nRightarrow R_0^m \), where \( k \in \{vi, xii, xviii\} \) and \( m \in \{8, 9\} \).

(f) \( \text{FR}_1(vi) \Rightarrow R_0^3 \), and so \( \text{FR}_1(k) \Rightarrow R_0^m \), where \( k \in \{vi, xii, xviii\} \) and \( m \in \{3, 4\} \).
(g) $FR_1(iv) \Rightarrow R_0^4$, and so $FR_1(k) \Rightarrow R_0^m$ where $k \in \{i-iv\}$ and $m \in \{1, 2, 3, 4\}$.

(h) $R_0^m \Rightarrow FR_1(k)$, where $k \in \{i, ii, ...., xvi\}$ and $m \in \{1, 2, ...., 9\}$.

Proof (a): Let $(X, t)$ be an $FR_1(xvi)$-fts and $x, y \in X$, $x \neq y$ such that $\overline{I}_y(x) = 0$. Therefore, $\exists \lambda \in t^c$ such that $\lambda(y) = 1$ and $\lambda(x) = 0$. Take $w = 1 - \lambda$. Now $w \in t$ such that $w(x) = 1$ and $w(y) = 0$. Since, $(X, t)$ is an $FR_1(xvi)$-fts, $\exists u, v \in t$ such that $u(x) = 1 = v(y)$ and $u \leq 1 - v$. Put, $\kappa = 1 - v \in t^c$. Now $\kappa(y) = 0$ and $\kappa(x) = 1$. Consequently, $\overline{I}_x(y) = 0$. Hence $(X, t)$ is $R_0^1$. □

Proof (b):
Example-1: Consider a fuzzy topological space $(X, t)$, where $X = \{x, y\}$, $u(x) = 0.5$, $u(y) = 0$ and $t = \{\{u\} \cup \{\text{constants}\}\}$. Clearly, $(X, t)$ is $FR_1(xiii)$ but it is not $R_0^5$. For $\overline{I}_x(y) = 1$ but $\overline{I}_y(x) < 1$. □

Proof (c): Let $(X, t)$ be an $FR_1(\nu)$-fts. Let $x, y \in X$, $x \neq y$, $\alpha \in I_0$ such that $\overline{aI}_x(y) < \alpha$. This implies that there exists $m \in t^c$ such that $m(x) = \alpha$ and $m(y) < \alpha$. Let $w = 1 - m \in t$. Then $w(x) \neq w(y)$. Since, $(X, t)$ is an $FR_1(\nu)$-fts, there exist $u, v \in t$ such that $u(x) > \alpha_1$, $v(y) > \alpha_2$, and $u \wedge v = 0 \forall \alpha_1, \alpha_2 \in I_0, 1$. Choose $\alpha_1, \alpha_2$ in such a way that $\alpha = \alpha_2$ and $\alpha_1 > 1 - \alpha$. Now $\alpha_1 y < v \leq 1 - u$. Therefore, $\overline{aI}_y \leq \overline{1-u} = 1 - u$ and so $\overline{\alpha_1 y}(x) \leq 1 - u(x) < 1 - \alpha_1 < \alpha$. Hence, $(X, t)$ is $R_0^8$. [Note 9]:

$(\forall \alpha \in I_0, \overline{aI}_x(y) = \alpha \Rightarrow \overline{aI}_y(x) = \alpha) \Leftrightarrow (\forall \alpha \in I_0, \overline{aI}_x(y) < \alpha \Rightarrow \overline{aI}_y(x) < \alpha)$] □

Proof (d): Let $(X, t)$ be an $FR_1(vi)$-fts. Let $x, y \in X$, $x \neq y$ and $w \in t$ such that $w(x) > w(y)$. Then, by $FR_1(vi)$ there exist $u, v \in t$ such that $u(x) > 0$, $v(y) > 0$ and $u \wedge v = 0$. Clearly, $v(y) > v(x)$. Hence, $(X, t)$ is $R_0^2$. [Note 9]: \{An fts $(X, t)$ is $R_0^2$} $\Leftrightarrow \{\forall x, y \in X, x \neq y, \text{if} \exists \text{a} t$-open set $\lambda$ such that $\lambda(y) < \lambda(x)$ then $\exists$ a $t$-open set $\mu$ such that $\mu(x) < \mu(y)$\}. □

Proof (e):
Example-2: Consider an fts $(X, t)$ where $X = \{x, y\}$, $t = \{\{u_1, u_2, u_3, u_4\} \cup \{\text{constants}\}\}$, $u_1(x) = u_1(y) = 0.6$, $u_2(y) = 0.7$, $u_3(x) = u_4(y) = 0$, $u_3(y) = 0.8$ and $u_4(x) = 0.4$. It can be checked that $(X, t)$ is $FR_1(vi)$. Let $m_k = 1 - u_k$, $k = 1, 2, 3, 4$. Now $m_1(x) = 0.4 = m_2(x)$, $m_3(x) = 1$, $m_4(x) = 0.6$, $m_1(y) = 0.4$, $m_2(y) = 0.3$, $m_3(y) = 0.2$.
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and \( m_d(y) = 1 \). Take \( \alpha = 0.4 \). Then \( \overline{\alpha l}(y) = 0.2 < \alpha \) . But \( \overline{\alpha l}(x) = 0.4 = \alpha \) . Therefore, \((X, t)\) is not \( R_0^8 \). □

**Proof (f):**

**Example-3:** Consider an fts \((X, t)\) where \( X = \{x, y\} \), \( u(x) = 0.6 \), \( u(y) = 0 = v(x) \) and \( v(y) = 0.4 \). Clearly, \((X, t)\) is \( FR_1(vi) \). Let \( \alpha = 0.5 \). Now \( \alpha < u(x) \). It can be checked that \( \overline{\alpha l}(y) = \alpha > u(y) \). Therefore, \( \overline{\alpha l}(y) \notin u \). Hence, \((X, t)\) is not \( R_3^0 \). □

**Proof (g):** Let \((X, t)\) be an \( FR_1(v) \)-fts. Let \( x \in X \), \( \lambda \in \ell \) and \( \alpha \in \ell \) such that \( \alpha \leq \lambda(x) \).

