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Ranking Fuzzy Numbers with Unified Integral Value and Comparative Reviews

S. Prasad*, A. Sinha

Department of Mathematics, BBMK University, Dhanbad - 828130, Jharkhand, India

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Abstract

Fuzzy numbers represent ambiguous numeric values; therefore, it is difficult to rank them according to their magnitude. In making decisions, it is important to rank the fuzzy numbers. Since fuzzy numbers in a fuzzy environment often measure alternatives, a comparison of these fuzzy numbers is, in fact, a comparison of alternatives. There are a lot of different types of methods for ranking fuzzy numbers that exist in the literature. Still, there is no single method superior to all others regarding discrimination and consistency. This paper proposes a method for ranking fuzzy numbers with a unified integral value that multiplies two discriminatory components, the mode area integral and a linear sum of the absolute values of the integrals of the left and the right limits of alpha-cut of the normalized form of a fuzzy number. The method can rank two or more fuzzy numbers simultaneously, regardless of their linear or nonlinear membership functions. Furthermore, the unified integral value consistently ranks fuzzy numbers and their images and symmetric fuzzy numbers with the same altitude. Various types of fuzzy numbers are used in examples for comparative studies and investigations.

Keywords: Ranking fuzzy number; Alpha-cut; Normalized fuzzy number; Integral value.

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1. Introduction

The concept of fuzzy sets, fuzzy subsets, fuzzy operations (fuzzy union, fuzzy intersection, and fuzzy compliment), cut-worthy sets, fuzzy sets induced by mappings, convexity, and boundedness of fuzzy sets were introduced by Zadeh [1] in 1965. The α-cut representation of fuzzy sets was explicitly formulated and explained by Zadeh [2] in 1971. The principle introduced in [1] under the heading "fuzzy sets induced by mappings" was further elaborated as an extension principle by Zadeh [3] in 1975 and explained its utility explicitly. Later, the notion of the α-cut representation of the fuzzy sets and the extension principle became instrumental in the evolution of fuzzy numbers and fuzzy arithmetic. Dubois and Prade [4] extended the usual algebraic operations on real numbers to fuzzy numbers using a fuzzification principle. The ranking of fuzzy numbers is an important feature for its use in real-world scenarios. Due to the left-right fuzziness in the fuzzy numbers, they do not always show a completely ordered set, as can be done with the

* *Corresponding author*: shpd1512@gmail.com

real numbers. To resolve the task of comparing fuzzy numbers, many authors have proposed various fuzzy ranking methods that give a linearly ordered set or ranking. The notion of ranking imprecise quantities represented as fuzzy sets was first proposed by Jain [5]. Yager [6-8] introduced the relevant concepts of ranking fuzzy numbers. Since then, many scholars have presented similar methods or their applications for ranking fuzzy numbers. Bortolan and Degani [9] reviewed some methods to rank the fuzzy numbers in 1985. S-H Chen [10] offered an approach for ranking fuzzy numbers by maximizing set and minimizing set concepts. Liou and Wang [11] in 1992 developed a ranking approach based on an integral value that also considers decision maker's attitudes concerning specific purposes. Choobineh and Li [12] defined an index by using the fuzzy number's left and right-side areas. Fortemps and Roubens [13] studied a method based on area compensation. Cheng [14] presented an approach for ranking fuzzy numbers by using the distance method. Chu and Tsao [15] proposed ranking fuzzy numbers between the centroid and original points. Deng *et al.* [16] suggested an area method using the radius of gyration. Abbasbandy and Asady [17] introduced an approach to rank the fuzzy numbers by sign distance. Asady and Zendehnam [18] suggested a method of distance minimization for ranking fuzzy numbers. Garcia and Lamata [19] proposed a modification to the Liou and Wang technique for ranking fuzzy numbers with an integral value which includes an index of modality that expresses the neutrality of the decision-maker. Based on Chu and Tsao's [15] method, Wang and Lee [20] proposed a revised way of ordering fuzzy integers with an area between the centroid point and the original point. Abbasbandy and Hajjari [21] defined the magnitude of the trapezoidal fuzzy numbers for their ranking based on the left and the right spreads. Asady [22] presented a revision in the distance minimization method [18]. Rao *et al.* [23] proposed a distance method for ranking fuzzy numbers based on the circumcenter of centroids and an index of modality. Nasseri *et al.* [24] presented a method using the fuzzy number's parametric form and the angle between the reference functions to rank the fuzzy numbers. Yu and Dat [25] presented an improved method for ranking fuzzy numbers with integral values based on areas to overcome the shortcomings of Liou and Wang [11]. Zhang *et al.* [26] proposed a new method for comparing fuzzy numbers based on a fuzzy probabilistic preference relation. Rezvani [27] proposed ranking generalized exponential fuzzy numbers based on variance. Chutia and Chutia [28] introduced a value and ambiguity-based method for ranking parametric forms of fuzzy numbers with their defuzzifiers are at different heights. Chutia [29] presented a modified epsilon-deviation degree method for ranking fuzzy numbers. Nguyen [30] defined a unified index by multiplying the centroid value and an index based on areas as two discriminatory components of a fuzzy number for their ranking and presented comparative reviews. Nasseri *et al.* [31] presented a review of the Rao *et al.* [23] ranking approach and explored some shortcomings. Chi and Yu [32] proposed ranking generalized fuzzy numbers using centroid and ranking index. Wu *et al.* [33] studied fuzzy risk analysis based on ranking generalized fuzzy numbers using ordered weighted averaging technique. Mao [34] discussed the ranking of fuzzy numbers using weighted distance. Rao [35] proposed a new method for ranking fuzzy numbers to overcome the limitations of Chen

[10] ranking approach. Wang [36] presented relative preference relation-based ranking triangular interval-valued fuzzy numbers. Dombi and Jonas [37] introduced a probability-based fuzzy relation comparing fuzzy numbers with trapezoidal membership functions. Hajjari [38] proposed a similarity measure index to calculate the degree of similarity of generalized trapezoidal fuzzy numbers. Barazandeh and Ghazanfari [39] presented a new method for ranking generalized fuzzy numbers by considering different left and right heights of the fuzzy numbers. Patra [40] proposed a fuzzy risk analysis method to rank generalized trapezoidal fuzzy numbers by considering their mean, area, and perimeter.

In contrast to the plurality of the ranking indices, there is a wide range of scope for further studies and investigations. For instance, the index approach of Yu and Dat [25] and Nguyen [30] that incorporates the decision maker's attitude leads to counterintuitive ranking results for the fuzzy numbers having different representative locations on the real axis. Numerical illustrations are given in Ex. 5.6. Based on the combination of the two discriminatory components, this paper proposes an approach for ranking fuzzy numbers with a unified integral value that multiplies the mode area integral and a linear sum in the shape of a convex combination of the absolute values of integrals of the left and the right limits of alpha-cut of the normalized form of a fuzzy number. The convex combination indicates optimism that reflects the decision maker's attitude and is referred to as attitude-incorporated left-right limits of the alpha-cut. The left and right limits integrals of an alpha-cut of a fuzzy number would be referred to as "the left integral and the right integral," respectively, in the extended work of this paper. The value of the integral of the right limit is used to reflect the optimistic attitude. The value of the integral of the left limit is used to reflect the pessimistic attitude. An intuitive ranking approach based on the height of the fuzzy number is further applied to differentiate the symmetrical fuzzy numbers having the same support and different height. Finally, several comparative examples are given to demonstrate the usages and advantages of the proposed ranking approach.

Apart from the introduction above, the forward task of our approach is organized into five sections as follows. Section 2 consists of a brief review of the basic concept of the fuzzy number. Section 3 presents the integrals of the left and the right limits of alpha-cut and the mode area integral of the fuzzy number, the proposed unified integral value, and the ranking procedure. In Section 4, the basic notion of the triangular and trapezoidal fuzzy numbers are recalled and the expression for the ranking index is presented in simplified form. Section 5 presents the comparative numerical examples to illustrate the proposed approach's consistency and intuitiveness strength and validate the superiority over existing methods. The concluding remarks of advantages according to theoretical proofs and comparative studies of the outlined approach are made at last in section 6.

2. Preliminaries

This section recalls some basic definitions and notations related to the present study, followed by [11].

2.1. *Normal fuzzy number*

A normal fuzzy number is a fuzzy subset A of the real line R with membership function $\mathcal{M}_A(x)$ which possesses the following conditions for $a, b, c, d \in R, (a \leq b \leq c \leq d)$:

(i) $\mathcal{M}_A(x)$ is a piece-wise continuous function from R to the closed interval $[0, 1]$.
(ii) $\mathcal{M}_A(x) = 0$, for all $x \in]-\infty, a]$.
(iii) $\mathcal{M}_A(x)$ is strictly increasing on $[a, b]$.
(iv) $\mathcal{M}_A(x) = 1$, for all $x \in [b, c]$.
(v) $\mathcal{M}_A(x)$ is strictly decreasing on $[c, d]$.
(vi) $\mathcal{M}_A(x) = 0$, for all $x \in [d, \infty[$.

