Publications

Available Online

J. Sci. Res. **12** (3), 327-339 (2020)

JOURNAL OF SCIENTIFIC RESEARCH

www.banglajol.info/index.php/JSR

Numerical Investigation of Two-Dimensional Oldroyd-B Fluid Flows Over A Straight Rectangular Domain

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Received 1 December 2019, accepted in final revised form 17 February 2020

Abstract

In this paper, non-Newtonian viscoelastic Oldroyd-B fluid flows in two-dimensional rectangular domain is numerically investigated, where the flow between two rigid walls is driven by a pressure difference along x-direction (horizontal). The numerical results of the nonlinear system of partial differential equations are obtained by decoupling the system into Navier-Stokes system and tensorial transport equation. Computational Fluid Dynamics (CFD) simulations are done by using the finite element method. The numerical simulations are presented in terms of the contours of velocity, pressure and extra stress tensor. The Hood-Taylor finite element method is used for the approximation of the velocity and the pressure while the discontinuous Galerkin method is used to approximate the stress tensor. All the meshes and simulations are carried out by the general finite element solver FreeFem++, which has been found as a potential tool to provide a reasonably good numerical simulations of complicated flow behavior.

Keywords: Oldroyd-B fluid; Navier-Stokes equations; Transport equations; CFD simulation; Finite element method.

doi: http://dx.doi.org/10.3329/jsr.v12i3.44970 J. Sci. Res. **12** (3), 327-339 (2020)

1. Introduction

The aim of this work is to analyze and simulate the incompressible non-Newtonian Oldroyd-B fluid flows [1-6], in a straight rectangular domain in case of two dimension. The constitutive equations for Oldroyd-B fluid is highly nonlinear system of partial differential equations (PDE), and therefore it is decoupled into two auxiliary problems, namely, Navier-Stokes equations and the tensorial transport equation. CFD describes the Oldroyd-B fluid flow in terms of these two auxiliary problems. In the present paper, we consider the two-dimensional flow of an Oldroyd-B fluid over a straight rectangular domain. The auxiliary problems are studied separately in our previous works [7-9].

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The Oldroyd-B model is a composed problem with three unknowns: the velocity components $\mathbf{u}$, the pressure p, and the viscoelastic extra stress tensor $\boldsymbol{\sigma}$. We use iterative scheme to solve this system. If $\boldsymbol{\sigma}$ is fixed, the model defines a Navier-Stokes systems in the variables $\mathbf{u}$ and p, which is analyzed using the Hood-Taylor finite element method for the approximation of the velocity and the pressure field $(\mathbf{u}, p)$ [8,9]. On the other hand, if $\mathbf{u}$ (and p) is fixed, then the model is transformed into a transport equation in the variable $\boldsymbol{\sigma}$. The approximation of $\boldsymbol{\sigma}$ is done by using discontinuous Galerkin finite element method [10-12], as discussed [7].

We briefly discussed the problem formulation, and mathematical and numerical analysis of Oldroyd-B model in the context of Hood-Taylor [13-15] and discontinuous Galerkin finite element method [10-12]. The variational formulation is discussed in terms of the auxiliary problems. To employ numerical simulations in Oldroyd-B model, Navier-Stokes problems is discretized using the Hood-Taylor finite elements and the discontinuous $\mathbb{P}_1$ elements ($\mathbb{P}_1 dc$) are used to discretize the transport problem. The finite element solver FreeFem++ [16] is used to obtain the approximation of the composed problem. The numerical results of the coupled Oldroyd-B problem are obtained both computationally and graphically by the numerical simulations of these two auxiliary problems implemented on the general finite element solver FreeFem++ [16].

2. Nomenclature

Before discussing the mathematical analysis of Oldroyd-B fluid flows with boundary conditions and variational formulation, we introduced some notations of different function spaces in the following table, details of which can be found [17,18].

$L^p(\Omega)$: The Lebesgue spaces	$H^m(\Omega)$: $W^{m,p}(\Omega)$, for $p=2$
$W^{m,p}(\Omega)$, : The standard Sobolev spaces where $m \geq 0$ be an integer and $1 \leq p \leq \infty$	$L^p(\Omega)$: $W^{0,p}(\Omega)$
$\lVert\cdot\rVert_{m,p}$: Norms of $W^{m,p}(\Omega)$	$C(\Omega)$: Vector space continuous functions

3. The Constitutive Equations for Oldroyd-B Model

Let Ω is a bounded, open and connected Lipschitz domain of $\mathbb{R}^d, d = 2,3$. Oldroyd-B model consists of the system of non-linear partial differential equations formed by the law of conservation of mass, the momentum equations and the form of transport equations, which can be written in the domain Ω as [19]

