

## **Compound Difference Anti-Synchronization between Hyper-Chaotic Systems of Fractional Order**

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### **Abstract**

In this article, the compound difference anti-synchronization between fractional order hyper-chaotic systems have been studied. Numerical simulations have been performed using MATLAB to verify the theoretical results on fractional order Xling, Vanderpol, Rikitake and Rabinovich hyper-chaotic systems.

*Keywords:* Compound difference anti-synchronization; Xling system; Vanderpol system; Rikitake system; Rabinovich system.

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### **1. Introduction**

CHAOS theory has been gaining popularity ever since the well-known Lorenz system was discovered. From then on, there has been no looking back in the growth and development of chaos theory. Chaos synchronization [1] was introduced by Pecora and Carroll in 1990. In synchronization two chaotic systems arising from different initial conditions is made stable by designing controllers. Where synchronizing [2-5] two chaotic systems is considered difficult, synchronizing more than two hyper-chaotic systems is in itself a big challenge.

Though fractional calculus is not new to mathematics, it has recently emerged most useful in modelling of processes and systems where integer order could not serve purpose. Motivated by the above discussions hyper-chaotic systems have been synchronized here. Numerical Simulations have been performed using MATLAB which verify the theoretical results.

### **2. Problem Formulation**

Consider three hyper-chaotic fractional order master systems and one chaotic fractional order slave system we formulate the compound difference anti-synchronization scheme. Let the scaling master system be:

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$$D^q x = f_1(x) \tag{1}$$

Let the base master systems be:

$$D^q y = f_2(y) \tag{2}$$

$$D^q z = f_3(z) \tag{3}$$

Let the slave system be:

$$D^q w = f_4(w) + v \tag{4}$$

Where  $x = (x_1, x_2, \dots, x_n)$ ,  $y = (y_1, y_2, \dots, y_n)$ ,  $z = (z_1, z_2, \dots, z_n)$  and  $w = (w_1, w_2, \dots, w_n)$  are state vectors of the respective systems,  $f_i, i = 1,2,3,4$  are continuous functions,  $0 < q < 1$  and  $v = (v_1, v_2, \dots, v_n)$ , is controller to be designed.

Defining the error as:

$$e = Aw + Bx(Cz - Dy)$$

where  $A = \text{diag}(\alpha_1, \alpha_2, \dots, \alpha_n)$ ,  $B = \text{diag}(\beta_1, \beta_2, \dots, \beta_n)$ ,  $C = \text{diag}(\gamma_1, \gamma_2, \dots, \gamma_n)$ ,  $D = \text{diag}(\delta_1, \delta_2, \dots, \delta_n)$  are suitably chosen diagonal matrices with  $A \neq 0$ .

To achieve the desired anti-synchronization we must have error tending to zero, i.e.

$$\text{Lim } ||e|| = 0 \text{ as } t \rightarrow \infty$$

i.e.,  $\text{Lim} || Aw + Bx(Cz - Dy) || = 0 \text{ as } t \rightarrow \infty$

Where  $|| \cdot ||$  represents the Euclidean norm.

We here define the controllers as:

$$v_i = \frac{\theta_i}{\alpha_i} - f_{4i} - \frac{K_i e_i}{\alpha_i} \tag{5}$$

Where  $\theta_i = \beta_i f_{1i}(\gamma_i z_i - \delta_i y_i) + \beta_i x_i(\gamma_i f_{3i} - \delta_i f_{4i})$  for  $i = 1,2, \dots, n$

**Theorem:** Systems (1)-(3) will be in compound difference anti-synchronization with (4) if the controllers are designed as in (5).