Conveniently, the fuzzy number in Def. 2.1 can be symbolized by $A = (a, b, c, d; 1)$ and its membership function $\mathcal{M}_A(x)$ can be expressed as

$$\mathcal{M}_A(x) = \begin{cases} \mathcal{M}_A^L(x)\,; & x \in [a, b] \\ 1; & x \in [b, c] \\ \mathcal{M}_A^R(x)\,; & x \in [c, d] \\ 0; & Otherwise \end{cases}, \tag{1}$$

Where $\mathcal{M}_A^L(x): [a, b] \longrightarrow [0, 1]$ and $\mathcal{M}_A^R(x): [c, d] \longrightarrow [0, 1]$ are respectively known as the left and the right membership functions of the fuzzy number A. $\mathcal{M}_A^L(x)$ is continuous and strictly increasing on $[a, b]$, whereas $\mathcal{M}_A^R(x)$ is continuous and strictly decreasing on $[c, d]$.

2.2. *Non-normal fuzzy number*

A non-normal fuzzy number is a fuzzy subset A of the real line R with membership function $\mathcal{M}_A(x)$ which possesses the following conditions for $a, b, c, d \in R, (a \leq b \leq c \leq d)$:

(i) $\mathcal{M}_A(x)$ is a piece-wise continuous function from R to the closed interval $[0, \omega]$ where ω is constant and $0 \leq \omega < 1$.
(ii) $\mathcal{M}_A(x) = 0$, for all $x \in]-\infty, a]$.
(iii) $\mathcal{M}_A(x)$ is strictly increasing on $[a, b]$.
(iv) $\mathcal{M}_A(x) = \omega$, for all $x \in [b, c]$.
(v) $\mathcal{M}_A(x)$ is strictly decreasing on $[c, d]$.
(vi) $\mathcal{M}_A(x) = 0$, for all $x \in [d, \infty[$.

Conveniently, the fuzzy number in Def. 2.2 can be symbolized by $A = (a, b, c, d; \omega)$, and its membership function $\mathcal{M}_A(x)$ can be expressed as

$$\mathcal{M}_A(x) = \begin{cases} \mathcal{M}_A^L(x)\,; & x \in [a, b] \\ \omega\,; & x \in [b, c] \\ \mathcal{M}_A^R(x)\,; & x \in [c, d] \\ 0; & Otherwise \end{cases}, \tag{2}$$

Where $\mathcal{M}_A^L(x): [a, b] \longrightarrow [0, \omega]$ and $\mathcal{M}_A^R(x): [c, d] \longrightarrow [0, \omega]$ are respectively known as the left and the right membership functions of the fuzzy number A. $\mathcal{M}_A^L(x)$ is continuous and strictly increasing on $[a, b]$, whereas $\mathcal{M}_A^R(x)$ is continuous and strictly decreasing on $[c, d]$.

The definition above in subsection 2.1 and 2.2 of normal and non-normal fuzzy numbers, respectively, can be summarized in a single definition as the generalized fuzzy number on replacing the inequality $0 \le \omega < 1$ by $0 \le \omega \le 1$ as follows:

2.3. *Generalized fuzzy number*

A fuzzy number (normal or non-normal) is a fuzzy subset A of the real line R with membership function $\mathcal{M}_A(x)$ which possesses the following conditions for $a, b, c, d \in R, (a \le b \le c \le d)$:

(i) $\mathcal{M}_A(x)$ is a piece-wise continuous function from R to the closed interval $[0, \omega]$ where ω is constant and $0 \le \omega \le 1$.
(ii) $\mathcal{M}_A(x) = 0$, for all $x \in]-\infty, a]$.
(iii) $\mathcal{M}_A(x)$ is strictly increasing on $[a, b]$.
(iv) $\mathcal{M}_A(x) = \omega$, for all $x \in [b, c]$.
(v) $\mathcal{M}_A(x)$ is strictly decreasing on $[c, d]$.
(vi) $\mathcal{M}_A(x) = 0$, for all $x \in [d, \infty[$.

Conveniently, the fuzzy number in Def. 2.3 can be symbolized by $A = (a, b, c, d; \omega)$, and its membership function $\mathcal{M}_A(x)$ can be expressed as

$$\mathcal{M}_A(x) = \begin{cases} \mathcal{M}_A^L(x); & x \in [a, b] \\ \omega; & x \in [b, c] \\ \mathcal{M}_A^R(x); & x \in [c, d] \\ 0; & Otherwise \end{cases}, \quad (3)$$

Where $\mathcal{M}_A^L(x): [a, b] \longrightarrow [0, \omega]$ and $\mathcal{M}_A^R(x): [c, d] \longrightarrow [0, \omega]$ are respectively known as the left and the right membership functions of the fuzzy number A. $\mathcal{M}_A^L(x)$ is continuous and strictly increasing on $[a, b]$, whereas $\mathcal{M}_A^R(x)$ is continuous and strictly decreasing on $[c, d]$.

2.4. *Image of a fuzzy number*

The image of a fuzzy number $A = (a, b, c, \mathrm{d};\ \omega)$ is defined by $A' = (-d, -c, -b, -a;\ \omega)$, where, $0 \le \omega \le 1$. Its membership function $\mathcal{M}_{A'}(x)$ can be expressed as

$$\mathcal{M}_{A'}(x) = \begin{cases} \mathcal{M}_{A'}^L(x); & x \in [-d, -c] \\ \omega; & x \in [-c, -b] \\ \mathcal{M}_{A'}^R(x); & x \in [-b, -a] \\ 0; & Otherwise \end{cases}, \quad (4)$$

2.5. *α –level set or α– cut*

The α –level set or α– cut of a fuzzy set $A = \{(x, \mathcal{M}_A(x)) \mid x \in U\}$ is a subset of U, defined by $A^\alpha = \{x \in U : \mathcal{M}_A(x) \ge \alpha\}$, $\alpha \in [0, 1]$, where U is the universal set, also known as the universe of discourse.

The α- cut of a fuzzy number $A = (a, b, c, d; \omega)$ is an interval $[A_l(\alpha),\ A_u(\alpha)]$ in R (the set of real numbers), where $A_l(\alpha) = a + (b-a)\frac{\alpha}{\omega}$ and $A_u(\alpha) = d - (d-c)\frac{\alpha}{\omega}$. $A_l(\alpha)$ and $A_u(\alpha)$ are respectively, called the lower and upper limit of the α- cut.

2.6. *Mode of a fuzzy number*

The α-cut of a fuzzy number $A = (a, b, c, d;\ \omega)$ for $\alpha = \omega$, is given by $A(\alpha) = [b, c]$, called the plateau of the fuzzy number. The mean of the plateau is called the mode of the fuzzy number, given by $M(A) = \frac{b+c}{2}$.

2.7. *Arithmetic operations*

The arithmetic operations of any two fuzzy numbers $A_i = (a_i, b_i, c_i, d_i;\ \omega_i)$ and $A_j = (a_j, b_j, c_j, d_j;\ \omega_j)$, $0 \le \omega_i, \omega_j \le 1$ are defined as follows:

(i) Addition of fuzzy numbers

$$A_i \oplus A_j = (a_i, b_i, c_i, d_i;\ \omega_i) \oplus (a_j, b_j, c_j, d_j;\ \omega_j)$$
$$= (a_i + a_j, b_i + b_j, c_i + c_j, d_i + d_j;\ \min\{\omega_i, \omega_j\}).$$

(ii) Subtraction of fuzzy numbers

$$A_i \ominus A_j = (a_i, b_i, c_i, d_i;\ \omega_i) \ominus (a_j, b_j, c_j, d_j;\ \omega_j)$$
$$= (a_i - a_j, b_i - b_j, c_i - c_j, d_i - d_j;\ \min\{\omega_i, \omega_j\}).$$

(iii) Multiplication of fuzzy numbers

$$A_i \otimes A_j = (a_i, b_i, c_i, d_i;\ \omega_i) \otimes (a_j, b_j, c_j, d_j;\ \omega_j)$$
$$= (a_i \times a_j, b_i \times b_j, c_i \times c_j, d_i \times d_j;\ \min\{\omega_i, \omega_j\}).$$

(iv) Division of fuzzy numbers

$$A_i \oslash A_j = (a_i, b_i, c_i, d_i;\ \omega_i) \oslash (a_j, b_j, c_j, d_j;\ \omega_j)$$
$$= \left(\frac{a_i}{d_j}, \frac{b_i}{c_j}, \frac{c_i}{b_j}, \frac{d_i}{a_j};\ \min\{\omega_i, \omega_j\}\right).$$

(v) Multiplication by a scalar $'k'$

$$kA_i = \begin{cases} (ka_i, kb_i, kc_i, kd_i;\ \omega_i)\ ; & if\ k \ge 0 \\ (kd_i, kc_i, kb_i, ka_i;\ \omega_i)\ ; & if\ k < 0 \end{cases}$$

Remark 1. Throughout our study, we will denote the normalized form of the fuzzy number A by $\bar{A}$ and the set of all fuzzy numbers by E.