$$\begin{cases} \rho \frac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u}.\nabla)\mathbf{u} - \mu_n \Delta \mathbf{u} + \nabla p = \nabla.\boldsymbol{\sigma} + \rho\, \mathbf{f} \text{ in } \Omega, \\ \nabla.\mathbf{u} = 0 \text{ in } \Omega, \\ \lambda_1 \left[\frac{\partial \boldsymbol{\sigma}}{\partial t} + (\mathbf{u}.\nabla)\boldsymbol{\sigma}\right] + \boldsymbol{\sigma} = 2\mu_e \mathbf{D}(\mathbf{u}) - \lambda_1[\boldsymbol{\sigma}\mathbf{W}(\mathbf{u}) - \mathbf{W}(\mathbf{u})\boldsymbol{\sigma} - \boldsymbol{\sigma}\mathbf{D}(\mathbf{u}) - \mathbf{D}(\mathbf{u})\boldsymbol{\sigma}] \text{ in } \Omega. \end{cases} \tag{1}$$

where $\mu_e = \mu - \mu_n$ is the coefficient of elastic viscosity, $\mu > 0$ is the dynamic viscosity coefficient expressing the fluid's resistance which offers to shear strain during the displacement $([\mu] = Pa\ s)$, with μ_n is the coefficient of Newtonian viscosity, λ_1 is the relaxation time of fluid (the measures of the time for which the fluid remembers the flow history), $\boldsymbol{\sigma}$ is the viscoelastic extra stress tensor, $\mathbf{D}(\mathbf{u}) = \frac{1}{2}[\nabla\mathbf{u} + (\nabla\mathbf{u})^t]$ is the rate of deformation tensor, and $\mathbf{W}(\mathbf{u}) = \frac{1}{2}[\nabla\mathbf{u} - (\nabla\mathbf{u})^t]$ is the rate of vorticity tensor.

Assuming $\mathbf{h}(\boldsymbol{\sigma}, \nabla\mathbf{u}) = 2\mu_e\mathbf{D}(\mathbf{u}) - \lambda_1[\boldsymbol{\sigma}\mathbf{W}(\mathbf{u}) - \mathbf{W}(\mathbf{u})\boldsymbol{\sigma} - \boldsymbol{\sigma}\mathbf{D}(\mathbf{u}) - \mathbf{D}(\mathbf{u})\boldsymbol{\sigma}]$
$= 2\mu_e\mathbf{D}(\mathbf{u}) + \lambda_1[(\nabla\mathbf{u})\boldsymbol{\sigma} + \boldsymbol{\sigma}(\nabla\mathbf{u})^{\mathrm{t}}]$,
the Oldroyd-B constitutive equations (1) can be written as

$$\begin{cases} \rho\frac{\partial\mathbf{u}}{\partial t} + \rho(\mathbf{u}.\nabla)\mathbf{u} - \mu_n\,\Delta\mathbf{u} + \nabla p = \nabla.\boldsymbol{\sigma} + \rho\,\mathbf{f} \text{ in } \Omega, \\ \nabla.\mathbf{u} = 0 \text{ in } \Omega, \\ \lambda_1\left[\frac{\partial\boldsymbol{\sigma}}{\partial t} + (\mathbf{u}.\nabla)\boldsymbol{\sigma}\right] + \boldsymbol{\sigma} = \mathbf{h}(\boldsymbol{\sigma}, \nabla\mathbf{u}), \text{ in } \Omega. \end{cases} \tag{2}$$

The above set of equations describes the behavior of an incompressible viscoelastic fluid of Oldroyd-B type, in the open subset Ω, where the fluid is homogeneous.

We observed that the conservation of momentum leads the symmetry properties of the tensor $\boldsymbol{\sigma}$, i.e., $\boldsymbol{\sigma}^t = \boldsymbol{\sigma}$.

The problem (2) is a mixed problem. The first two equations form a parabolic system for $(\boldsymbol{u}, p)$ which is in the form of Navier-stokes equation. The last equation has a hyperbolic characteristic which is in the form of Transport equation.

In case of steady flow, $\boldsymbol{u}$ is independent of time and then $\frac{\partial\mathbf{u}}{\partial t} = 0$. So, the Oldroyd-B constitutive equations in case of steady flow is a non-linear system of partial differential equations (PDE) of a combined elliptic-hyperbolic type

$$\begin{cases} \rho(\mathbf{u}.\nabla)\mathbf{u} - \mu_n\,\Delta\mathbf{u} + \nabla p = \nabla.\boldsymbol{\sigma} + \rho\,\mathbf{f}, \text{in } \Omega, \\ \nabla.\mathbf{u} = 0, \text{in } \Omega, \\ \lambda_1(\mathbf{u}.\nabla)\boldsymbol{\sigma} + \boldsymbol{\sigma} = \mathbf{h}(\boldsymbol{\sigma}, \nabla\mathbf{u}), \text{in } \Omega. \end{cases} \tag{3}$$

3.1. Non-dimensional governing equations

To obtain a system of dimensionless variables, we discuss some scaling properties of three equations of the system (1) to introduce Reynolds number Re and Weissenberg number We that respectively measures the effect of viscosity and elasticity on the flow.

Let L be the characteristic length, U represents a characteristic velocity of the flow and $\mu = \mu_e + \mu_n$ be the viscosity coefficient. We transform the system (1) into dimensionless form by changing variables and by introducing the following dimensionless quantities:

$$\mathbf{x} = \frac{\mathbf{x}'}{L}, \quad t = \frac{t'}{T} = \frac{Ut'}{L}, \quad \mathbf{u} = \frac{\mathbf{u}'}{U}, \quad \boldsymbol{\sigma} = \frac{\boldsymbol{\sigma}'L}{\mu\,U}, \quad p = \frac{p'L}{\mu\,U} \quad \mathbf{f} = \frac{\mathbf{f}'L^2}{\mu\,U}$$
$$Re = \rho\frac{UL}{\mu} = \frac{UL}{\nu}, We = \lambda_1\frac{U}{L}$$

where the symbol ' is attached to dimensional parameters.

