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Fixed Point for Weakly Compatible Maps Satisfying Generalized Contraction Principle

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Abstract

We prove common fixed point theorems for a pair of weakly compatible mappings satisfying a generalized contraction principle by using a control function and implicit relation. We also establish invariant approximation result as an application of the result.

Keywords: Common fixed point; Weakly compatible maps; Control function; Invariant approximation.

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1. Introduction.

Sessa [1] introduced weakly commuted mappings which was generalized as compatible mappings by Jungck [2]. The notion of *R*-weakly commuting mappings was coined by Pant [3]. The term called weakly compatible mappings was defined by Jungck and Rhoades [4].

Generalization for weakly contractive mapping in Hilbert space was proved by several authors [5- 9]. Generalization for weakly contractive mapping in complete metric space was proved by Rhoades [7].

Fixed point theorems for a self mapping by altering distances between the points and using a control function were proved by Park [10] and Khan *et al*. [11]. Sastry [8] extended the concept for weakly commuting pairs of self mappings and proved common fixed point theorem in a complete metric space by using the control function.

Dutta and Choudhury [6] obtained a fixed point result by generalizing the concept of control function and the weakly contractive mapping. Jungck [12] proved a common fixed point theorem for commuting mappings generalizing the Banach's contraction principle.

The existence of invariant approximation using fixed point theorem was established by Meinardus [13] which was generalized by Brosowski [14]. Subrahmanyam [15] and Singh

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[16] relaxed the linearity of the mapping and the convexity of the set of best approximants. Also the existence of invariant approximation using fixed point theorem was generalized by several authors [17-21].

The main purpose of this paper is to obtain common fixed point for weakly compatible mappings satisfying a more general weak contractive condition using implicit relation. As an application we have established best approximation result.

2 Preliminaries

We recall the definitions and results that will be needed in the sequel.

Definition 2.1 A sequence $\{x_n\}$ in a metric space (X,d) is said to be convergent to a point $x \in X$, denoted by $\lim_{n\to\infty} x_n = x$, if $\lim_{n\to\infty} d(x_n, x) = 0$.

Definition 2.2 A sequence $\{x_n\}$ in a metric space (X,d) is said to be Cauchy sequence if $\lim_{t\to\infty} d(x_n, x_m) = 0$ for all $n, m > t$.

Definition 2.3 A metric space (X,d) is said to be complete if every Cauchy sequence in X is convergent.

Definition 2.4 Let f and T be two self-maps on a set X. Maps f and T are said to be commuting if $fTx = Tfx$ for all $x \in X$.

Definition 2.5 Let f and T be self-maps on a set X. If $fx = Tx$, for some x in X then x is called coincidence point of f and T.

Definition 2.6 Let f and T be two self-maps defined on a set X. Then f and T are said to be weakly compatible if they commute at coincidence points. That is, if $fu = Tu$ for some $u \in X$, then $fTu = Tfu$.

Proposition 2.1 Let f and T be weakly compatible self mappings of a set X. If f and T have a unique point of coincidence, that is, $w = fx = Tx$, then w is the unique common fixed point of f and T.

Definition 2.7 Let f and T be self mappings of a nonempty subset M of a metric space X. The mapping T is called f-contraction mapping, if there exists a real number $0 \le k < 1$ such that $d(Tx, Ty) \le kd(fx, fy)$ for all $x, y \in M$.

Definition 2.8 [11] A control function φ is defined as $\varphi : R^+ \to R^+$ which is continuous at zero, monotonically increasing and $\varphi(t) = 0$ if and only if $t = 0$.

Definition 2.9 [5] A self mapping T of a metric space (X, d) is said to be weakly contractive with respect to a self mapping $f : X \to X$, if for each $x, y \in X$, $d(Tx, Ty) \le d(fx, fy) - \varphi(d(fx, fy))$, where $\varphi : [0,\infty) \to [0,\infty)$ is a continuous and non decreasing function such that φ is positive on $(0,\infty)$, $\varphi(0) = 0$ and $\lim_{t\to\infty} \varphi(t) = \infty$

Proposition 2.2 [6] Let (X, d) be a complete metric space and $T : X \to X$ be a self mapping satisfying

$$\varphi(d(Tx,Ty)) \le \varphi(d(fx,fy)) - \phi(d(fx,fy))$$

where $\varphi,\phi:[0,\infty)\to[0,\infty)$ are both continuous and monotonic increasing function with $\varphi(x)=0=\phi(x)$ if and only if $x=0$.
Then T has a unique fixed point.

