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Exact Solution of Schrödinger Equation with Inverted Woods-Saxon and Manning-Rosen Potential

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Abstract

We have analytically solved the radial Schrödinger equation with inverted Woods-Saxon and Manning-Rosen Potentials. With the ansatz for the wave function, we obtain the generalized wave function and the negative energy spectrum for the system.

Keywords: Inverted Woods-Saxon Potentials; Manning-Rosen Potential; Schrödinger Equation.

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1. Introduction

The exact solutions of the Schrödinger wave equation (SWE) are very important because of the understanding of Physics that can only be brought by such solutions [1-4]. These solutions are valuable tools in checking and improving models and numerical methods being introduced for solving complicated physical problems at least in some limiting cases [5-6]. However, the exact solution of SWE for central potentials has generated much interest in recent years. These potentials in questions are the parabolic-like potential [7-8], the Eckart potential [4, 8, 9], the Rosen-Morse potential [10], the Fermi-step potential [4, 9], the Scarf barrier [11] and the Morse potential [12]. Various methods exist that have been adopted for the solution of the above mentioned potential. One of such method is the analytical solution of the radial Schrödinger equation which is of high importance in non-relativistic quantum mechanics; because the wave function contains all necessary information for full description of a quantum system [13-20]. The SWE can be solved exactly for only few cases of potential for all n and l.

However, the radial SWE for the Woods-Saxon potential were exactly solvable for $l \neq 0$, analytically [1,7], but Flugge [4] obtain an exact wave function and the energy eigen-values at $l = 0$ using graphical method. Woods-Saxon potential is one of the important short-range potential in Physics. The Woods-Saxon potential plays an essential

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role in microscopic physics, since it can be used to describe the interaction of a nuclear with a heavy nucleus [21-25]. Woods and Saxon introduced this potential to study elastic scattering of 20 MeV protons by a heavy nuclei [22].

Recently, an alternative method known as the Nikiforov-Uvarov (NU) method [26] was proposed for solving SWE. Therefore, the solution of radial SWE for Woods-Saxon potential of $l \neq 0$ using NU method has been reported in the literature [1]. The exact solution of SWE for the modified form of generalized Woods-Saxon potential for $l \neq 0$ have been studied analytically [1, 27].

In this article, we solve the radial SWE for the inverted Woods-Saxon and Manning-Rosen potential using the analytical method [1, 24-25, 27] and obtain the energy eigen-values and corresponding eigen function for arbitrary *l*-values.

2. Woods-Saxon, modified Woods-Saxon and the inverted Woods-Saxon and Manning-Rosen potentials

The Standard Woods-Saxon potential [1, 7, 22, 24, 27] is defined by

$$V(r) = \frac{V_0}{1+e^{\frac{(r-R_0)}{a}}} + \frac{V_1}{\left(1+e^{\frac{(r-R_0)}{a}}\right)^2}, \qquad a << R_0 \tag{1}$$

where V_0 and V_1 are the nuclear depth, R_0 is the width of the potential and "a" is the surface thickness. This potential was used for description of interaction of a nucleon with a heavy nuclear. Pahlavani *et al.* [27] modified the Woods-Saxon potential by adding two terms in the form:

$$V_{\text{mod}}(r) = \frac{V_0}{\left(1+e^{\left(\frac{r-R_0}{a}\right)}\right)} + \frac{\tau}{\left(1+e^{\left(\frac{r-R_0}{a}\right)}\right)^2} + \mu \coth\left(\frac{r-R_0}{a}\right) + \eta \coth^2\left(\frac{r-R_0}{a}\right). \tag{2}$$

where V_0, τ, μ and η are real parameters. Pahlavani *et al.* [27] in their paper noted that the third and fourth terms in Eq. (2) within the limit $(r - R_0) << a_0$ transformed to $1/r$ and $1/r^2$ corresponding to the Colombian repulsive potential and its square respectively. We present in Fig. 1 and Fig. 2 the plots of the standard Woods-Saxon and the modified Woods-Saxon potentials as a function of r.

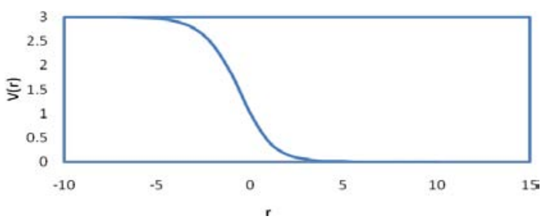

Fig. 1. A plot of Woods-Saxon potential.