**Proof:** We define the compound difference anti-synchronization error as:

$$e_i = \alpha_i w_i + \beta_i x_i(\gamma_i z_i - \delta_i y_i) \tag{6}$$

Differentiating (6) we get the error dynamical system as:

$$D^q e_i = \alpha_i D^q w_i + \beta_i D^q x_i(\gamma_i z_i - \delta_i y_i) + \beta_i x_i(\gamma_i D^q z_i - \delta_i D^q y_i) \tag{7}$$

Substituting the values of the derivatives and applying the designed controller, the error dynamical system simplifies to:

$$D^q e_i = -K_i e_i \tag{8}$$

Next, we consider the Lyapunov function as:

$$V(e(t)) = \frac{1}{2}(e_1^2 + e_2^2 + e_3^2 + e_4^2) \tag{9}$$

Differentiating we get

$$D^q (V(e(t))) \leq (e_1 D^q e_1 + e_2 D^q e_2 + e_3 D^q e_3 + e_4 D^q e_4) \tag{10}$$

Substituting (8) into (10), we get

$$\begin{aligned} D^q (V(e(t))) &\leq (e_1(-K_1 e_1) + e_2(-K_2 e_2) + e_3(-K_3 e_3) + e_4(-K_4 e_4)) \\ &= -K_1 e_1^2 - K_2 e_2^2 - K_3 e_3^2 - K_4 e_4^2 \end{aligned} \tag{11}$$

i.e.  $V(e(t))$  is positive definite function with a negative definite derivative. Hence, by Lyapunov Stability Theory we have that error tends to zero, implying desired anti-synchronization has been achieved.

**Note:** We have taken Caputo’s version of fractional derivative in our paper.

### 3. System Description

#### 3.1. Scaling master system

We consider the fractional order hyper-chaotic Xling system as the scaling master system given by:

*Fractional Order Hyper-Chaotic Xling System*

$$\begin{aligned} \frac{d^q x_1}{dt^q} &= a_1(x_2 - x_1) + x_4 \\ \frac{d^q x_2}{dt^q} &= a_2 x_1 + x_1 x_3 - x_4 \\ \frac{d^q x_3}{dt^q} &= -a_3 x_3 - a_4 x_1^2 \\ \frac{d^q x_4}{dt^q} &= a_3 x_1 \end{aligned} \tag{12}$$

Here  $x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$  are state variables and  $a_1, a_2, a_3, a_4 \in \mathbb{R}$  are parameters. For parameter values  $a_1 = 10, a_2 = 40, a_3 = 2.5, a_4 = 4$  and initial conditions of state variables as (1,2,3,4) the system shows chaotic behavior as displayed in Fig. 1 (a).

#### 3.2. Base master systems

Next we consider the hyperchaotic fractional order Rabinovich and Rikitake chaotic systems.

*Fractional Order Hyper-Chaotic Rabinovich System*

$$\begin{aligned} \frac{d^q y_1}{dt^q} &= -c_1 y_1 + c_2 y_2 + y_2 y_3 \\ \frac{d^q y_2}{dt^q} &= c_2 y_1 - y_2 + y_4 - y_1 y_3 \\ \frac{d^q y_3}{dt^q} &= -y_3 + y_1 y_2 \\ \frac{d^q y_4}{dt^q} &= -c_3 y_2 \end{aligned} \tag{13}$$

Here  $y = (y_1, y_2, y_3, y_4) \in \mathbb{R}^4$  are state variables and  $c_1, c_2, c_3 \in \mathbb{R}$  are parameters. For parameter values  $c_1 = 34, c_2 = 6.75, c_3 = 2$  and initial conditions of state variables as (5.5,-1.25,8.4,2.75) the system shows chaotic behavior as displayed in Fig. 1 (b).

*Fractional Order Hyper-Chaotic Rikitake System*

$$\begin{aligned} \frac{d^q z_1}{dt^q} &= -z_1 + z_2 z_3 - d_1 z_4 \\ \frac{d^q z_2}{dt^q} &= -z_2 + z_1(z_3 - d_2) - d_1 z_4 \\ \frac{d^q z_3}{dt^q} &= d_2 - z_1 z_2 \\ \frac{d^q z_4}{dt^q} &= d_3 z_2 \end{aligned} \tag{14}$$

Here  $z = (z_1, z_2, z_3, z_4) \in \mathbb{R}^4$  are state variables and  $d_1, d_2, d_3 \in \mathbb{R}$  are parameters. For parameter values  $d_1 = 1.7, d_2 = 1, d_3 = 0.7$  and initial conditions of state variables as  $(3.5, 1.7, -4.5, 2.8)$  the system shows chaotic behavior as displayed in Fig. 1 (c).