Suppose \( \overline{\alpha l}(x) \notin \lambda \). This implies that there exist \( y \in X \), \( x \neq y \) such that \( \overline{\alpha l}(y) > \lambda(y) \). Thus \( \lambda(x) \neq \lambda(y) \). Hence there exist \( p, q \in \ell \) such that \( p(x) = 1 = q(y) \) and \( p \leq 1 - q \). Put \( m = 1 - p \) and \( n = 1 - q \). Now \( m, n \in \ell \) such that \( m(x) = 0 = n(y) \) and \( m(y) = 1 = n(x) \). Therefore, \( \overline{\alpha l}(x) \leq \overline{\lambda l}(y) \leq n(y) = 0 \), which is a contradiction. Therefore, \( \overline{\alpha l}(x) \leq \lambda \). Hence \((X, t)\) is \( R_4^0 \). □

**Proof (h):**

**Example-4** [13]: Let \( X \) be an infinite set. For \( x, y \in X \), we define \( U_{xy} \in I^X \) as follows:

\[
U_{xy}(z) = \begin{cases} 
0 & \text{if } z \in \{x, y\} \\
1 & \text{if } z \notin \{x, y\} 
\end{cases}
\]

Let \( t \) be the fuzzy topology on \( X \) generated by \( \{U_{xy} : x, y \in X\} \). It can be checked that if \( x \neq y \), \( \overline{\lambda l}(y) = 0 \). Therefore, \((X, t)\) is \( R_0^4 \), \( R_0^7 \) and \( R_0^9 \). But \((X, t)\) is neither \( FR_1(xvii) \) nor \( FR_1(xviii) \) as there exist no \( u, v \in t \) such that \( u \leq 1 - v \). Therefore, \((X, t)\) is not \( FR_1(k), k \in \{i, ii, ..., xviii\} \). □

5. Fuzzy regular axioms

In this section, we recall five definitions of fuzzy regular axioms from [7, 8], and we show that, the well known topological property \((regular \Rightarrow R_1)\) is not true, in general, for fts.

**Definition-5.1:** An fts \((X, t)\) is called

(a) \( FR(i) \) if and only if \( \alpha \in \ell_0, \lambda \in \ell, x \in X \) and \( \alpha \leq 1 - \lambda(x) \) imply that there exist \( u, v \in t \) such that \( \alpha \leq u(x), \lambda \leq v \) and \( u \leq 1 - v \).

(b) \( FR(ii) \) if and only if \( \alpha \in \ell_0, \lambda \in \ell, x \in X \) and \( \alpha \leq 1 - \lambda(x) \) imply that there exist \( u, v \in t \) such that \( \alpha \leq u(x), \lambda \leq v \) and \( u \leq 1 - v \).
(c) $FR(iii)$ if and only if each $u \in t$ is a supremum of $u_j, j \in J$, where $\forall j, u_j \in t$ and $u_j \leq u$.

(d) $FR(iv)$ if and only if $\lambda \in \mathcal{T}, x \in X$ and $\lambda(x) = 0$ imply that there exist $u, v \in t$ such that $u(x) = 1$, $\lambda \leq v$ and $u \leq 1 - v$.

(e) $FR(v)$ if and only if $\lambda \in \mathcal{T}, x \in X$ and $1 - \lambda(x) > 0$ imply that there exist $u, v \in t$ such that $u(x) > 0$, $\lambda \leq v$ and $u \leq 1 - v$.

Note-1 [7, 8]: Let $x \in X$ and $\lambda$ be a fuzzy set in $X$. Then for $\alpha \in I_0$, “$\alpha \leq \lambda(x)$” means $\alpha < \lambda(x)$ if $\alpha \neq 1$ and $\lambda(x) = 1$ if $\alpha = 1$.

Note-2 [7, 8]: The following implications exist among $FR(i)$, $FR(ii)$, …, $FR(v)$:

$$FR(i) \Rightarrow FR(ii) \Rightarrow FR(iii) \Rightarrow FR(v) \Downarrow FR(iv)$$

For proof see [7, 8, 10]. □

Example-5: Let $X = \{x, y, z\}$. For every pair $x, y \in X$ we define $U_{xy} \in I^X$ as follows:

$U_{xy}(x) = 1$, $U_{xy}(y) = 0$ and $U_{xy}(z) = 0.5$. Let $t$ be the fuzzy topology on $X$ generated by $\left\{U_{xy} \in I^X : x, y \in X\right\}$. Now it can be easily verified that $(X, t)$ is $FR(i)$. But $(X, t)$ is neither $FR_x(xvi)$ nor $FR_x(xvii)$, since there exist no $u, v \in t$ such that $u \wedge v = 0$. Therefore, $FR(k) \not\Rightarrow FR\left(m\right), k \in \{i, ii, \ldots, v\}$ and $m \in \{i, ii, \ldots, xvii\}$. Thus we see that the property (regular $\Rightarrow R_1$) is not true, in general, for fts. □

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