3. The Unified Integral Value

Based on the integrals, the unified integral value, which multiplies the mode area integral and attitude incorporated left, and right limits of the α-cut of a fuzzy number, is presented in this section.

3.1. *Left-right integrals and mode integral*

Let $A = (a, b, c, d; \omega), 0 \leq \omega \leq 1$ be any fuzzy number. The normalized form of A is given by $\bar{A} = (a, b, c, d; 1)$ and the α- cut of $\bar{A}$ is $[\bar{A}_l(\alpha),\ \bar{A}_u(\alpha)]$, where $\bar{A}_l(\alpha) = a + (b-a)\alpha$ and $\bar{A}_u(\alpha) = d - (d-c)\alpha$. We define,

(i) The left integral of A as

$$I_L(A) = \int_0^1 \bar{A}_l(\alpha)\, d\alpha, \tag{5}$$

(ii) The right integral of A as

$$I_R(A) = \int_0^1 \bar{A}_u(\alpha)\, d\alpha, \tag{6}$$

(iii) The mode integral of A as

$$I_M(A) = \int_0^1 M(\bar{A})\, d\alpha, \tag{7}$$

3.2. *Attitude incorporated the left-right limits of the alpha-cut*

Now we define decision maker's approach, called attitude-incorporated left and right limits of the α- cut of a fuzzy number, symbolized by AI_A^η.

$$AI_A^\eta = \eta|I_R(A)| + (1-\eta)|I_L(A)|, \tag{8}$$

Where $\eta \in [0, 1]$ represents the level of optimism of a decision-maker. The value of $\eta = 0.5 \in [0, 1]$ represents a neutral approach of the decision-maker. The higher value of η, $(\eta > 0.5)$ indicates a higher degree of optimism and represents the optimistic approach of a decision-maker. The lower value of $\eta, (\eta < 0.5)$ indicates a lower degree of optimism and represents the pessimistic approach of a decision-maker. The two extreme values $\eta = 0$ and $\eta = 1$ indicate, respectively, the completely pessimistic and completely optimistic approach of the decision-maker.

Let us define the unified integral value of a fuzzy number A as a discriminatory tool (symbolized by UI_A^η) by multiplying the two different discriminatory components of the fuzzy number to boost the power of discrimination as

$$UI_A^\eta = [I_M(A) + \varepsilon][\eta|I_R(A)| + (1-\eta)|I_L(A)|], \tag{9}$$

Where ε is zero when $I_M(A) \neq 0$, otherwise it is a suitably small quantifiable positive rational number, taken for comparing fuzzy numbers for which $I_M(A) = 0$.

Remark 2. Let $E = \{A_1, A_2, A_3, \ldots\ldots\ A_n\}$ is the set of fuzzy numbers whose membership function is defined by Eq. (3). Then, for any two fuzzy numbers $A_i = (a_i, b_i, c_i, \mathrm{d}_i;\ \omega_i)$, $A_j = (a_j, b_j, c_j, \mathrm{d}_j;\ \omega_j) \in E$, the ranking decision can be made by using the unified integral value of the fuzzy number defined in Eq. (9) in the sense of a ranking function as follows,

(i) $UI_{A_i}^\eta > UI_{A_j}^\eta$, then $A_i \succ A_j$.
(ii) $UI_{A_i}^\eta < UI_{A_j}^\eta$, then $A_i \prec A_j$.
(iii) $UI_{A_i}^\eta = UI_{A_j}^\eta$, then the intuitive ranking approach based on height is applied to discriminate the fuzzy numbers as follows:

(a) If $\omega_i > \omega_j$, then $A_i \succ A_j$.
(b) If $\omega_i < \omega_j$, then $A_i \prec A_j$.
(c) If $\omega_i = \omega_j$, then $A_i \sim A_j$.

We now prove the consistency properties of the unified integral value for ranking fuzzy numbers and their images. Without loss of generality, $I_M(A) \neq 0$ is considered in the following theorems.

Theorem 1. Let $A_i' = (-d_i, -c_i, -b_i, -a_i;\ \omega_i)$, is the image of the fuzzy number $A_i = (a_i, b_i, c_i, \mathrm{d}_i;\ \omega_i)$, $i = 1, 2, \ . \ . \ . \ n$, respectively. Then,

(i) $I_M(A_i') = -I_M(A_i)$

(ii) $|I_L(A_i')| = |I_R(A_i)|$ and $|I_R(A_i')| = |I_L(A_i)|$.

(iii) $UI_{A_i'}^{\eta} = -UI_{A_i}^{(1-\eta)}$ and $UI_{A_i'}^{(1-\eta)} = -UI_{A_i}^{\eta}$

Proof. (i) From Eq. (7),

$$I_M(A_i') = \int_0^1 M(\bar{A}_i')\, d\alpha = \int_0^1 \frac{-b_i - c_i}{2}\, d\alpha$$
$$= -\int_0^1 \frac{b_i + c_i}{2}\, d\alpha$$
$$= -\int_0^1 M(\bar{A}_i)\, d\alpha = -I_M(A_i).$$

(ii) From Eq. (5) and Eq. (6), we have

$$|I_L(A_i')| = \left|\int_0^1 \bar{A}_{i_l}'(\alpha)\, d\alpha\right| = \left|\int_0^1 [-d_i + (-c_i + d_i)\alpha]\, d\alpha\right|$$
$$= \left|\int_0^1 -[d_i + (c_i - d_i)\alpha] d\alpha\right|$$
$$= \left|\int_0^1 [d_i + (c_i - d_i)\alpha]\, d\alpha\right|$$
$$= |I_R(A_i)|.$$

Also, $$|I_R(A_i')| = \left|\int_0^1 \bar{A}_{i_u}'(\alpha)\, d\alpha\right| = \left|\int_0^1 \{-a_i - (-a_i + b_i)\alpha\}\, d\alpha\right|$$
$$= \left|\int_0^1 -\{a_i + (b_i - a_i)\alpha\}\, d\alpha\right|$$
$$= \left|\int_0^1 \{a_i + (b_i - a_i)\alpha\}\, d\alpha\right|$$
$$= |I_L(A_i)|.$$

(iii) From Eq. (9), we have

$$UI_{A_i'}^{\eta} = I_M(A_i')[\eta|I_R(A_i')| + (1-\eta)|I_L(A_i')|]$$
$$= -I_M(A_i)[\eta|I_L(A_i)| + (1-\eta)|I_R(A_i)|]$$
$$= -I_M(A_i)\big[(1-\eta)|I_R(A_i)| + \big(1 - (1-\eta)\big)|I_L(A_i)|\big]$$
$$= -UI_{A_i}^{(1-\eta)}$$

Also, $$UI_{A_i'}^{(1-\eta)} = I_M(A_i')\big[(1-\eta)|I_R(A_i')| + \big(1-(1-\eta)\big)|I_L(A_i')|\big]$$
$$= -I_M(A_i)[(1-\eta)|I_L(A_i)| + \eta|I_R(A_i)|]$$
$$= -UI_{A_i}^{\eta}.$$ Hence, the proof.

Theorem 2. Let A'_k = ($-a_k$, $-b_k$, $-c_k$, $-d_k$; ω_k) is the image of the fuzzy number A_k = (a_k, b_k, c_k, d_k; ω_k), $k = 1, 2, \ldots n$, respectively. Then for $i, j \in k, (i \neq j)$

(i) $UI^{\eta}_{A_i} > UI^{\eta}_{A_j}$ if and only if $UI^{(1-\eta)}_{A'_i} < UI^{(1-\eta)}_{A'_j}$

(ii) $UI^{\eta}_{A_i} < UI^{\eta}_{A_j}$ if and only if $UI^{(1-\eta)}_{A'_i} > UI^{(1-\eta)}_{A'_j}$

Proof. (i) We let $UI^{\eta}_{A_i} > UI^{\eta}_{A_j}$

Then, by Theorem 1, $-UI^{(1-\eta)}_{A'_i} > -UI^{(1-\eta)}_{A'_j}$

This implies $UI^{(1-\eta)}_{A'_i} < UI^{(1-\eta)}_{A'_j}$

(ii) We let $UI^{\eta}_{A_i} < UI^{\eta}_{A_j}$

Then, by Theorem 1, $-UI^{(1-\eta)}_{A'_i} < -UI^{(1-\eta)}_{A'_j}$

This implies $UI^{(1-\eta)}_{A'_i} > UI^{(1-\eta)}_{A'_j}$ This completes the proof.

Remark 3. Let A_k = (a_k, b_k, c_k, d_k; ω_k), $k = \overline{1, n}$ are the fuzzy numbers and A'_k = ($-a_k$, $-b_k$, $-c_k$, $-d_k$; ω_k) are their respective images. Then, by the ranking algorithm in Remark 2 and the Theorem 2, the following statements can be made for pair-wise comparison of the fuzzy numbers A_i ; A_j and their respective images A'_i ; A'_j for $i, j \in k$.