Implicit relations

Let $\mathbf{F}^*$ be the set of continuous functions $F(t_1,t_2,t_3,t_4):[0,\infty)^4\to[0,\infty)$ satisfying the following conditions:

(F_1) F is non decreasing in variables t_1.
(F_2) For $u\ge 0$, $v\ge 0$ $F(u+v,0,u,v)\le u$
(F_3) $F(u,u,0,0)\le u$ and $F(0,u,0,u)\le u$, $\forall u>0$.

3. Main Result

In this section we prove a common fixed point theorem for a pair of weakly compatible mappings in complete metric spaces by using a control function and implicit relation.

Theorem 3.1 Let (X,d) be a complete metric space. Suppose that the mappings T and f are two self-maps of X satisfying the following conditions:

(i) $T(X)\subseteq f(X)$.
(ii) $T(X)$ is complete subspace of X.
(iii) $\varphi(d(Tx,Ty))\le\varphi(M(x,y))-\phi(M(x,y))$
where $M(x,y)=F\{d(Tx,fy),d(Ty,fx),d(Tx,fx),d(Ty,fy)\}$
and $\varphi:[0,\infty)\to[0,\infty)$ is continuous and monotonic increasing function and $\phi:[0,\infty)\to[0,\infty)$ is continuous and monotonic decreasing function with $\varphi(x)=0=\phi(x)$ if and only if x = 0 and $F\in F^*$.
(iv) The pair (T,f) is weakly compatible.

Then f and g have a unique common fixed point.

Proof : let x_0 be an arbitrary point of X.

Since $T(X)\subseteq f(X)$, we can choose x_n and x_{n+1} in X such that,

$Tx_n = fx_{n+1}$ $n=0,1,2...$

By using (iii) we have,

$$\varphi(d(Tx_{n+1},Tx_n)) \le \varphi(M(x_{n+1},x_n))-\phi(M(x_{n+1},x_n)) \tag{1}$$

where $M(x_{n+1},x_n)=F\{d(Tx_{n+1},fx_n),d(Tx_n,fx_{n+1}),d(Tx_{n+1},fx_{n+1}),d(Tx_n,fx_n)\}$

$$=F\{d(Tx_{n+1},Tx_{n-1}),d(Tx_n,Tx_n),d(Tx_{n+1},Tx_n),d(Tx_n,Tx_{n-1})\}$$

$$\le F\{(d(Tx_{n+1},Tx_n)+d(Tx_n,Tx_{n-1})),0,d(Tx_{n+1},Tx_n),d(Tx_n,Tx_{n-1})\}$$

Thus form (F_2) we have, $M(x_{n+1}, x_n) \le d(Tx_n, Tx_{n-1})$

Thus from (1)

$$\varphi(d(Tx_{n+1}, Tx_n)) \le \varphi(d(Tx_n, Tx_{n-1})) - \phi(d(Tx_n, Tx_{n-1})) \quad (2)$$

$$\varphi(d(Tx_{n+1}, Tx_n)) \le \varphi(d(Tx_n, Tx_{n-1}))$$

Now φ is monotonic increasing function. This implies that, the sequence $\{d(Tx_{n+1}, Tx_n)\}$ is monotonic decreasing.

Hence there exists a real number, say $r \ge 0$ such that $\lim_{n\to\infty} d(Tx_{n+1}, Tx_n) = r$.

Therefore as $n \to \infty$, equation (2) implies that

$$\varphi(r) \le \varphi(r) - \phi(r)$$

So that $\phi(r) \le 0$, which is possible only if $r = 0$.

Thus $\lim_{n\to\infty} d(Tx_{n+1}, Tx_n) = 0$

Now we show that $\{Tx_n\}$ is a Cauchy sequence.