The dimensionless form of the system (1) can be written as

$$\begin{cases} Re[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u}] + \nabla p = (1-\lambda)\,\Delta \mathbf{u} + \nabla.\boldsymbol{\sigma} + \mathbf{f}, \text{in } \Omega, \\ \nabla \cdot \mathbf{u} = 0, \text{in } \Omega, \\ We\frac{D_a \boldsymbol{\sigma}}{Dt} + \boldsymbol{\sigma} = 2\lambda\, \mathbf{D}(\mathbf{u}), \text{ in } \Omega. \end{cases} \tag{4}$$

where $\lambda = 1 - \frac{\lambda_2}{\lambda_1} = \frac{\mu_e}{\mu_e + \mu_n}$, is the retardation parameter $(0 < \lambda < 1)$, and $\lambda_2 \geq 0$ is the retardation time of fluid.

Reynolds and Weissenberg numbers are the dimensionless numbers. Small values of We means that the fluid is little elastic and small values of Re means that the fluid is very viscous.
In case of stationary motion, the problem can be written as follows :
Find the quantities $\boldsymbol{\sigma}$, $\mathbf{u}$ and p defined in Ω such that

$$\begin{cases} Re[(\mathbf{u}.\nabla)\mathbf{u}] + \nabla p = (1-\lambda)\Delta \mathbf{u} + \nabla.\boldsymbol{\sigma} + \mathbf{f} \text{ in } \Omega, \\ \nabla.\mathbf{u} = 0 \text{ in } \Omega, \\ We[(\mathbf{u}.\nabla)\boldsymbol{\sigma}] + \boldsymbol{\sigma} = 2\lambda\, \mathbf{D}(\mathbf{u}) + We[(\nabla \mathbf{u})\boldsymbol{\sigma} + \boldsymbol{\sigma}(\nabla \mathbf{u})^{t}] \text{ in } \Omega. \end{cases} \tag{5}$$

The above equation is composed of a Navier-Stokes like system for $(\boldsymbol{u}, p)$ and a transport equation for extra stress tensor $\boldsymbol{\sigma}$.

3.2. Boundary conditions

The system of equations (3) has to be used with some boundary conditions. For a connected flow domain $\Omega \subset R^2$, the required boundary conditions are the following:

(i) Dirichlet boundary conditions for the velocity on the boundary $\partial\Omega$

$$\mathbf{u} = \mathbf{g} \text{ on } \partial\Omega$$

with compatibility condition

$$\int_{\partial\Omega} \mathbf{g}.\mathbf{n} = 0,$$

where $\boldsymbol{n}$ is the unit outward normal vector to Ω at the boundary $\partial\Omega$. For homogeneous case, $\boldsymbol{g} = 0$.

(ii) For the stress, a condition on the upstream boundary section

$$\partial\Omega^- = \{\mathbf{x} \in \partial\Omega : \mathbf{u}(\mathbf{x}).\mathbf{n}(\mathbf{x}) < 0\}$$

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_{\partial\Omega} \text{ on } \partial\Omega^-.$$

With the homogeneous Dirichlet boundary conditions, the Oldroyd-B fluid model problem is well-posed [20].

With the homogeneous Dirichlet boundary conditions defined over Ω, the problem of determining the extra stress tensor $\boldsymbol{\sigma}$, the velocity $\boldsymbol{u}$ and the pressure p satisfying the Oldroyd-B constitutive equations can be reformulated as follows:
Find the quantities $\boldsymbol{\sigma}$, $\mathbf{u}$ and p defined in Ω such that

$$\begin{cases} \rho(\mathbf{u}.\nabla)\mathbf{u} - \mu_n\, \Delta \mathbf{u} + \nabla p = \nabla.\boldsymbol{\sigma} + \rho\, \mathbf{f}, \text{in } \Omega, \\ \nabla.\mathbf{u} = 0, \text{in } \Omega, \\ \lambda_1(\mathbf{u}.\nabla)\boldsymbol{\sigma} + \boldsymbol{\sigma} = \mathbf{h}(\boldsymbol{\sigma}, \nabla \mathbf{u}), \text{in } \Omega, \\ \mathbf{u} = 0, \text{on } \partial\Omega \end{cases} \tag{6}$$

subject to the boundary condition (ii).

For non-dimensional case, the problem (6) can be read as the form of (5) as follows:

Find the non-dimensional quantities, still denoted by $\boldsymbol{\sigma}$, $\boldsymbol{u}$ and p, defined in defined in Ω such that

$$\begin{cases} Re[(\mathbf{u}.\nabla)\mathbf{u}] + \nabla p = (1-\lambda)\Delta\mathbf{u} + \nabla.\boldsymbol{\sigma} + \mathbf{f}, \text{in } \Omega, \\ \nabla.\mathbf{u} = 0 \text{ in } \Omega, \\ We[(\mathbf{u}.\nabla)\boldsymbol{\sigma}] + \boldsymbol{\sigma} = 2\lambda\,\mathbf{D}(\mathbf{u}) + We[(\nabla\mathbf{u})\boldsymbol{\sigma} + \boldsymbol{\sigma}(\nabla\mathbf{u})^t], \text{in } \Omega. \end{cases} \tag{7}$$

subject to the boundary conditions in (i) and (ii).