(i) $A_i \succ A_j$ if and only if $A'_i \prec A'_j$.

(ii) $A_i \prec A_j$ if and only if $A'_i \succ A'_j$.

(iii) $A_i \sim A_j$ if and only if $A'_i \sim A'_j$.

Theorem 3. Consider the symmetric fuzzy numbers A_k = (a_k, b_k, c_k, d_k; ω_k), $k = 1, 2, \ldots n$ and their images A'_k = ($-a_k$, $-b_k$, $-c_k$, $-d_k$; ω_k). Then, by using the unified integral value from Eq. (9) for any pair of symmetric fuzzy numbers A_i and A_j, $i, j \in k$,

(i) $\eta \in [0, 0.5) \Rightarrow A_i \prec A_j \ (A'_i \succ A'_j)$

(ii) $\eta = 0.5 \Rightarrow A_i \sim A_j \ (A'_i \sim A'_j)$

(iii) $\eta \in (0.5, 1] \Rightarrow A_i \succ A_j \ (A'_i \prec A'_j)$

Proof. Since, $A_i = (a_i, b_i, c_i, d_i;\ \omega_i)$ and $A_j = (a_j, b_j, c_j, d_j;\ \omega_j)$, $i, j = \overline{1, n}$ are symmetric fuzzy numbers, therefore, we must have $a_i + d_i = a_j + d_j$ and $b_i + c_i = b_j + c_j$. Without loss of generality, we may assume that $a_i < a_j \ (d_i > d_j)$ and $b_i < b_j \ (c_i > c_j)$

Therefore, from Eq. (7) $I_M(A_i) = \int_0^1 \frac{b_i + c_i}{2}\, d\alpha = \int_0^1 \frac{b_j + c_j}{2}\, d\alpha = I_M(A_j)$.

From Eq. (8), the attitude incorporated left-right integrals of A_i and A_j are respectively,

$AI^{\eta}_{A_i} = \eta |I_R(A_i)| + (1-\eta)|I_L(A_i)|$

and $AI^{\eta}_{A_j} = \eta |I_R(A_j)| + (1-\eta)|I_L(A_j)|$

Where $|I_L(A_i)| = \left|\int_0^1 \bar{A}_{i_l}(\alpha)\, d\alpha\right|$

$|I_R(A_i)| = \left|\int_0^1 \bar{A}_{i_u}(\alpha)\, d\alpha\right|$

$|I_L(A_j)| = \left|\int_0^1 \bar{A}_{j_l}(\alpha)\, d\alpha\right|$

and $|I_R(A_j)| = \left|\int_0^1 \bar{A}_{j_u}(\alpha)\, d\alpha\right|$

Due to the symmetry of A_i *and* A_j we have,

(a) $\eta \in [0, 0.5) \Leftrightarrow AI_{A_i}^{\eta} < AI_{A_j}^{\eta}$

(b) $\eta = 0.5 \Leftrightarrow AI_{A_i}^{\eta} = AI_{A_j}^{\eta}$

and (c) $\eta \in (0.5, 1] \Leftrightarrow AI_{A_i}^{\eta} > AI_{A_j}^{\eta}$

Using results above, Eq. (9) and Remark 2, 3, we have

(i) $\eta \in [0, 0.5) \Rightarrow UI_{A_i}^{\eta} < UI_{A_j}^{\eta} \Rightarrow A_i \prec A_j \;(A_i' \succ A_j')$

(ii) $\eta = 0.5 \Rightarrow UI_{A_i}^{\eta} = UI_{A_j}^{\eta} \Rightarrow A_i \sim A_j \;(A_i' \sim A_j')$

(iii) $\eta \in (0.5, 1] \Rightarrow UI_{A_i}^{\eta} > UI_{A_j}^{\eta} \Rightarrow A_i \succ A_j \;(A_i' \prec A_j')$

Hence the proof.

4. Unified Integral Value of the Triangular and Trapezoidal Fuzzy Numbers

In this section, the formulae for the unified integral value of triangular and trapezoidal fuzzy numbers are derived in simplified form after a quick overview of their basic notion [11].

4.1. *Triangular fuzzy number*

A fuzzy number A is a triangular fuzzy number, if for $0 \le \omega \le 1$, its membership function $\mathcal{M}_A(x)$ is given by

$$\mathcal{M}_A(x) = \begin{cases} \omega\frac{x-a}{b-a}\,; & x \in [a,b] \\ \omega\,; & x = b \\ \omega\frac{x-c}{b-c}\,; & x \in [b,c] \\ 0; & Otherwise \end{cases}, \tag{10}$$

The triangular fuzzy number as defined above is denoted by $A = (a, b, c; \omega)$
In the case of triangular fuzzy number $A = (a, b, c; \omega)$, we obtain the unified integral values as follows.

The normalized form of the fuzzy number $A = (a, b, c; \omega)$ is $\bar{A} = (a, b, c; 1)$ and the α- cut of $\bar{A} = (a, b, c; 1)$ is given by $[\bar{A}_l(\alpha),\; \bar{A}_u(\alpha)]$.
Where $\bar{A}_l(\alpha) = a + (b-a)\alpha$ and $\bar{A}_u(\alpha) = c - (c-b)\alpha$,
The mode of $\bar{A} = (a, b, c; 1)$ is given by $M(\bar{A}) = b$,

Using Eq. (5), Eq. (6), and Eq. (7), we obtain the integrals

$$I_L(A) = \int_0^1 \bar{A}_l(\alpha)\, d\alpha = \int_0^1 [a + (b-a)\alpha]\, d\alpha = \frac{(a+b)}{2},$$

$$I_R(A) = \int_0^1 \bar{A}_u(\alpha)\, d\alpha = \int_0^1 [c - (c-b)\alpha]\, d\alpha = \frac{(b+c)}{2},$$

$$I_M(A) = \int_0^1 M(\bar{A})\, d\alpha = \int_0^1 b\, d\alpha = b,$$

Using Eq. (9), the unified integral value of the triangular fuzzy number $A = (a, b, c; \omega)$ is obtained by

$$UI_A^{\eta} = (b + \varepsilon).\left[\eta.\left|\frac{(b+c)}{2}\right| + (1-\eta).\left|\frac{(a+b)}{2}\right|\right], \tag{11}$$

4.2. *Trapezoidal fuzzy number*

A fuzzy number $A = (a, b, c, d; \omega)$ is a trapezoidal fuzzy number, if for $0 \le \omega \le 1$, its membership function $\mathcal{M}_A(x)$ is given by

$$\mathcal{M}_A(x) = \begin{cases} \omega\frac{x-a}{b-a}; & x \in [a,b] \\ \omega; & x \in [b,c] \\ \omega\frac{x-d}{c-d}; & x \in [c,d] \\ 0; & Otherwise \end{cases} \tag{12}$$

The trapezoidal fuzzy number as defined above is denoted by $A = (a, b, c, d; \omega)$

In the case of trapezoidal fuzzy number $A = (a, b, c, d; \omega)$, we obtain the unified integral values as follows.

The normalized form of the fuzzy number $A = (a, b, c, d; \omega)$ is $\bar{A} = (a, b, c, d; 1)$ and the α- cut of $\bar{A} = (a, b, c, d; 1)$ is given by $[\bar{A}_l(\alpha),\ \bar{A}_u(\alpha)]$.

where $\bar{A}_l(\alpha) = a + (b-a)\alpha$ and $\bar{A}_u(\alpha) = d - (d-c)\alpha$,

The mode of $\bar{A} = (a, b, c, d; 1)$ is given by $M(\bar{A}) = \frac{b+c}{2}$,

Using Eq. (5), Eq. (6), and Eq. (7), we obtain the following integrals as

$$I_L(A) = \int_0^1 \bar{A}_l(\alpha)\, d\alpha = \int_0^1 [a + (b-a)\alpha]\, d\alpha = \frac{(a+b)}{2},$$

$$I_R(A) = \int_0^1 \bar{A}_u(\alpha)\, d\alpha = \int_0^1 [d - (d-c)\alpha]\, d\alpha = \frac{(c+d)}{2},$$

$$I_M(A) = \int_0^1 M(\bar{A})\, d\alpha = \int_0^1 \frac{b+c}{2}\, d\alpha = \frac{b+c}{2},$$

Using Eq. (9), the unified integral value of the trapezoidal fuzzy number $A = (a, b, c, d; \omega)$ is obtained as

$$UI_A^{\eta} = \left(\frac{b+c}{2} + \varepsilon\right).\left[\eta.\left|\frac{(c+d)}{2}\right| + (1-\eta).\left|\frac{(a+b)}{2}\right|\right], \tag{13}$$

5. Comparative Numerical Examples

In this section, several fuzzy-number examples are used to compare ranking outputs between the proposed approach and some up-to-date representative methods from the literature, standard for a wide range of fuzzy number comparative studies and investigations. Based on Theorems 1, 2, and Remarks 2, 3, the unified integral value of the fuzzy number fulfills the consistency property for ranking the fuzzy numbers and their associated images. Therefore, for the shake of brevity, the ranking values of the images are not shown in the comparative tables. In Examples 5.1 to 5.8, the detailed explanations of the performance of some existing indices and other methods in contrast with the proposed index are subsequently described.