### 3.3. Slave system

We consider the slave system as the hyperchaotic fractional order Vanderpol system. *Fractional Order Hyper-Chaotic Vanderpol System*

$$\begin{aligned}
 \frac{d^q w_1}{dt^q} &= w_2 \\
 \frac{d^q w_2}{dt^q} &= -(b_1 + b_2 w_3)w_1 - (b_1 + b_2 w_3)w_1^3 - b_3 w_2 + b_4 w_3 \\
 \frac{d^q w_3}{dt^q} &= w_4 \\
 \frac{d^q w_4}{dt^q} &= -w_3 + b_5 w_1 + b_6 w_4(1 - w_3^2)
 \end{aligned} \tag{15}$$

Here  $w = (w_1, w_2, w_3, w_4) \in \mathbb{R}^4$  are state variables and  $b_1, b_2, b_3, b_4, b_5, b_6 \in \mathbb{R}$  are parameters.

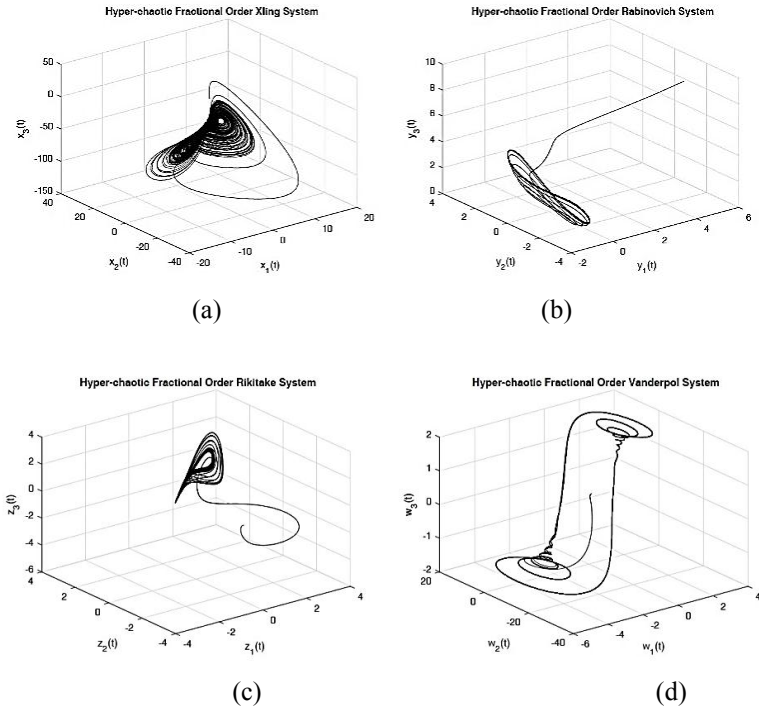


Fig. 1. Phase portraits of the scaling master system, base master system I, base master system -II, slave system in (a)  $x_1 - x_2 - x_3$  space, (b)  $y_1 - y_2 - y_3$  space, (c)  $z_1 - z_2 - z_3$  space and (d)  $w_1 - w_2 - w_3$  space

For parameter values  $b_1 = 10, b_2 = 3, b_3 = 0.4, b_4 = 70, b_5 = 5, b_6 = 0.1$  and initial conditions of state variables as  $(0.1, -0.5, 0.1, -0.5)$  the system shows chaotic behavior as displayed in Fig. 1 (d).