3.3. Variational formulation and discretization of the problem

In this work, we considered an incompressible viscoelastic fluid confined into a rectangular domain Ω with fixed boundary $\partial\Omega$. Mathematically, we had written the steady Oldroyd-B equations with the Dirichlet boundary conditions, i.e., $\boldsymbol{u} = \boldsymbol{u}_0$ such that $\boldsymbol{u}_0.\boldsymbol{n} = 0$ on $\partial\Omega$. So, given an external force field $\boldsymbol{f} \in \boldsymbol{H}^{-1}(\Omega)$ and $0 < \lambda < 1$ the viscoelastic fraction of the viscosity, the steady Oldroyd-B problem is defined by

$$\begin{cases} Re[(\mathbf{u}.\nabla)\mathbf{u}] + \nabla p = (1-\lambda)\Delta\mathbf{u} + \nabla.\boldsymbol{\sigma} + \mathbf{f}, \text{ in } \Omega, \\ \nabla.\mathbf{u} = 0, \text{in } \Omega, \\ We[(\mathbf{u}.\nabla)\boldsymbol{\sigma}] + \boldsymbol{\sigma} = 2\lambda\,\mathbf{D}(\mathbf{u}) + We[(\nabla\mathbf{u})\boldsymbol{\sigma} + \boldsymbol{\sigma}(\nabla\mathbf{u})^t], \text{ in } \Omega \\ \mathbf{u} = \mathbf{u}_0\,, \mathbf{u}_0.\mathbf{n} = 0, \text{ on } \partial\Omega \end{cases} \tag{8}$$

Considering $\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega)$, $q \in L_0^2(\Omega)$, and $\boldsymbol{\tau} \in \boldsymbol{L}_s^2(\Omega)$ as arbitrary test functions, and taking the scalar product between the momentum equation and $\boldsymbol{v}$, between the transport equation and $\boldsymbol{\tau}$, and multiplying the continuity equation by q and finally integrating all of them over Ω, we obtain the variational form to Oldroyd-B problem as [21],

Given $\boldsymbol{f} \in \boldsymbol{H}^{-1}(\Omega)$, find $(\boldsymbol{u}, p, \boldsymbol{\sigma}) \in \boldsymbol{H}_0^1(\Omega) \times L_0^2(\Omega) \times \boldsymbol{L}_s^2(\Omega)$ such that

$$\begin{cases} \int_\Omega Re[(\mathbf{u}.\nabla)\mathbf{u}.\mathbf{v}] + \int_\Omega (1-\lambda)\nabla\mathbf{u}:\nabla\mathbf{v} + \int_\Omega \sigma:\nabla\mathbf{v} - \int_\Omega p\nabla\cdot\mathbf{v} = +\int_\Omega \mathbf{f}.\mathbf{v} \\ \int_\Omega q\nabla.\mathbf{u} = 0 \\ \int_\Omega \boldsymbol{\sigma}:\boldsymbol{\tau} + \int_\Omega We[(\mathbf{u}.\nabla)\boldsymbol{\sigma}]:\boldsymbol{\tau} - \int_\Omega We[(\nabla\mathbf{u})\boldsymbol{\sigma} + \boldsymbol{\sigma}(\nabla\mathbf{u})^t]:\tau = \int_\Omega 2\lambda\,\mathbf{D}(\mathbf{u}):\boldsymbol{\tau} \end{cases} \tag{9}$$

for all $(\mathbf{v}, p, \boldsymbol{\tau}) \in \boldsymbol{H}_0^1(\Omega) \times L_0^2(\Omega) \times \boldsymbol{L}_s^2(\Omega)$

To approach the hyperbolic transport problem we consider the Discontinuous Galerkin Method and to approach the elliptic Navier-Stokes problem we consider the Hood-Taylor finite element method.

Let $\mathcal{T}_h, h > 0$, where, h is discretization parameter, be a non-degenerated regular triangulation of Ω such that $\overline{\Omega} = \bigcup_{K\in\mathcal{T}_h} K$.