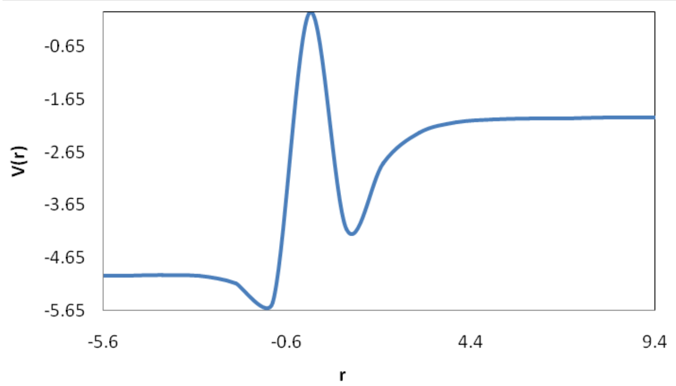

Fig. 2. A plot of modified Woods-Saxon potential.

Based on the argument of Pahlavani *et al.* [27] and without loss of generality, we write the inverted Woods-Saxon and Manning-Rosen potentials as

$$V_{in}(r)=\frac{V_0}{\left(1+\left(\frac{r-R_0}{a}\right)\right)}+\frac{V_1}{\left(1+\left(\frac{r-R_0}{a}\right)\right)^2}+V_2\tanh\left(\frac{r-R_0}{a}\right)+V_3\tanh^2\left(\frac{r-R_0}{a}\right). \qquad (3)$$

where V_0, V_1, V_2 and V_3 are potential depths. In Fig. 3, the inverted Woods-Saxon and Manning-Rosen potential is plotted as a function of *r* for V_0 = 5, 10 and 15 MeV, V_1 = 1, 5 and 10 MeV, $V_2 = V_3$ = 1 MeV and $a = R_0$ = 1 fm by comparison with Fig. 2 shows that $V_{in}(r)$ accounted for both positive and negative potentials unlike Woods-Saxon that accounted for only negative potentials.

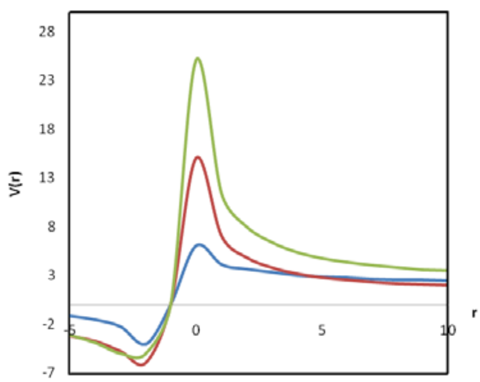

Fig. 3. Inverted Woods-Saxon and Manning-Rosen potential.

3. Exact Solution of Inverted Woods-Saxon and Manning-Rosen Potential

The radial Schrödinger equation with Eq. (3) is

$$\frac{-\hbar^2}{2m}\left[\frac{d^2}{dr^2}\psi_{nl}(r)+\frac{2}{r}\frac{d\psi_{nl}(r)}{dr}-\frac{L^2}{\hbar^2 r^2}\psi_{nl}(r)\right]$$

$$+\left[\frac{V_0}{1+\left(\frac{r-R_0}{a}\right)}+\frac{V_1}{\left(1+\left(\frac{r-R_0}{a}\right)\right)^2}+V_2\tanh\left(\frac{r-R_0}{a}\right)+V_3\tanh^2\left(\frac{r-R_0}{a}\right)\right]\psi_{nl}=E_n(r)\psi_{nl}(r) \quad (4)$$

Introducing a new function

$$\psi_{nl}(r)=\frac{\varphi_{nl}}{r}, \quad (5)$$

takes Eq. (4) into the form

$$\frac{-\hbar^2}{2m}\frac{d^2}{dr^2}\varphi_{nl}(r)+\frac{V_0}{\left(1+\left(\frac{r-R_0}{a}\right)\right)}\varphi_{nl}(r)+\frac{V_1}{\left(1+\left(\frac{r-R_0}{a}\right)\right)^2}\varphi_{nl}(r)$$

$$+V_2\tanh\left(\frac{r-R_0}{a}\right)\varphi_{nl}(r)+V_3\tanh^2\left(\frac{r-R_0}{a}\right)\varphi_{nl}(r) = E_{nl}(r)\psi_{nl}(r) \quad (6)$$

We transform Eq. (6) into one dimension by making the transformation,

$$x=\coth\left(\frac{r}{a}\right) \quad r=a(r-R_0), \quad (7)$$

and this reduces Eq. (6) to

$$\left(1-x^2\right)\frac{d^2\varphi_{nl}}{dx^2}-\frac{2xd\varphi_{nl}}{dx}+\frac{2L^2}{mx^2\left(1-x^2\right)}\varphi_{nl}(x)$$

$$+\frac{2mV_0a^2}{\hbar^2(1+x)}\varphi_{nl}(x)+\frac{2mV_1a^2}{\hbar^2(1+x)}\varphi_{nl}(x)+\frac{2mV_2a^2}{\hbar^2}\frac{x}{(1-x^2)}\varphi_{nl}(x)+\frac{2mV_3a^2}{\hbar^2(1+x)}\varphi_{nl}(x)$$