#### 4. Compound Difference Anti-Synchronization between Hyperchaoticsystems of Fractional Orders

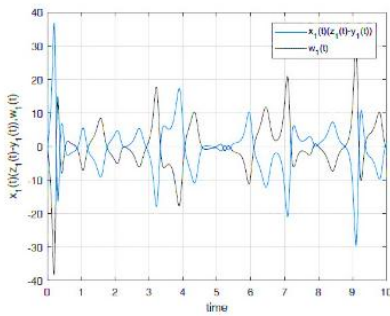
##### 4.1. Numerical simulations and discussions

Corresponding to master system (1)-(3) and slave system (4), the slave system with control functions is given as:

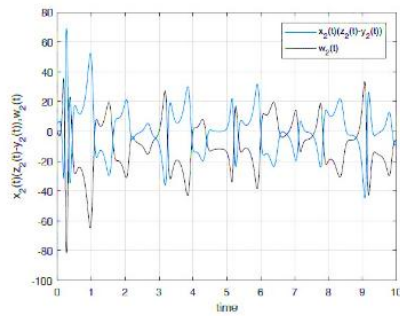
$$\begin{aligned} \frac{d^q w_1}{dt^q} &= w_2 + v_1 \\ \frac{d^q w_2}{dt^q} &= -(b_1 + b_2 w_3)w_1 - (b_1 + b_2 w_3)w_1^3 - b_3 w_2 + b_4 w_3 + v_2 \\ \frac{d^q w_3}{dt^q} &= w_4 + v_3 \\ \frac{d^q w_4}{dt^q} &= -w_3 + b_6 w_1 + b_5 w_4(1 - w_3^2) + v_4 \end{aligned} \tag{16}$$

We define the error given by (6) as:

$$\begin{aligned} e_1 &= \alpha_1 w_1 + \beta_1 x_1(\gamma_1 z_1 - \delta_1 y_1) \\ e_2 &= \alpha_2 w_2 + \beta_2 x_2(\gamma_2 z_2 - \delta_2 y_2) \\ e_3 &= \alpha_3 w_3 + \beta_3 x_3(\gamma_3 z_3 - \delta_3 y_3) \\ e_4 &= \alpha_4 w_4 + \beta_4 x_4(\gamma_4 z_4 - \delta_4 y_4) \end{aligned} \tag{17}$$



(a)



(b)

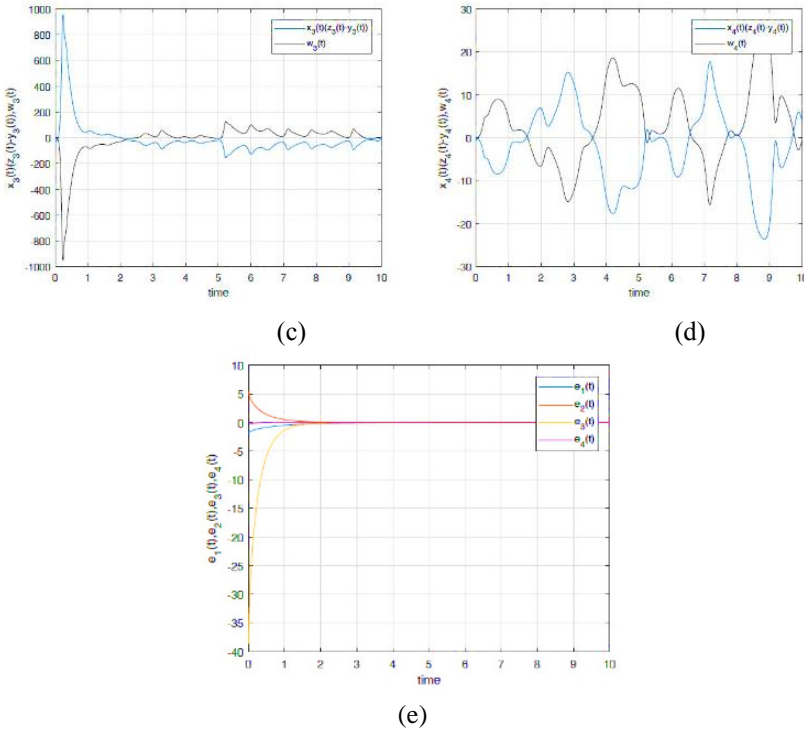


Fig. 2. Anti-synchronized trajectories of compound of master systems with slave system and error plot.