Let $\mathbf{V}_h$ and Q_h be two finite-dimensional spaces for the velocity and the pressure, respectively, such that $\mathbf{V}_h \in \mathbf{H}^1(\Omega)$ and $Q_h \in L_0^2(\Omega)$. We defined the pair of

discrete space $\mathbf{V}_h^0 = \mathbf{V}_h \cap \mathbf{H}_0^1(\Omega)$ and $M_h = Q_h \cap L_0^2(\Omega)$ which correspond to Hood-Taylor finite element method. We also consider the space
$\mathbf{T}_h = \{\sigma_h \in \mathbf{T} \cap \mathbf{C}(\Omega) | \boldsymbol{\sigma}_{h|K} \in \mathbb{p}_1,\ \forall K \in \mathcal{T}_h \subset \mathrm{S}_1^{\mathrm{d}\times\mathrm{d}},$

$T = \{\ \boldsymbol{\sigma} \in\ L^2(\Omega) | \mathbf{u} \cdot \nabla \boldsymbol{\sigma} \in\ L^2(\Omega), \sigma_{12} = \sigma_{21}\}.$
In these spaces, the Oldroyd-B model is approached by the following problem:
Given $\boldsymbol{f} \in \boldsymbol{H}^{-1}(\Omega)$, find $(\boldsymbol{u}_h, p_h, \sigma_h) \in \boldsymbol{V}_h^0 \times M_h \times \boldsymbol{T}_h$ such that

$$\begin{cases} a(\mathbf{u}_h, \mathbf{v}_h) + c(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) = (\nabla . \boldsymbol{\sigma}_h +\ \mathbf{f}, \mathbf{v}_h)\ , \forall \mathbf{v}_h \in \mathbf{V}_h^0 \\ b(\mathbf{u}_h, q_h) = 0, \forall q_h \in M_h, \\ (\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + (\mathbf{u}_h \cdot \nabla . \boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + (\phi_{\partial K}^*(\boldsymbol{\sigma}_h), \boldsymbol{\tau}_h) = (\mathbf{h}(\nabla \mathbf{u}, \boldsymbol{\sigma}), \boldsymbol{\tau}_h), \quad \forall\, \boldsymbol{\tau}_h \in \mathbf{W} \end{cases} \tag{10}$$

with the interface and boundary adjoint-fluxes

$$\phi^{*,i}(\boldsymbol{\sigma}_h) | \partial K = \left(\alpha |\mathbf{u} \cdot \mathbf{n}_\mathrm{k}| - \tfrac{1}{2} \mathbf{u} \cdot \mathbf{n}_\mathrm{k}\right) \big[[\boldsymbol{\sigma}_h]\big] \partial K, \tag{11}$$

$$\phi^{*,\partial}(\boldsymbol{\sigma}_h) | \partial K = -|\mathbf{u} \cdot \mathbf{n}| \boldsymbol{\sigma}_h \chi_{\partial\Omega^-} \tag{12}$$

where $\alpha > 0$ is a parameter and $\chi_{\partial\Omega^-}$denotes the characteristic function of $\partial\Omega^-$.

The nondimensional approach problem can be written as follows:
Given $\boldsymbol{f} \in \boldsymbol{H}^{-1}(\Omega)$, find $(\boldsymbol{u}_h, p_h, \sigma_h) \in \boldsymbol{V}_h^0 \times M_h \times \boldsymbol{T}_h$ such that

$$\begin{cases} \int_\Omega Re[(\mathbf{u}_h . \nabla)\mathbf{u}_h . \mathbf{v}_h] + \int_\Omega (1-\lambda) \nabla \mathbf{u}_h : \nabla \mathbf{v}_h + \int_\Omega \sigma_h : \nabla \mathbf{v}_h - \int_\Omega p_h \nabla . \mathbf{v}_h = \int_\Omega \mathbf{f} . \mathbf{v}_h \\ \int_\Omega q_h \nabla . \mathbf{u}_h = 0, \\ \int_\Omega \sigma_h : \tau_h + \int_\Omega We[(\mathbf{u}_h . \nabla)\sigma_h] : \tau_h - \int_\Omega We[(\nabla \mathbf{u}_h)\sigma_h\ + \sigma_h (\nabla \mathbf{u}_h)^\mathrm{t}] : \tau_h + (\phi_{\partial K}^*(\boldsymbol{\sigma}_h), \boldsymbol{\tau}_h) = \int_\Omega 2\lambda\, \mathbf{D}(\mathbf{u}_h) : \tau_h, \end{cases} \tag{13}$$

$\forall \mathbf{v}_h \in \mathbf{V}_h^0, \forall q_h \in M_h, \forall\, \tau_h \in \mathbf{T}_h$.

3.4. Algorithm to solve the discrete Oldroyd-B problem

To solve this elliptic-hyperbolic system, we applied the decoupled technique. The extra-stress tensor is computed separately from the kinematic equations. From a fixed value for the velocity (and pressure) the extra-stress tensor is evaluated solving the tensorial transport equation with fixed-point method. Then the velocity field and pressure are updated with the current extra-stress tensor whose components are treated as known body forces, solving the resulting Navier-Stokes equation by the Newton-Raphson method. This procedure is iterated.