Example 5.1. Consider the ranking of two triangular fuzzy numbers $A_1 = (1, 4, 5)$ and $A_2 = (2, 3, 6)$, taken from Nguyen [30], which are congruent and overlapped as displayed in Fig. 1. with their associated images A'_1 and A'_2 are shown on the left of the membership axis. It is challenging for intuition to distinguish these fuzzy numbers and their associated images due to overlapping after flipping and sliding movement. Using formulae in Eq. (11), the unified integral values for these triangular fuzzy numbers at different levels of optimism $\eta \in [0, 1]$ are obtained and displayed in Table 1. On account of Theorem 1, Theorem 2, and Remark 3, the ranking result is $A_1 \succ A_2$ $(A'_1 \prec A'_2)$ at any arbitrary level of optimism of $\eta \in [0, 1]$. This ranking result is quite logical and reasonable, supported by S-H Chen [10], Abbasbandy and Hajjari [21], Chutia and Chutia [28], and Chutia [29]. Yager [6], Cheng [14], Rezvani [27], and the unified index of Nguyen [30] advocates different consequences as $A_1 \prec A_2$. The ranking indexes of Liou and Wang [11] and Yu and Dat [25] are inconsistent in distinguishing these fuzzy numbers and their associated images and infer $A_1 \sim A_2$ $(A'_1 \sim A'_2)$. Yu and Dat [25] further advocates the median value (*Me*) of the fuzzy numbers to make a preference and found $A_1 \prec A_2$. Different approaches of Yager [8], Choobineh and Li [12], Fortemps and Roubens [13], Chu and Tsao [15], Abbasbandy and Asady [17], Asady and Zendehnam [18], and Nasseri *et al*. [24] are also inconsistent and infer $A_1 \sim A_2$. Hence, the proposed approach can lead to rank the fuzzy numbers precisely.

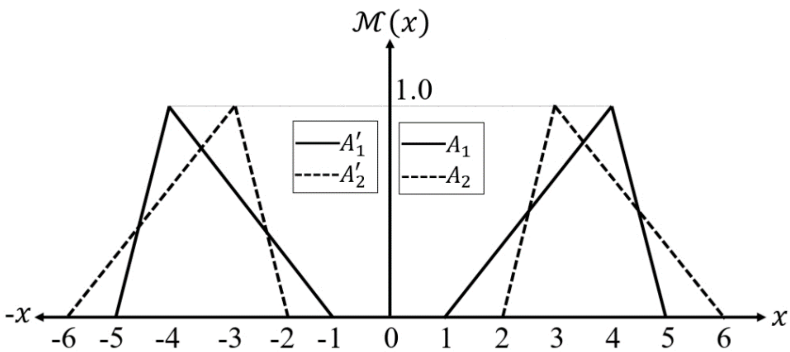

Fig. 1. Visual representation of fuzzy numbers and their images of Ex. 5.1.

Table 1. Ranking results for fuzzy numbers of Ex. 5.1.

η	$UI_{A_1}^{\eta}$	$UI_{A_2}^{\eta}$	Ranking results	
0.0	10.00	7.50	$A_1 \succ A_2$	$A_1' \prec A_2'$
0.1	10.80	8.10	$A_1 \succ A_2$	$A_1' \prec A_2'$
0.2	11.60	8.70	$A_1 \succ A_2$	$A_1' \prec A_2'$
0.3	12.40	9.30	$A_1 \succ A_2$	$A_1' \prec A_2'$
0.4	13.20	9.90	$A_1 \succ A_2$	$A_1' \prec A_2'$
0.5	14.00	10.50	$A_1 \succ A_2$	$A_1' \prec A_2'$
0.6	14.80	11.10	$A_1 \succ A_2$	$A_1' \prec A_2'$
0.7	15.60	11.70	$A_1 \succ A_2$	$A_1' \prec A_2'$
0.8	16.40	12.30	$A_1 \succ A_2$	$A_1' \prec A_2'$
0.9	17.20	12.90	$A_1 \succ A_2$	$A_1' \prec A_2'$
1.0	18.00	13.50	$A_1 \succ A_2$	$A_1' \prec A_2'$

Example 5.2. Consider three triangular fuzzy numbers $A_1 = (5, 6, 7)$, $A_2 = (5.9, 6, 7)$ and $A_3 = (6, 6, 7)$, taken from Nguyen [30], which have the same vertex and identical right spread as visualized in Fig. 2. with their associated images A_1' and A_2' are shown on the left of the membership axis. Intuitively, the size of the left fuzziness can easily distinguish them. Hence, for the fuzzy numbers and their associated images, the logical ranking is $A_1 \prec A_2 \prec A_3$ ($A_1' \succ A_2' \succ A_3'$). Therefore, this example is suitable to justify the intuition and consistency performance of the proposed unified integral value. Using Eq. (11), the unified integral value of these fuzzy triangular numbers is obtained and displayed in Table 2. On account of Theorem 1, Theorem 2, and Remark 3, the ranking results are $A_1 \prec A_2 \prec A_3$ ($A_1' \succ A_2' \succ A_3'$) except at the level of optimism $\eta = 1$, where the ranking results appear $A_1 \sim A_2 \sim A_3$ ($A_1' \sim A_2' \sim A_3'$). These results are quite reasonable because the effect of left fuzziness vanishes for $\eta = 1$. The ranking results of the index approaches of Liou and Wang [11] and Yu and Dat [25] are in total support. Different approaches of Yager [8], Chen [10], Cheng [14], Abbasbandy and Asady [17], Asady and Zendehnam [18], Deng *et al*. [16], Abbasbandy and Hajjari [21], and Nasseri *et al*. [24] also infer $A_1 \prec A_2 \prec A_3$. As a result, this example exhibits the great discrimination strength of the proposed ranking approach.

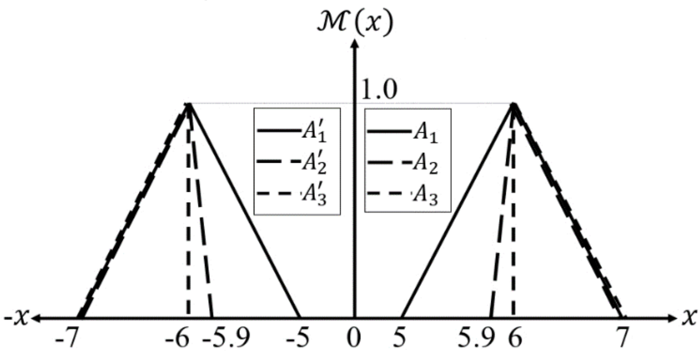

Fig. 2. Visual representation of fuzzy numbers and their images of Ex. 5.2.

Table 2. Ranking results for fuzzy numbers of Ex. 5.2.

η	$UI^{\eta}_{A_1}$	$UI^{\eta}_{A_2}$	$UI^{\eta}_{A_3}$	Ranking results	
0.0	33.00	35.70	36.00	$A_1 \prec A_2 \prec A_3$	$A'_1 \succ A'_2 \succ A'_3$
0.1	33.60	36.03	36.30	$A_1 \prec A_2 \prec A_3$	$A'_1 \succ A'_2 \succ A'_3$
0.2	34.20	36.36	36.60	$A_1 \prec A_2 \prec A_3$	$A'_1 \succ A'_2 \succ A'_3$
0.3	34.80	36.69	36.90	$A_1 \prec A_2 \prec A_3$	$A'_1 \succ A'_2 \succ A'_3$
0.4	35.40	37.02	37.20	$A_1 \prec A_2 \prec A_3$	$A'_1 \succ A'_2 \succ A'_3$
0.5	36.00	37.35	37.50	$A_1 \prec A_2 \prec A_3$	$A'_1 \succ A'_2 \succ A'_3$
0.6	36.60	37.68	37.80	$A_1 \prec A_2 \prec A_3$	$A'_1 \succ A'_2 \succ A'_3$
0.7	37.20	38.01	38.10	$A_1 \prec A_2 \prec A_3$	$A'_1 \succ A'_2 \succ A'_3$
0.8	37.80	38.34	38.40	$A_1 \prec A_2 \prec A_3$	$A'_1 \succ A'_2 \succ A'_3$
0.9	38.40	38.67	38.70	$A_1 \prec A_2 \prec A_3$	$A'_1 \succ A'_2 \succ A'_3$
1.0	39.00	39.00	39.00	$A_1 \sim A_2 \sim A_3$	$A'_1 \sim A'_2 \sim A'_3$