$$-\frac{2mEa^2}{\hbar^2\left(1-x^2\right)}\varphi_{nl}(x)=0 \tag{8}$$

Thus by introducing the following dimensionless parameters,

$$\varepsilon=-\frac{2mEa^2}{\hbar^2}<0, E<0, \qquad \sigma=\frac{2mV_2a^2}{\hbar^2}$$

$$\beta=\frac{2mV_0a^2}{\hbar^2}, \beta>0, \delta=\frac{2mV_2a^2}{\hbar^2} \qquad r=\frac{2mV_2a^2}{\hbar^2}, \tag{9}$$

lead to Jacobi differential equation given as

$$\left(1-x^2\right)\frac{d^2\varphi_{nl}(x)}{dx^2}-\frac{2xd\varphi_{nl}(x)}{dx}+\frac{2l(l+1)}{x^2\left(1-x^2\right)}\varphi_{nl}(x)$$

$$+\frac{\beta}{(1+x)}\varphi_{nl}(x)+\frac{\gamma(1-x)}{(1+x)}\varphi_{nl}(x)+\sigma\frac{(1-x)}{(1+x)}\varphi_{nl}(x)$$

$$+\frac{\delta x^2}{\left(1-x^2\right)}\varphi_{nl}(x)+\frac{\varepsilon}{\left(1-x^2\right)}\varphi_{nl}(x)=0, \tag{10}$$

where $L=\sqrt{l(l+1)}$.

We seek the solution to Eq. (10) by choosing an ansatz for the wave function in the form.

$$\varphi_{nl}(x)=U(x)w(x), \tag{11}$$

where *w(x)* is approximated as

$$w(x)=w_0(x)-i\ln\left[1+iw_0(x)\right]. \tag{12}$$

Substituting Eq. (11) and (12) into Eq. (10) and after a little algebra, we get

$$\left(1-x^2\right)U''(x)+\left[2\left(1-x^2\right)\left\{\frac{w_0'(x)}{w_0(x)\left(1-\frac{i}{w_0(x)}\ln(1+iw_0(x)\right)}\right.\right.$$

$$+\frac{2w_0'(x)}{w_0(x)\left(1-\frac{i}{w_0}\ln\left(1+iw_0(x)\right)\right)}-2x\Bigg\}\Bigg]U'(x)+\left[\frac{\left(1-x^2\right)}{\left(1-\frac{i}{w_0(x)}\ln(1+iw_0(x))\right)}\left\{\frac{w_0''(x)}{w_0(x)}\right.\right.$$

$$-\frac{w_0''(x)}{w_0(x)(1+iw_0(x))}-\frac{iw'^2{}_0(x)}{w_0(x)(1+iw_0(x))^2}$$

$$\left.\left.-\frac{iw_0'(x)}{(1+iw_0(x))^2}\right\}-\frac{w_0'(x)}{w_0(x)}\left(1-\frac{1}{w_0(x)(1+iw_0(x))}\right)2x\right]U(x)+$$

$$+\left[\frac{2l(l+1)}{x^2\left(1-x^2\right)}+\frac{\beta}{1+x}+\frac{\gamma(1-x)}{1+x}+\frac{\sigma x}{\left(1-x^2\right)}+\frac{\delta x^2}{\left(1-x^2\right)}+\frac{\varepsilon}{\left(1-x^2\right)}\right]U(x)=0 \tag{13}$$

The standard Jacobi Polynomials $P_n^{\alpha,\beta}<P_n^{\alpha,\beta}(x)$ satisfy the equation [28]

$$\left(1-x^2\right)P_n''^{\alpha,\beta}(x)+\left[\beta-\alpha-(\alpha+\beta+2)x\right]P_n'^{\alpha,\beta}$$
$$+n\left(n+\alpha+\beta+1\right)P_n^{\alpha,\beta}(x)=0, \tag{14}$$

where $$P_n^{\alpha,\beta}=\frac{\left(-1\right)}{2^n n!}(1-x)^{-\alpha}(1+\alpha)^{-\beta}\frac{d^n}{dx^n}\left[(1-\alpha)^{\alpha+n}(1+\alpha)^{\beta+n}\right]. \tag{15}$$

On substitution of Eq. (15) into Eq. (14) yields [27, 28]

$$\left(1-x^2\right)P_{n,l}''^{\alpha,\beta}(x)+\left[\beta-\alpha-\left(\alpha+\beta+2\right)x\right]P_{n,l}'^{\alpha,\beta}(x)$$

$$+\left[n(\alpha+\beta+n+1)-l\frac{\left(\alpha+\beta+l+(\alpha-\beta)x\right)}{1-x^2}\right]P_{n,l}^{\alpha,\beta}(x)=0 \tag{16}$$