- Given $(\mathbf{u}_h^n, p_h^n, \boldsymbol{\sigma}_h^n)$ the approach solution of iteration n, find $(\mathbf{u}_h^{n+1}, p_h^{n+1})$, the solution of

$$\int_\Omega Re\big[(\mathbf{u}_h^{n+1} . \nabla)\mathbf{u}_h^{n+1} . \mathbf{v}_h\big] + \int_\Omega (1-\lambda) \nabla \mathbf{u}_h^{n+1} : \nabla \mathbf{v}_h - \int_\Omega p_h^{n+1} \nabla . \mathbf{v}_h = \int_\Omega \boldsymbol{\sigma}_h^n : \nabla \mathbf{v}_h + \int_\Omega \mathbf{f} . \mathbf{v}_h$$

- Given $\boldsymbol{u}_h^{n+1}$ and $\boldsymbol{\sigma}_h^n = \boldsymbol{\sigma}_h^{n_0}$, find the solution $\boldsymbol{\sigma}^* = \boldsymbol{\sigma}_h^{n_k+1}$ of

$$\int_{\Omega} \boldsymbol{\sigma}_h^{n_{k+1}} : \tau_h + We \int_{\Omega} [(\mathbf{u}_h^{n+1}.\nabla)\boldsymbol{\sigma}_h^{n_{k+1}}] : \tau_h + (\phi_{\partial K}^{*}(\boldsymbol{\sigma}_h^{n_{k+1}}), \tau_h)$$
$$= \int_{\Omega} We[\boldsymbol{\sigma}_h^{n_k}(\nabla \mathbf{u}_h^{k+1}) + \nabla \mathbf{u}_h^{k+1} \boldsymbol{\sigma}_h^{n_k}] : \tau_h + \int_{\Omega} 2\lambda\, \mathbf{D}(\mathbf{u}_h^{k+1}) : \tau_h \,, k \geq 0$$

4. Numerical Results and Discussions

This section is concerned with the application of the finite element method to obtain the numerical results for non-Newtonian viscoelastic Oldroyds-B fluid flows. By the implementation of the finite element method in our own script in FreeFem++, we obtained the numerical solutions of the Oldroyd-B problem.

Here, we considered the Oldroyd-B flow between two rigid walls where the flow is driven by a pressure difference along x-direction. This flow is laminar and referred as Poiseuille flows.

The velocity is uniaxial and has a parabolic profile and we supposed that $\frac{\partial p}{\partial x} = 1$. The analytic solution for the kinematic is given by [22],

$$u_1(x,y) = y\,(1-y) \text{ (means } \mu = 0.125\ Pa\ s)$$
$$u_2(x,y) = 0$$
$$p(x,y) = x + C \text{ (} C \text{ is a constant)}$$

Substituting the velocity in the transport equations, we obtain by simple calculations, the components of the tensor as the functions of u_1 which can be written as

$$\sigma_{11} = 2\lambda\, We \left(\frac{\partial u_1}{\partial y}\right)^2$$
$$\sigma_{12} = \lambda \frac{\partial u_1}{\partial y}$$
$$\sigma_{22} = 0 \qquad (14)$$

We considered the fluid is confined into a domain $\Omega = [0,10] \times [0,1]$. The no-slip conditions on the two rigid walls are given by $u_1 = 0, u_2 = 0$. We assumeed that $u_2 = 0$ everywhere at the inlet and $u_1(x,y) = y\,(1-y)$ as the exact solution. At outlet we imposed $u_2 = 0$. The condition for stress tensor on the upstream boundary section $\partial\Omega^-$ agrees with the exact solution (14).

The problem has been solved using four grids obtained by successive refinements dividing each triangle into four new triangles starting with a coarse mesh with 344 elements (Fig. 1).

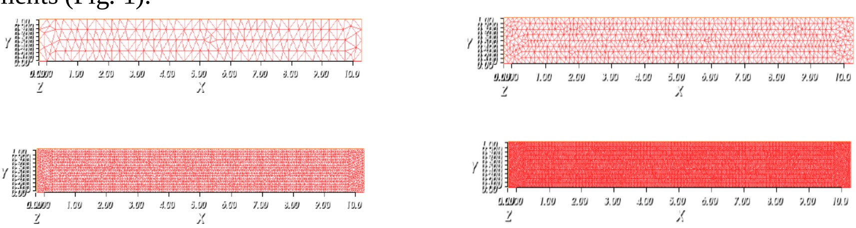

Fig. 1. Different meshes used. From the left to the right and top to bottom: mesh with 344 elements, 1374 elements, 5410 elements, 22654 elements, respectively.

We consider the problem with $Re = 1$, $We = 1$ and $\lambda = 0.1$.

The Table 1 characterizes the mesh through the diameter $\boldsymbol{h}$, number of elements, degree of freedoms.

Table 1. Characterizations of the grids.

Grid	h	No. of elements	$\mathbb{p}_1$ nodes	$\mathbb{p}_2$ nodes	$\mathbb{p}_1$ dc nodes
Grid 1	0.372678	344	777	217	1032
Grid 2	0.18815	1374	2925	776	4122
Grid 3	0.10233	5410	11173	2882	16230
Grid 4	0.0511443	22654	46013	11680	67962

In each case, we evaluated the error of fluid velocity in $\boldsymbol{H}^1$-norm and the error of the components of the tensor in L^2-norm which are respectively defined by

$$err_u = \|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{H}^1(\Omega)}$$

and

$$err_{\sigma_{ij}} = \left\|\sigma_{ij} - \sigma_{h,ij}\right\|_{L^2(\Omega)}, i,j = 1, 2.$$

The results obtained for $\boldsymbol{u}$ and $\sigma_{ij}, i,j = 1, 2$ over the different meshes are present in the following table (Table 2).

Table 2. Error of the velocity field and tensor components.