Example 5.3. Consider the three fuzzy numbers $A_1 = (1, 3, 5)$, $A_2 = (2, 3, 4)$ and $A_3 = (1, 4, 6)$ as shown in Fig. 3. The vertex and right fuzziness of A_3 are right way out of A_1 and A_2. Therefore, intuition prefer A_3 to A_1 and A_2. The intuition is not clear to distinguish A_1 and A_2 due to symmetry about the line $x = 3$. Using Eq. (11), the unified integral values are investigated for these fuzzy numbers and displayed in Table 3. Using Theorem 1, Theorem 2, and Remark 3, the ranking order found as A_3 leads to A_1 and A_2 irrespective of the value of $\eta \in [0, 1]$, which confirms the human intuition. The ranking order of A_1 and A_2 and their images are as follows: For $\eta = 0.5$, $A_1 \sim A_2$ $(A'_1 \sim A'_2)$; For $\eta \in [0, 0.5)$, $A_1 \prec A_2$ $(A'_1 \succ A'_2)$ and for $\eta \in (0.5, 1]$, $A_2 \prec A_1$ $(A'_2 \succ A'_1)$. This result verifies Theorem 3. The unified index procedure of Nguyen [30] advocates the same ranking results for all values of $\eta \in [0, 1]$. The integral value procedure of Liou and Wang [11] and Yu and Dat [25] yield the same ranking outputs except at $\eta = 0$, where they produce $A_1 \prec A_2 \sim A_3$. Other several Methods, Yager [6, 8], Fortemps and Roubens [13], Cheng [14], Chu and Tsao [15], Asady and Zendehnam [18], Abbasbandy and Hajjari [21] are inconsistent to distinguish the fuzzy numbers A_1 and A_2 and produce $A_1 \sim A_2$. Hence, the proposed ranking approach gives intuitive ranking results for the fuzzy numbers and their images and also shows consistency.

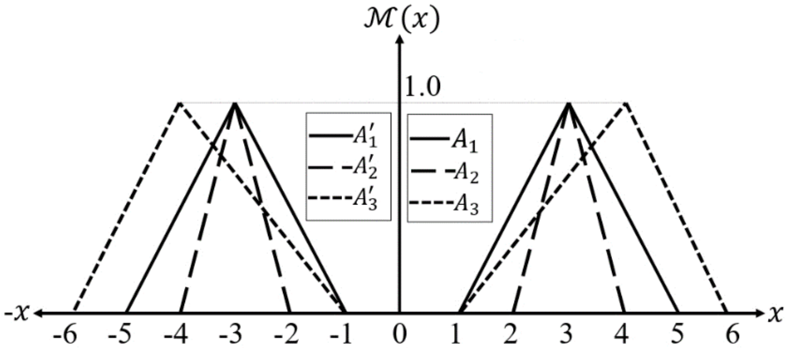

Fig. 3. Visual representation of fuzzy numbers and their images of Ex. 5.3.

Table 3. Ranking results for fuzzy numbers of Ex. 5.3.

η	$UI^{\eta}_{A_1}$	$UI^{\eta}_{A_2}$	$UI^{\eta}_{A_3}$	Ranking results	
0.0	6.00	7.50	10.00	$A_1 \prec A_2 \prec A_3$	$A'_1 \succ A'_2 \succ A'_3$
0.1	6.60	7.80	11.00	$A_1 \prec A_2 \prec A_3$	$A'_1 \succ A'_2 \succ A'_3$
0.2	7.20	8.10	12.00	$A_1 \prec A_2 \prec A_3$	$A'_1 \succ A'_2 \succ A'_3$
0.3	7.80	8.40	13.00	$A_1 \prec A_2 \prec A_3$	$A'_1 \succ A'_2 \succ A'_3$
0.4	8.40	8.70	14.00	$A_1 \prec A_2 \prec A_3$	$A'_1 \succ A'_2 \succ A'_3$
0.5	**9.00**	**9.0**	15.00	$\boldsymbol{A_1} \sim \boldsymbol{A_2} \prec A_3$	$\boldsymbol{A'_1} \sim \boldsymbol{A'_2} \succ A'_3$
0.6	9.60	9.30	16.00	$A_2 \prec A_1 \prec A_3$	$A'_2 \succ A'_1 \succ A'_3$
0.7	10.20	9.60	17.00	$A_2 \prec A_1 \prec A_3$	$A'_2 \succ A'_1 \succ A'_3$
0.8	10.80	9.90	18.00	$A_2 \prec A_1 \prec A_3$	$A'_2 \succ A'_1 \succ A'_3$
0.9	11.40	10.20	19.00	$A_2 \prec A_1 \prec A_3$	$A'_2 \succ A'_1 \succ A'_3$
1.0	12.00	10.50	20.00	$A_2 \prec A_1 \prec A_3$	$A'_2 \succ A'_1 \succ A'_3$

Example 5.4. Consider a triangular fuzzy number $A_1 = (1, 5, 5)$ overlapped with a trapezoidal fuzzy number $A_2 = (2, 3, 5, 5)$ as shown in Fig. 4 with their images A'_1 and A'_2 are in the left of the membership axis. Intuitively, the ranking outcome is not clear. Several existing measures in the literature have shown conflicting consequences. Yager [6], Cheng [14] and Deng *et al*. [16] infers $A_1 \prec A_2$ while Yager [8], S-H Chen [10], Choobineh and Li [12], Fortemps and Roubens [13], Chu and Tsao [15], Abbasbandy and Asady [17], Asady and Zendehnam [18], Abbasbandy and Hajjari [21] and Nasseri *et al*. [24] demonstrate for $A_1 \succ A_2$. Therefore, first, we investigate the unified integral values for these triangular and trapezoidal fuzzy numbers by using Eq. (11) and Eq. (13), displayed in Table 4. Using Theorem 1, Theorem 2, and remark 3, the ranking results are obtained as $A_1 \succ A_2$ $(A'_1 \prec A'_2)$ irrespective of the value of $\eta \in [0, 1]$. The ranking index approaches of Liou and Wang [11], Yu and Dat [25] yield the same ranking outputs for $0 \leq \eta \leq 0.9$, but they fail to discriminate the fuzzy numbers at the index of optimism $\eta = 1$. The unified index procedure of Nguyen [30] produces the same ranking consequences for $0 \leq \eta \leq 0.8$, and it produces $A_1 \prec A_2$ $(A'_1 \succ A'_2)$ for $0.9 \leq \eta \leq 1$. Hence, the proposed approach can lead to rank the triangular and trapezoidal fuzzy numbers conveniently.

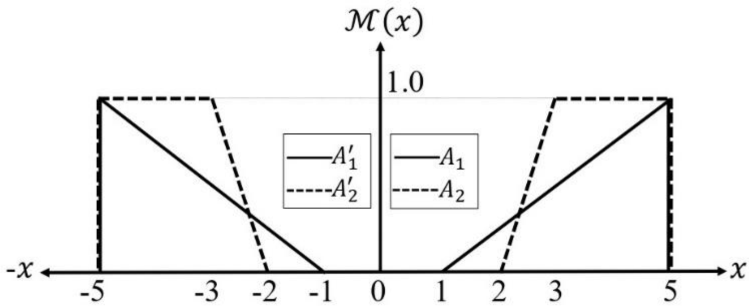

Fig. 4. Visual representation of fuzzy numbers and their images of Ex. 5.4.

Table 4. Ranking results for fuzzy numbers of Ex. 5.4.

η	$UI_{A_1}^{\eta}$	$UI_{A_2}^{\eta}$	Ranking results	
0.0	15.00	10.00	$A_1 \succ A_2$	$A'_1 \prec A'_2$
0.1	16.00	11.00	$A_1 \succ A_2$	$A'_1 \prec A'_2$
0.2	17.00	12.00	$A_1 \succ A_2$	$A'_1 \prec A'_2$
0.3	18.00	13.00	$A_1 \succ A_2$	$A'_1 \prec A'_2$
0.4	19.00	14.00	$A_1 \succ A_2$	$A'_1 \prec A'_2$
0.5	20.00	15.00	$A_1 \succ A_2$	$A'_1 \prec A'_2$
0.6	21.00	16.00	$A_1 \succ A_2$	$A'_1 \prec A'_2$
0.7	22.00	17.00	$A_1 \succ A_2$	$A'_1 \prec A'_2$
0.8	23.00	18.00	$A_1 \succ A_2$	$A'_1 \prec A'_2$
0.9	24.00	19.00	$A_1 \succ A_2$	$A'_1 \prec A'_2$
1.0	25.00	20.00	$A_1 \succ A_2$	$A'_1 \prec A'_2$