Let if possible we assume that $\{Tx_n\}$ is not a Cauchy sequence

Then there exists an $\varepsilon > 0$ and subsequences $\{n_i\}$ and $\{m_i\}$ such that

$m_i < n_i < m_{i+1}$ and

$$d(Tx_{m_i}, Tx_{n_i}) \ge \varepsilon \text{ and } d(Tx_{m_i}, Tx_{n_i-1}) < \varepsilon \quad (3)$$

So that $\varepsilon \le d(Tx_{m_i}, Tx_{n_i}) \le d(Tx_{m_i}, Tx_{n_i-1}) + d(Tx_{n_i-1}, Tx_{n_i}) < \varepsilon + d(Tx_{n_i-1}, Tx_{n_i})$

Therefore $\lim_{i\to\infty} d(Tx_{m_i}, Tx_{n_i}) = \varepsilon$

Now $d(Tx_{m_i}, Tx_{n_i}) \le d(Tx_{m_i}, Tx_{m_i-1}) + d(Tx_{m_i-1}, Tx_{n_i-1}) + d(Tx_{n_i-1}, Tx_{n_i})$

$d(Tx_{m_i-1}, Tx_{n_i-1}) \le d(Tx_{m_i-1}, Tx_{m_i}) + d(Tx_{m_i}, Tx_{n_i}) + d(Tx_{n_i}, Tx_{n_i-1})$

By taking limit as $i \to \infty$, we get $\lim_{i\to\infty} d(Tx_{m_i-1}, Tx_{n_i-1}) = \varepsilon$

Now by (iii) and (3)

$$\varphi(\varepsilon) \le \varphi(d(Tx_{m_i}, Tx_{n_i})) \le \varphi(M(x_{m_i}, x_{n_i})) - \phi(M(x_{m_i}, x_{n_i}))$$

where $M(x_{m_i}, x_{n_i}) = F\{d(Tx_{m_i}, fx_{n_i}), d(Tx_{n_i}, fx_{m_i}), d(Tx_{m_i}, fx_{m_i}), d(Tx_{n_i}, fx_{n_i})\}$

$$= F\{d(Tx_{m_i}, Tx_{n_i-1}), d(Tx_{n_i}, Tx_{m_i-1}), d(Tx_{m_i}, Tx_{m_i-1}), d(Tx_{n_i}, Tx_{n_i-1})\}$$

By taking limit as $i \to \infty$, we get $\lim_{i\to\infty} M(x_{m_i}, x_{n_i}) = F(\varepsilon, \varepsilon, 0, 0)$

Thus form (F_3) we have, $\lim_{i\to\infty} M(x_{m_i}, x_{n_i}) \le \varepsilon$

Therefore $\varphi(\varepsilon) \le \varphi(\varepsilon) - \phi(\varepsilon)$ and hence $\phi(\varepsilon) \le 0$

This is a contradiction because as $\phi:[0,\infty)\to[0,\infty)$, we must have $\phi(\varepsilon)\geq 0$ and $\phi(\varepsilon)=0$ if and only if $\varepsilon=0$ but $\varepsilon>0$.
Hence our supposition is false.
Thus $\{Tx_n\}$ is a Cauchy sequence in $T(X)$.
But by (ii), $T(X)$ is a complete subset of X, there exists a point q in X such that

$$\lim_{n\to\infty} Tx_n = q$$

Since $T(X)\subseteq f(X)$, there exists a point $p\in X$ such that $q = fp$.

Now from (iii), $\varphi(d(Tx_n,Tp))\leq\varphi(M(x_n,p))-\phi(M(x_n,p))$

where $M(x_n,p)=F\{d(Tx_n,fp),d(Tp,fx_n),d(Tx_n,fx_n),d(Tp,fp)\}$

$$=F\{d(Tx_n,fp),d(Tp,Tx_{n-1}),d(Tx_n,Tx_{n-1}),d(Tp,fp)\}$$

$$=F\{d(q,q),d(Tp,q),d(q,q),d(Tp,q)\} \text{ (By taking limit as } n\to\infty)$$

$$=F\{0,d(Tp,q),0,d(Tp,q)\}$$

Thus form (F_3) we have, $M(x_n,p)\leq d(Tp,q)$

Therefore $\varphi(d(q,Tp))\leq\varphi(d(Tp,q))-\phi(d(Tp,q))$

Which implies that $\phi(d(Tp,q))\leq 0$ and this is possible only if $Tp = q$.