Comparing Eq. (14) and Eq. (13) give

$$\frac{6\left(1-x^2\right)w_0'(x)}{w_0(x)\left[1-\frac{i}{w_0}(x)\ln\left(1+iw_0(x)\right)\right]}=\beta-\alpha, \tag{17}$$

and solving Eq, (17) results

$$w_0(x)=(1-x)^{\frac{\beta+\alpha}{3}}(1+x)^{\frac{\beta-\alpha}{3}}-\left(\frac{\beta-\alpha}{3}\right)\int\frac{\frac{i}{w_0(x)\ln\left(1+iw_0(x)\right)dx}}{\left(1-x^2\right)} \tag{18}$$

The integral in the RHS of Eq. (18) can be solved by noting the expansion,

$$\ln(1+x) = x + \frac{x^2}{2!} + \cdots \tag{19}$$

To first order correction, Eq. (18) becomes

$$w_0(x) = w_0(0)(1+x)^{\frac{\beta-\alpha}{3}}(1-x)^{\frac{\beta+\alpha}{3}}, \tag{20}$$

where $w_0(0)$ is the normalization constant Eq. (20) is the exact result obtained by Pahlavani *et al.* [27]. Taylor expansion of the argument in the integral of Eq. (18) up to second order gives the solution as

$$w_0(x) = w_0(0)(1+x)^{\frac{\beta-\alpha}{3}}(1-x)^{\frac{\beta+\alpha}{3}} + \frac{w_0^2(1)}{2}\ln(1-x)^{\frac{\beta-\alpha}{3}} - \frac{w_0^2(-1)}{2}\ln(1+x)^{\frac{\beta-\alpha}{3}} \tag{21}$$

Based on the symmetry and he odd or even property of the system, we may approximate the term

$$w_0^2(-1) = -w_0^2(1), \tag{22}$$

and Eq. (21) becomes

$$w_0(x) = w_0(0)(1-x)^{\frac{\beta+\alpha}{3}}(1+x)^{\frac{\beta-\alpha}{3}} + \frac{w_0^2(1)}{2}(1-x)^{\frac{\beta+\alpha}{3}}(1+x)^{\frac{\beta-\alpha}{3}} \tag{23}$$

Using Eq. (12), we obtain the wave function $w(x)$ as

$$w(x) = w_0(0)(1-x)^{\frac{\beta+\alpha}{3}}(1+x)^{\frac{\beta-\alpha}{3}} + \frac{w_0^2(1)}{2}(1-x)^{\frac{\beta-\alpha}{3}}$$
$$-i\ln\left[w_0(0)(1-x)^{\frac{\beta+\alpha}{3}}(1+x)^{\frac{\beta-\alpha}{3}}\right] - i\ln\left[1+\frac{1}{2}\frac{w_0^2(1)}{w_0(0)}\right] \tag{24}$$

where the last term is the normalization constant. With Eqs. (5), (7), (11) and (24), we obtain the bound states eigen function of the Schrödinger equation for the inverted Woods-Saxon and Manning-Rosen potential as

$$\psi_{nl}(r) = w_0(0)\left(1-\coth\frac{r}{a}\right)^{\frac{\beta+\alpha}{3}}\left(1+\coth\frac{r}{a}\right)^{\frac{\beta-\alpha}{3}} P_{n,l}^{\alpha,\beta}(r)$$

$$+\frac{w_0^2}{2}(0)\left(1-\coth\frac{r}{a}\right)^{\frac{\beta-\alpha}{3}}\left(1+\coth\frac{r}{a}\right)^{\frac{\beta+\alpha}{3}}P_{n,l}^{\alpha,\beta}(r)$$
$$-iA_0\ln\left[\left(1-\coth\frac{r}{a}\right)^{\frac{\beta+\alpha}{3}}\left(1+\coth\frac{r}{a}\right)^{\frac{\beta-\alpha}{3}}\right]P_{n,l}^{\alpha,\beta}(r)$$

where

$$A_0 = w_0(0)\ln\left[1+\frac{1}{2}\frac{w_0^2(1)}{w_0(0)}\right], \tag{26}$$

is the normalization constant. The eigen value associated with the wave function is obtained using Eqs. (9), (14) and (16) as:

$$E_{nl} = \frac{-\hbar^2 a^2}{2m}\left[(\alpha+1)^2 - l(l+1)\right] \tag{27}$$

3. Conclusion

We have used the analytical method to obtain the eigen function and the energy spectrum of the inverted Woods-Saxon and Manning-Rosen potential for $l \neq 0$, with an ansaltz for the wave function. We generalized the result to second order terms, and the first order term of the wave function reduces to the inverted form result reported in ref. [27], while up to second order term gives the result reported in ref. [25].

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