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Does the Solution to the Non-linear Diophantine Equation $3^x + 35^y = Z^2$ Exist?

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Abstract

This paper investigates the solutions (if any) of the Diophantine equation $3^x + 35^y = Z^2$, where x, y, and z are whole numbers. Diophantine equations are drawing the attention of researchers in diversified fields over the years. These are equations that have more unknowns than a number of equations. Diophantine equations are found in cryptography, chemistry, trigonometry, astronomy, and abstract algebra. The absence of any generalized method by which each Diophantine equation can be solved is a challenge for researchers. In the present communication, it is found with the help of congruence theory and Catalan's conjecture that the Diophantine equation $3^x + 35^y = Z^2$ has only two solutions of (x, y, z) as $(1, 0, 2)$ and $(0, 1, 6)$ in non-negative integers.

Keywords: Exponential diophantine equation; Catalan's conjecture; Modulo system; Congruence.

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1. Introduction

Diophantine equations are such equations that have more unknowns than a number of equations. These equations of the form $a^x + b^y = c^z$ are a widely studied field of research by applying different methods for solutions [1-4]. Diophantine equations are used to solve projective curves [5], chemical equations [6], and public-key cryptography [7]. The irrationality of a given number can be proved with the help of the Diophantine equation [8,9]. Different types of puzzles and riddles, e.g., the monkey and coconut puzzle and Mahavira's puzzle, have the solutions by considering these puzzles in Diophantine equations of first and second degree [10].

The Diophantine equation of the form of $2^x + 5^y = z^2$ is studied by Acu [11], who shows that this equation has two solutions in non-negative integers. Sroysang shows that the Diophantine equations $3^x + 5^y = z^2$ and $3^x + 17^y = z^2$ have unique solution (1, 0, 2) [12, 13]. Asthana and Singh find four non-negative integers solutions to the Diophantine equation $3^x + 13^y = z^2$ [14]. It is also found that the Diophantine equation $3^x + 117^y = z^2$ has exactly four solutions as (1, 0, 2), (3, 1, 12), (7, 1, 48) and (7, 2, 126)

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respectively [15]. It is shown that the Diophantine equation $3^x + 85^y = z^2$ has unique solution (1, 0, 2) for non-negative integers (*x*, *y*, *z*) [16]. Two Diophantine equations $3^x + 91^y = z^2$ and $3^x + 19^y = z^2$ are studied by Rabago, and it is established that these equations have two solutions for non-negative integers (*x*, *y*, *z*) [17].

The solutions of the Diophantine equation $n^x + (n + 5m)^y = z^{2k}$ are given by Gupta and Kumar [18]. All the solutions of the hyperbolic equation $x^2 - (\mu^2 - \mu)y^2 - (4\mu + 2)x + (6\mu^2 - 6\mu)y - (5\mu - 13) = 0$ for $\mu \geq 2$ are found by Kannan *et al*. [19]. It is also shown that the Diophantine equation $[p_1^c] + [p_2^c] + [p_3^c] = N$ is solvable in prime variables p_1, p_2 and p_3 with some conditions on p_i [20]. A series of explicit formulas for the integer solutions (x, y, z) of the Diophantine equation $x^2 - Dy^2 = K^Z$ is reported based on some conditions [21]. The solutions of the Diophantine equation F_{n+1}^x-$F_{n-1}^x = F_m^y$ where F_k denotes the kth terms of the Fibonacci sequence are found by Gómez et al. [22]. All positive solutions of the Diophantine equation $cx^2 + p^{2m} = 4y^n$ for some conditions are found by Chakraborty *et al*. [23]. It is proved that for all positive integers *n*, the Diophantine equation $(x/y) + p(y/z) + (z/w) + p(w/x) = 8np$, where $p = 1$ or p is a prime congruent to 1 (mod 8), does not have a solution in non-negative integers [24]. A heuristic list of the positive integer solutions *x*, *y* , and *z* of the Diophantine equation $a^x + (ab + 1)^y = b^z$ is described by Miyazaki et al. [25]. All the explicit solutions of the Diophantine equation $F_{n+1}^x - F_{n-1}^x = F_m$ are also found using lower bounds for linear forms in logarithms and properties of continued fractions [26]. Aggarwal and Kumar examine the Diophantine equation $13^{2m} + (6r + 1)^n = Z^2$ and find that these equations have no integer solutions [27]. The conditional solution of the Diophantine equation $(7^k - 1)^x + (7^k)^y = z^2$ is examined by Rahmawati *et al.* [28]. Burshtein finds the conditional solution of the Diophantine equation $p^x + (p + 4)^y = z^2$ as $(p, x, y, z) = (3, 2, 1, 4)$ [29].

The Diophantine equations of the form $3^x + p^y = z^2$ have drawn the attention of the researchers [12-17]. The present article is focused on one such kind of Diophantine equation. The aim of this article is to discuss the existence (if any) of the solution of the non-linear Diophantine equation $3^x + 35^y = Z^2$, where *x*, *y*, and Z are non-negative integers. The paper is organized as follows: In sec. 2, Catalan's conjecture, along with two lemmas, would be considered. The main theorem, along with its proof, is presented in sec. 3. Conclusion with the result is given in sec. 4.