Example 5.5. Consider a trapezoidal fuzzy number $A_1 = (0, 2, 4, 6)$ mingled with two triangular fuzzy numbers $A_2 = (0, 3, 6)$ and $A_3 = (-1, 0, 2)$, taken from Nguyen [30], Their membership functions are visualized in Fig. 5. Here, A_3 is in left from A_1 and A_2. Therefore, intuitively, A_3 will be smaller than A_1 and A_2. A_1 and A_2 are of equal height and equal left and right spreads and symmetrical about the line $x = 3$. Therefore, intuition is not clear to determine their preference. Hence, the task is to find the preference of the trapezoidal fuzzy number A_1 and triangular fuzzy number A_2. Using Eq. (11) and Eq. (13), the unified integral values for these fuzzy numbers are obtained and listed in Table 5. Since the core of A_3 is at the original point, the value of ε is taken suitably small as $\varepsilon = 0.2$. On account of Remark 2, A_3 (A'_3) found smallest (largest) irrespective of the index of optimism $\eta \in [0, 1]$, which confirms the human intuition. On account of Theorem 1, Theorem 2, and Remark 3, the ranking results of A_1 and A_2 are as follows: the value $\eta = 0.5$ implies $A_1 \sim A_2$ $(A'_1 \sim A'_2)$ which divides as $\eta \in [0, 0.5) \Longrightarrow A_1 \prec A_2$ $(A'_1 \succ A'_2)$ and $\eta \in (0.5, 1] \Longrightarrow A_2 \prec A_1$ $(A'_2 \succ A'_1)$. This ranking results of A_1 and A_2 verifies the Theorem 3. The integral value approaches of Liou and Wang [11] and Yu and Dat [25] and the unified index approach of Nguyen [30] produce the same ranking results. Hence, this example judged the performance of the proposed approach.

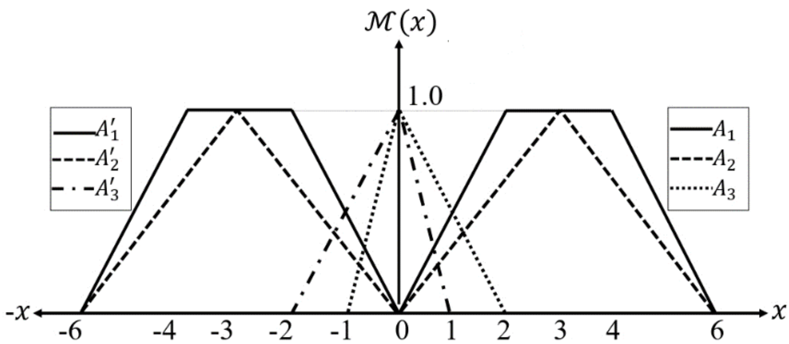

Fig. 5. Visual representation of fuzzy numbers and their images of Ex. 5.5.

Table 5. Ranking results for fuzzy numbers of Ex. 5.5.

η	$UI^{\eta}_{A_1}$	$UI^{\eta}_{A_2}$	$UI^{\eta}_{A_3}$	Ranking results	
0.0	3.00	4.50	0.10	$A_3 \prec A_1 \prec A_2$	$A'_3 \succ A'_1 \succ A'_2$
0.1	4.20	5.40	0.11	$A_3 \prec A_1 \prec A_2$	$A'_3 \succ A'_1 \succ A'_2$
0.2	5.40	6.30	0.12	$A_3 \prec A_1 \prec A_2$	$A'_3 \succ A'_1 \succ A'_2$
0.3	6.60	7.20	0.13	$A_3 \prec A_1 \prec A_2$	$A'_3 \succ A'_1 \succ A'_2$
0.4	7.80	8.10	0.14	$A_3 \prec A_1 \prec A_2$	$A'_3 \succ A'_1 \succ A'_2$
0.5	**9.00**	**9.00**	0.15	$A_3 \prec \boldsymbol{A_1} \sim \boldsymbol{A_2}$	$A'_3 \succ \boldsymbol{A'_1} \sim \boldsymbol{A'_2}$
0.6	10.20	9.90	0.16	$A_3 \prec A_2 \prec A_1$	$A'_3 \succ A'_2 \succ A'_1$
0.7	11.40	10.80	0.17	$A_3 \prec A_2 \prec A_1$	$A'_3 \succ A'_2 \succ A'_1$
0.8	12.60	11.70	0.18	$A_3 \prec A_2 \prec A_1$	$A'_3 \succ A'_2 \succ A'_1$
0.9	13.80	12.60	0.19	$A_3 \prec A_2 \prec A_1$	$A'_3 \succ A'_2 \succ A'_1$
1.0	15.00	13.50	0.20	$A_3 \prec A_2 \prec A_1$	$A'_3 \succ A'_2 \succ A'_1$

Example 5.6. Consider the three trapezoidal fuzzy numbers $A_1 = (5, 7, 9, 10; 1)$, $A_2 = (6, 7, 9, 10; 0.6)$ and $A_3 = (7, 8, 9, 10; 0.4)$ of different altitudes and the same right spread, taken from Liou and Wang [11]. The visual representation of their membership functions is shown in Fig. 6. Intuitively, one can make a preference $A_1 \prec A_2 \prec A_3$ $(A'_1 \succ A'_2 \succ A'_3)$ based on the representative locations of these fuzzy numbers on the real axis. Using Eq. (13), the unified integral values are obtained and displayed in Table 6. On account of Theorem 1, Theorem 2, and Remark 3, the ranking outcome is $A_1 \prec A_2 \prec A_3$ $(A'_1 \succ A'_2 \succ A'_3)$ except at an entirely optimistic level $(\eta = 1)$, where it produces $A_1 \sim A_2 \prec A_3$ $(A'_1 \sim A'_2 \succ A'_3)$ for $\eta = 1$. Liou and Wang [11] advocates the same ranking results. The ranking index of Yu and Dat [25] gives counterintuitive conclusions. He further employed median value (Me) of the fuzzy numbers and obtained intuitive ranking as $A_1 \prec A_2 \prec A_3$. The unified index of Nguyen [30] gives counterintuitive ranking results as $A_1 \succ A_2 \succ A_3$ $(A'_1 \prec A'_2 \prec A'_3)$ irrespective of the index of optimism, $\eta \in [0, 1]$. The two different approaches, Chu and Tsao [15] and Deng *et al*. [16], have used these fuzzy numbers in their paper, and they also concluded counterintuitive consequences as $A_1 \succ A_2 \succ A_3$. Wang and Lee [20] introduced a revision in the approach of Chu and Tsao [15] and have an intuitive preference for fuzzy numbers. As a result, the proposed method is a reliable strategy for generating intuitive ranking order.

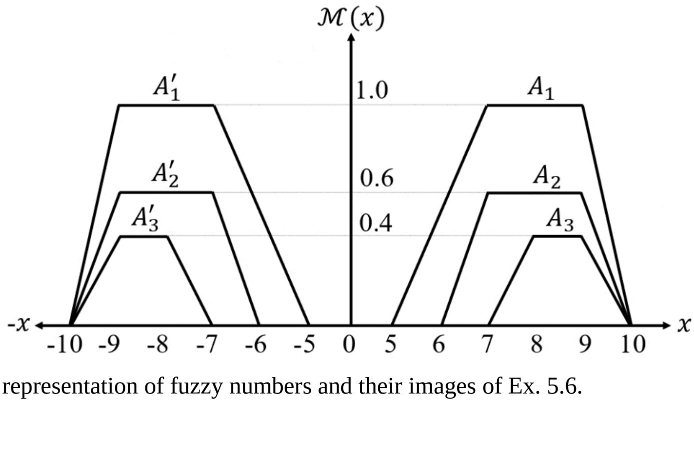

Fig. 6. Visual representation of fuzzy numbers and their images of Ex. 5.6.

Table 6. Ranking results for fuzzy numbers of Ex. 5.6.

η	$UI_{A_1}^{\eta}$	$UI_{A_2}^{\eta}$	$UI_{A_3}^{\eta}$	Ranking results	
0.0	48.00	52.00	63.75	$A_1 \prec A_2 \prec A_3$	$A_1' \succ A_2' \succ A_3'$
0.1	50.80	54.40	65.45	$A_1 \prec A_2 \prec A_3$	$A_1' \succ A_2' \succ A_3'$
0.2	53.60	56.80	67.15	$A_1 \prec A_2 \prec A_3$	$A_1' \succ A_2' \succ A_3'$
0.3	56.40	59.20	68.85	$A_1 \prec A_2 \prec A_3$	$A_1' \succ A_2' \succ A_3'$
0.4	59.20	61.60	70.55	$A_1 \prec A_2 \prec A_3$	$A_1' \succ A_2' \succ A_3'$
0.5	62.00	64.00	72.25	$A_1 \prec A_2 \prec A_3$	$A_1' \succ A_2' \succ A_3'$
0.6	64.80	66.40	73.95	$A_1 \prec A_2 \prec A_3$	$A_1' \succ A_2' \succ A_3'$
0.7	67.60	68.80	75.65	$A_1 \prec A_2 \prec A_3$	$A_1' \succ A_2' \succ A_3'$
0.8	70.40	71.20	77.35	$A_1 \prec A_2 \prec A_3$	$A_1' \succ A_2' \succ A_3'$
0.9	73.20	73.60	79.05	$A_1 \prec A_2 \prec A_3$	$A_1' \succ A_2' \succ A_3'$
1.0	76.00	76.00	80.75	$A_1 \sim A_2 \prec A_3$	$A_1' \sim A_2' \succ A_3'$