Error	Grid 1	Grid 2	Grid 3	Grid 4	Slope of the log-log plot
err_u	0.0016779	0.00065528	0.00034032	0.00017678	1.1279
$err_{\sigma_{11}}$	0.025702	0.00615	0.0015003	0.0003415	2.18775
$err_{\sigma_{12}}$	7.8211×10^{-5}	3.4432×10^{-5}	1.7488×10^{-5}	8.5971×10^{-6}	1.1117
$err_{\sigma_{22}}$	4.7276×10^{-5}	2.7917×10^{-5}	1.0495×10^{-5}	5.3389×10^{-6}	1.142

The good convergence of results for all kinematic can be confirmed by the slope value, which gives us the rate of convergence to the exact solution with respect to the corresponding norms. We used the least squares approximation to find the slope of the log-log plot of the error of the velocity and the tensor components. The following plots (Figs. 2-5) show the error curves:

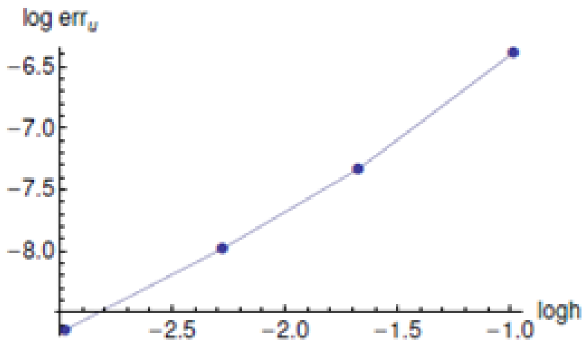

Fig. 2. Log-log plot of the error of the velocity.

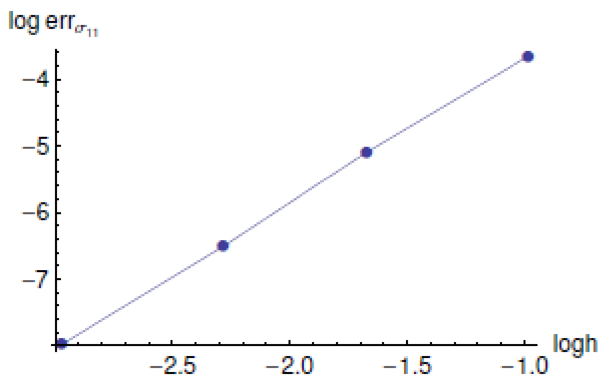

Fig. 3. Log-log plot of the error of the component σ_{11} of the tensor.

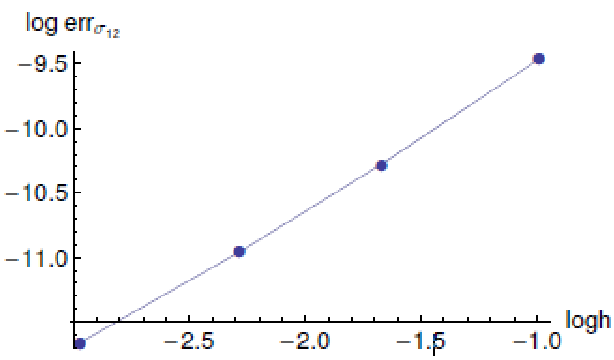

Fig. 4. Log-log plot of *the error of* the component σ_{12} of the tensor.

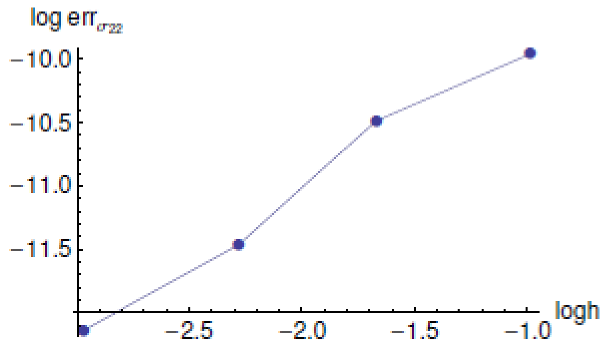

Fig. 5. Log-log plot of the error of the component σ_{22} of the tensor.

The values for the rate of convergence (the slope) which we obtained guarantees the errors for all the variables approaches to zero as h tends to zero, which we expected theoretically. We are in conditions to affirm that the numerical solution converges to the exact solution. So, the algorithm is convergent, and we observed that the behavior of exact and numerical solutions is approximately same.

The approach solution obtained with 22654 elements is illustrated graphically (with color scales) in Figs. 6-11.

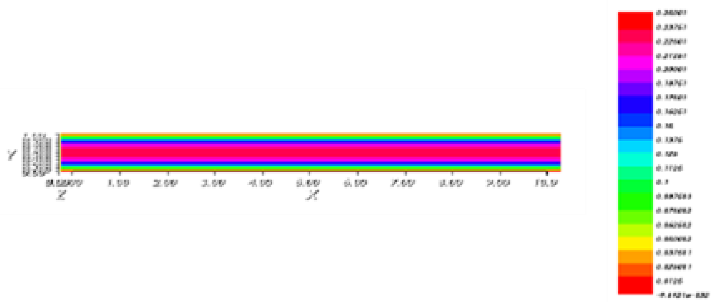

Fig. 6. Contours of the first component of the velocity.