Thus $Tp = q = fp$ and hence p is the coincidence point of T and f.

Since T and f are weakly compatible, they commute at their coincidence point.
i.e., $Tfp = fTp$ which implies that $Tq = fq$.

Again from (iii), $\varphi(d(Tq,Tp))\leq\varphi(M(q,p))-\phi(M(q,p))$

Where $M(q,p)=F\{d(Tq,fp),d(Tp,fq),d(Tq,fq),d(Tp,fp)\}$

$$=F\{d(Tq,q),d(q,Tq),d(Tq,Tq),d(q,q)\}$$

$$=F\{d(Tq,q),d(q,Tq),0,0\}$$

Thus form (F_3) we have, $M(q,p)\leq d(Tq,q)$

Therefore $\varphi(d(q,Tq))\leq\varphi(d(Tq,q))-\phi(d(Tq,q))$

Which implies that $\phi(d(Tq,q))\leq 0$ and this is possible only if $Tq = q$.

Thus $Tq = q = fq$ and hence q is the common fixed point of T and f.

Uniqueness: For uniqueness of q let if possible, we assume that q and t,($q\neq t$) are common fixed points of f and T
from (iii), $\varphi(d(Tq,Tt))\leq\varphi(M(q,t))-\phi(M(q,t))$

where $M(q,t)=F\{d(Tq,ft),d(Tt,fq),d(Tq,fq),d(Tt,ft)\}$

$$=F\{d(q,t),d(q,t),0,0\}$$

Thus form (F_3) we have, $M(q,t)\leq d(q,t)$

Therefore $\varphi(d(q,t))\leq\varphi(d(q,t))-\phi(d(q,t))$

Which implies that $\phi(d(q,t)) \le 0$ and this is possible only if $q = t$.
Hence the theorem. □

Application as best approximation

Definition 3.1 Let M be a nonempty subset of a metric space (X,d). The set of best M-approximants to $u \in X$, denoted as $P_M(u)$ is defined by

$$P_M(u) = \{y \in M : d(y,u) = dist(u,M)\}$$

where $dist(u,M) = \inf\{d(x,u) : x \in M\}$.

Theorem 3.2 Let T and f be self mappings of a metric space (X,d). Suppose that $u \in X$, T and f satisfy following condition

$$\varphi(d(Tx,Ty)) \le \varphi(M(x,y)) - \phi(M(x,y))$$

where $M(x,y) = F\{d(Tx,fy), d(Ty,fx), d(Tx,fx), d(Ty,fy)\}$ and $\varphi, \phi : [0,\infty) \to [0,\infty)$ are both monotonic increasing function with $\varphi(x) = 0 = \phi(x)$ if and only if $x = 0$.

T leaves f-invariant compact subset M of closed subspace $f(X)$ as invariant. For each $b \in P_M(u)$, let $d(x,Tb) < f(x,fb)$ and $fb \in P_M(u)$. If T and f are weakly compatible, then u has a best approximation in M which is also a common fixed point of T and f.

Proof: Let $u \in F(T) \cap F(f)$

Since M is a compact subset of $f(X)$, $P_M(u) \neq \phi$.

We claim that $T(P_M(u)) \subseteq f(P_M(u))$,

Let if possible we assume that, there exists $b \in P_M(u)$ such that $Tb \notin f(P_M(u))$.

Now $d(u,fb) = dist(u,M) \le d(u,Tb) < d(u,fb)$ which is a contradiction.

Hence $T(P_M(u)) \subseteq f(P_M(u))$.

Now $f(P_M(u))$ being a closed subset of a complete space is complete.

Hence $P_M(u) \cap F(T) \cap F(f)$ is singleton. □

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