Example 5.7. Consider the two sets of crisp numbers which are considered by Nguyen [30]. The first set consists $A_1 = (1, 1, 1, 1; 0.5)$ and $A_2 = (1, 1, 1, 1; 1.0)$ and the second set consists $B_1 = (0.1, 0.1, 0.1, 0.1; 0.8)$ and $B_2 = (-0.1, -0.1, -0.1, -0.1; 1)$. These crisp numbers can be visualized in Fig. 7. Using Eq. (9), the unified integral values for these crisp numbers are obtained and displayed in Table 7, where $UI_{A_1}^{\eta}$ and $UI_{A_2}^{\eta}$ are found equal irrespective of the index of optimism $\eta \in [0, 1]$, therefore, A_1 and A_2 are ranked according to their height as $A_1 \prec A_2$ by Remark 2. From Table 7, $UI_{B_1}^{\eta}$ and $UI_{B_2}^{\eta}$ are scored as $UI_{B_1}^{\eta} > UI_{B_2}^{\eta}$ irrespective of $\eta \in [0, 1]$, therefore, B_1 and B_2 are ranked as $B_1 \succ B_2$ by Remark 2. Rezvani [27], Chutia and Chutia [28], and Nguyen [30] all come up with the same ranking results, indicating that the proposed method may be utilized with crisp numbers as well.

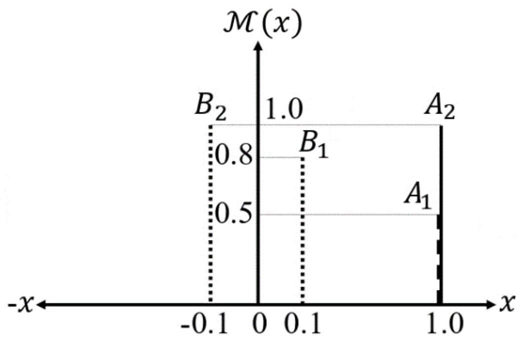

Fig. 7. Visual representation of crisp numbers of Ex. 5.7.

Table 7. Ranking results for crisp numbers of Ex. 5.7.

η	$UI_{A_1}^{\eta}$	$UI_{A_2}^{\eta}$	$UI_{B_1}^{\eta}$	$UI_{B_2}^{\eta}$	Ranking results	
0.0	1	1	0.01	-0.01	$A_1 \prec A_2$	$B_1 \succ B_2$
0.1	1	1	0.01	-0.01	$A_1 \prec A_2$	$B_1 \succ B_2$
0.2	1	1	0.01	-0.01	$A_1 \prec A_2$	$B_1 \succ B_2$
0.3	1	1	0.01	-0.01	$A_1 \prec A_2$	$B_1 \succ B_2$
0.4	1	1	0.01	-0.01	$A_1 \prec A_2$	$B_1 \succ B_2$

0.5	1	1	0.01	-0.01	$A_1 \prec A_2$	$B_1 \succ B_2$
0.6	1	1	0.01	-0.01	$A_1 \prec A_2$	$B_1 \succ B_2$
0.7	1	1	0.01	-0.01	$A_1 \prec A_2$	$B_1 \succ B_2$
0.8	1	1	0.01	-0.01	$A_1 \prec A_2$	$B_1 \succ B_2$
0.9	1	1	0.01	-0.01	$A_1 \prec A_2$	$B_1 \succ B_2$
1.0	1	1	0.01	-0.01	$A_1 \prec A_2$	$B_1 \succ B_2$

Example 5.8. Considering a triangular fuzzy number $A_1 = (1, 2, 5; 1)$ and a general fuzzy number $A_2 = (1, 2, 2, 4; 1)$ with non-linear membership function given by

$$\mathcal{M}_{A_2}(x) = \begin{cases} \sqrt{1-(x-2)^2}\ ; & 1 \le x \le 2 \\ \sqrt{1-\frac{1}{4}(x-2)^2}\ ; & 2 \le x \le 4 \\ 0; & otherwise \end{cases}$$

taken from Liou and Wang [11]. The visual representation of their membership functions is shown in Fig. 8. For the fuzzy number A_2, we have

$$\bar{A}_{2_l}(\alpha) = 2 - \sqrt{(1-\alpha^2)}\ ,\ \ \bar{A}_{2_u}(\alpha) = 2 + 2\sqrt{(1-\alpha^2)}\ ; \qquad 0 \le \alpha \le 1$$

and $\quad M(\bar{A}_2) = 2.$

Therefore, $I_L(A_2) = \int_0^1 \bar{A}_{2_l}(\alpha)\, d\alpha = \int_0^1 \left(2 - \sqrt{(1-\alpha^2)}\right) d\alpha = 1.2146$

$$I_R(A_2) = \int_0^1 \bar{A}_{2_u}(\alpha)\, d\alpha = \int_0^1 \left(2 + 2\sqrt{(1-\alpha^2)}\right) d\alpha = 3.5708$$

$$I_M(A_2) = \int_0^1 M(\bar{A}_2)\, d\alpha = 2$$

Using Eq. (9) and Eq. (11), the unified integral values are obtained and displayed in Table 8. On account of Theorem 1, Theorem 2, and Remark 3, the ranking outcomes can be viewed as, $A_1 \succ A_2\ (A_1' \prec A_2')$ for $0 \le \eta \le 0.8$ and $A_1 \prec A_2\ (A_1' \succ A_2')$ for $\eta = 0.9$ and $\eta = 1$. The index approach of Liou and Wang [11] and Yu and Dat [25] produce the same ranking results. Nguyen [30] infers $A_1 \succ A_2\ (A_1' \prec A_2')$ irrespective of the index of optimism $\eta \in [0, 1]$. Hence, in addition to triangular and trapezoidal fuzzy numbers, the proposed method can also rank fuzzy numbers with nonlinear membership functions.

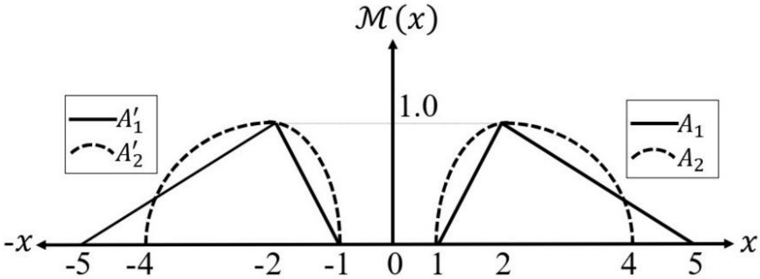

Fig. 8. Visual representation of fuzzy numbers of Ex. 5.8.

Table 8. Ranking results for fuzzy numbers of Ex. 5.8.

η	$UI_{A_1}^{\eta}$	$UI_{A_2}^{\eta}$	Ranking results	
0.0	3.000	2.429	$A_1 \succ A_2$	$A_1' \prec A_2'$
0.1	3.400	2.900	$A_1 \succ A_2$	$A_1' \prec A_2'$
0.2	3.800	3.372	$A_1 \succ A_2$	$A_1' \prec A_2'$
0.3	4.200	3.843	$A_1 \succ A_2$	$A_1' \prec A_2'$
0.4	4.600	4.314	$A_1 \succ A_2$	$A_1' \prec A_2'$
0.5	5.000	4.785	$A_1 \succ A_2$	$A_1' \prec A_2'$
0.6	5.400	5.256	$A_1 \succ A_2$	$A_1' \prec A_2'$
0.7	5.800	5.728	$A_1 \succ A_2$	$A_1' \prec A_2'$
0.8	6.200	6.199	$A_1 \succ A_2$	$A_1' \prec A_2'$
0.9	6.600	6.670	$A_1 \prec A_2$	$A_1' \succ A_2'$
1.0	7.000	7.142	$A_1 \prec A_2$	$A_1' \succ A_2'$

6. Conclusion

The fuzzy-number ranking solutions are obstructed by their counterintuitive nature, computational complexities, and lack of consistency. To minimize these ranking obstacles, this paper proposes a unified integral value as a discriminatory tool that multiplies the mode area integral and a linear sum in the shape of a convex combination of the absolute values of the integrals of the left and the right limits of α- cut of the normalized form of a fuzzy number. The unified integral value has four advantages for ranking fuzzy numbers according to theoretical proofs and comparative studies. Firstly, the ranking results support human intuition. Secondly, it demonstrates computational easiness irrespective of the types of fuzzy numbers. Thirdly, the unified integral value explains ranking conflicts in the literature based on optimism and the decision maker's attitude. Finally, the consistency property of the unified integral value can be used to rank the fuzzy numbers and their partnered images and symmetric fuzzy numbers with the same elevation. This property is significant for the accurate matching and recovery of information and evidence in the fields of computer vision and image pattern recognition.

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