2. Preliminaries

Catalan's conjecture [30], conjectured in 1844, was proved by Mihailescu [31] in 2004. In this section, Catalan's conjecture is used to prove the following Lemmas.

2.1. *Proposition*

$(3, 2, 2, 3)$ is a unique solution of (a, b, x, y) for the Diophantine equation $a^x - b^y = 1$, where *a*, *b*, *x*, and *y* are integers with $min\{a, b, x, y\} > 1$ [30, 31].

2.2. *Lemma*

$(1,2)$ is a unique solution of (x,z) for the Diophantine equation $3^x + 1 = Z^2$, where *x* and *z* are non-negative integers.

Proof: Suppose that there are non-negative integers *x* and *Z* such that $3^x + 1 = Z^2$. If $x = 0$, then $3^0 + 1 = 2 = Z^2$, which is impossible. It follows that $x \geq 1$. Thus, $Z^2 = 3^x + 1 \geq 3^1 + 1 = 4$. Therefore, $Z \geq 2$. Now the equation $3^x + 1 = Z^2$ can be written as $Z^2 - 3^x = 1$. By proposition 2.1, $Z = 2$ for $x = 1$. Hence, $(1,2)$ is a unique solution of (x,z) for the Diophantine equation $3^x + 1 = Z^2$.

2.3. *Lemma*

$(1,6)$ is a unique solution of (y,z) for the Diophantine equation $1 + 35^y = Z^2$, where *y* and *z* are non-negative integers.

Proof: Suppose that there are non-negative integers *y* and *Z* such that $1 + 35^y = Z^2$. If $y = 0$, then $35^0 + 1 = 2 = Z^2$, which is impossible. It follows that $y \geq 1$. Thus, $Z^2 = 35^y + 1 \geq 35^1 + 1 = 36$. Therefore $Z \geq 6$. Now the equation $35^y + 1 = Z^2$ can be written as $Z^2 - 35^y = 1$. By proposition 2.1, $Z = 6$ for $y = 1$. Hence, $(1,6)$ is a unique solution of (y,z) for the Diophantine equation $1 + 35^y = Z^2$.

3. Result and Discussion

In this section, it will be proved that the Diophantine equation $3^x + 35^y = Z^2$ has two unique non-negative integer solutions.

3.1. *Theorem*

The non-linear Diophantine equation $3^x + 35^y = Z^2$, where *x*, *y*, and *z* are non-negative integers has only two non-negative integer solutions of (x,y,z) as $(1,0,2)$ and $(0,1,6)$.

Proof: Here, three cases will be considered.

Case – I: If $x = 0$, then from Lemma 2.3, it can be concluded that the solution of the Diophantine equation $3^x + 35^y = Z^2$ for (x,y,z) is $(0,1,6)$.

Case – II: If $y = 0$, then by Lemma 2.2, it can be concluded that the solution of the Diophantine equation $3^x + 35^y = Z^2$ for (x,y,z) is $(1,0,2)$.

Case – III: If $x, y \geq 1$, then 3^x and 35^y both are odd. Thus Z^2 is even. So *Z* is even. Now it can be shown that $3^x \equiv 3 \ (mod\ 4)$ for the odd values of *x*, and $3^x \equiv 1 \ (mod\ 4)$ for the even values of *x*. Similarly, $35^y \equiv 3 \ (mod\ 4)$ for the odd values of *y* and $35^y \equiv 1 \ (mod\ 4)$ for the even values of *y*. It can be summarized and presented in Table 1 given below.

Table 1. Representation of expected solution of the Diophantine equation $3^x + 35^y = Z^2$for the odd and even values of x and y.

Case	3^x	35^y	$3^x + 35^y$	$Z^2 = 0, 1$ mod 4
Odd	$3\ (mod\ 4)$	$3\ (mod\ 4)$	$2\ (mod\ 4)$	$Z^2 \not\equiv 2$ (has no solution)
Even	$1\ (mod\ 4)$	$1\ (mod\ 4)$	$2\ (mod\ 4)$	$Z^2 \not\equiv 2$ (has no solution)

Therefore, it can be concluded from Table 1 that the Diophantine equation has no solution for $x, y \geq 1$.

4. Conclusion

There are various types of Diophantine equations. No universal method is helpful in finding all possible solutions (if exists) to Diophantine equations. Congruence theory and Catalan's conjecture are often employed to find solutions to some special types of Diophantine equations. In the present communication, these tools have also been utilized to obtain the solution to the Diophantine equation $3^x + 35^y = Z^2$. It is found that the Diophantine equation $3^x + 35^y = Z^2$ has solutions for non-negative integers *x*, *y*, and Z as $(1, 0, 2)$ and $(0, 1, 6)$. Researchers in a variety of subjects may find the results useful.

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