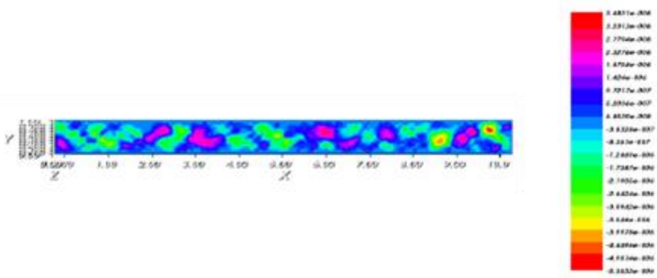

Fig. 7. Contours of the second component of the velocity.

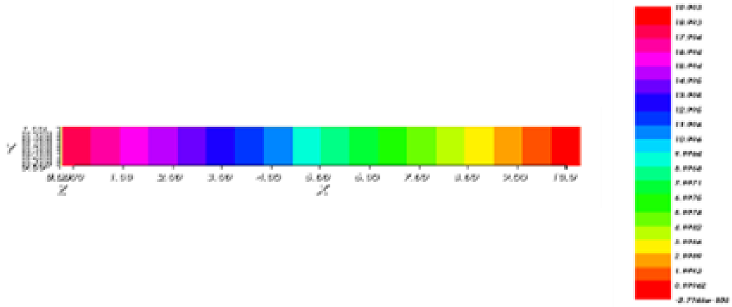

Fig. 8. Contours of the pressure.

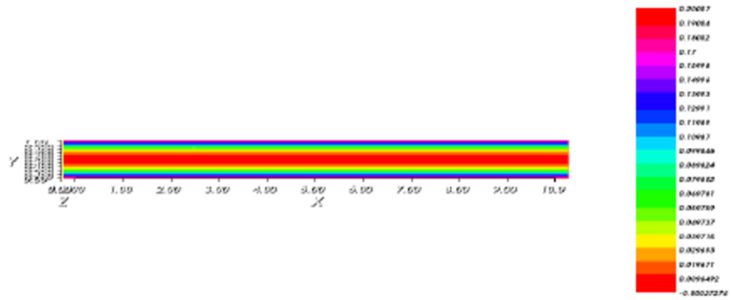

Fig. 9. Contours of the first component of the tensor.

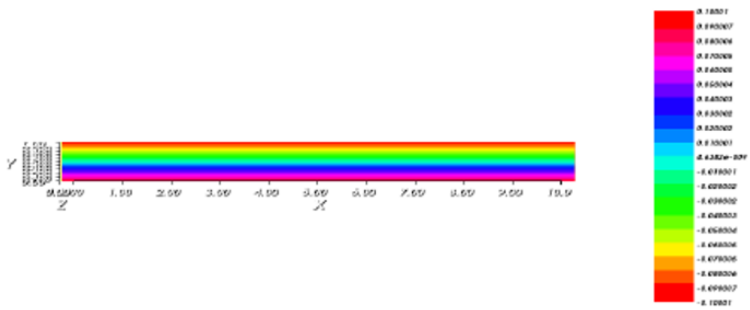

Fig. 10. Contours of the second component of the tensor.

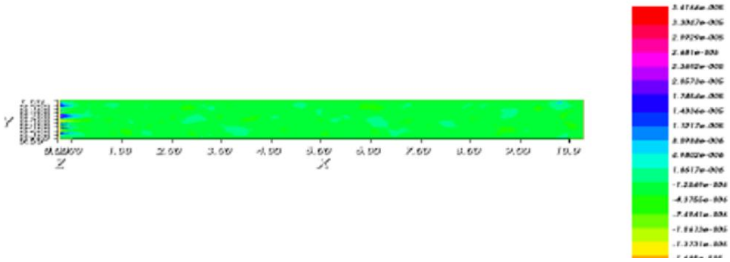

Fig. 11. Contours of the third component of the tensor.

So, from all the above numerical and graphical results, we observed that the approximate solution converges to the exact solution with respect to the corresponding norms.

5. Conclusion

The main goal of this work was the mathematical and numerical study of the non-linear system of partial differential equations that model the motion of incompressible non-Newtonian fluids of Oldroyd-B type, in dimension 2, in case of steady flow over a rectangular domain. The numerical simulations to the Oldroyd-B problem were obtained computationally by the implementation of the finite element methods (continuous for kinematic and discontinuous for the extra stress tensor) in a script of FreeFem++. This mixed problem of elliptic-hyperbolic type was decoupled into two auxiliary problems, namely, the Navier-Stokes system and the tensorial transport problem. Based on the numerical techniques described for both the auxiliary problems, the approximation of the solution of the Oldroyd-B problem was obtained.

Here, the numerical results have been obtained by considering the benchmark problem over a rectangular domain. We have observed the solutions both graphically and computationally, and also have observed the behavior of the solutions in the rectangular domain. We have found a good agreement between the approximate and exact solutions which confirms the convergency of the algorithm suggesting the capability of providing the better numerical approximations of Oldroyd-B model using FreeFem++.

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