The error dynamical system is given by:

$$\begin{aligned}
 \frac{d^q w_1}{dt^q} &= \alpha_1 \frac{d^q w_1}{dt^q} + \beta_1 \frac{d^q x_1}{dt^q} (\gamma_1 z_1 - \delta_1 y_1) + \beta_1 x_1 (\gamma_1 \frac{d^q z_1}{dt^q} - \delta_1 \frac{d^q y_1}{dt^q}) \\
 \frac{d^q w_2}{dt^q} &= \alpha_2 \frac{d^q w_2}{dt^q} + \beta_2 \frac{d^q x_2}{dt^q} (\gamma_2 z_2 - \delta_2 y_2) + \beta_2 x_2 (\gamma_2 \frac{d^q z_2}{dt^q} - \delta_2 \frac{d^q y_2}{dt^q}) \\
 \frac{d^q w_3}{dt^q} &= \alpha_3 \frac{d^q w_3}{dt^q} + \beta_3 \frac{d^q x_3}{dt^q} (\gamma_3 z_3 - \delta_3 y_3) + \beta_3 x_3 (\gamma_3 \frac{d^q z_3}{dt^q} - \delta_3 \frac{d^q y_3}{dt^q}) \\
 \frac{d^q w_4}{dt^q} &= \alpha_4 \frac{d^q w_4}{dt^q} + \beta_4 \frac{d^q x_4}{dt^q} (\gamma_4 z_4 - \delta_4 y_4) + \beta_4 x_4 (\gamma_4 \frac{d^q z_4}{dt^q} - \delta_4 \frac{d^q y_4}{dt^q})
 \end{aligned} \tag{18}$$

Substituting values of the derivatives from (12)-(15) and designing controllers as:

$$v_1 = \frac{\Theta_1}{\alpha_1} - f_{41} - \frac{K_1 e_1}{\alpha_1}$$

Where  $\Theta_1 = \beta_1 f_{11} (\gamma_1 z_1 - \delta_1 y_1) + \beta_1 x_1 (\gamma_1 f_{31} - \delta_1 f_{41})$

$$v_2 = \frac{\Theta_2}{\alpha_2} - f_{42} - \frac{K_2 e_2}{\alpha_2}$$

Where  $\Theta_2 = \beta_2 f_{12} (\gamma_2 z_2 - \delta_2 y_2) + \beta_2 x_2 (\gamma_2 f_{32} - \delta_2 f_{42})$  (19)

$$v_3 = \frac{\Theta_3}{\alpha_3} - f_{43} - \frac{K_3 e_3}{\alpha_3}$$

Where  $\Theta_3 = \beta_3 f_{13} (\gamma_3 z_3 - \delta_3 y_3) + \beta_3 x_3 (\gamma_3 f_{33} - \delta_3 f_{43})$

$$v_4 = \frac{\Theta_4}{\alpha_4} - f_{44} - \frac{K_4 e_4}{\alpha_4}$$

Where  $\Theta_4 = \beta_4 f_{14}(\gamma_4 z_4 - \delta_4 y_4) + \beta_4 x_4(\gamma_4 f_{34} - \delta_4 f_{44})$

The error dynamical system simplifies to:

$$\begin{aligned} \frac{d^q e_1}{dt^q} &= -K_1 e_1 \\ \frac{d^q e_2}{dt^q} &= -K_2 e_2 \\ \frac{d^q e_3}{dt^q} &= -K_3 e_3 \\ \frac{d^q e_4}{dt^q} &= -K_4 e_4 \end{aligned} \tag{20}$$

Next, we consider the lyapunov function as:

$$V(e(t)) = \frac{1}{2} (e_1^2 + e_2^2 + e_3^2 + e_4^2)$$

Differentiating we get

$$\begin{aligned} D^q (V(e(t))) &\leq (e_1 D^q e_1 + e_2 D^q e_2 + e_3 D^q e_3 + e_4 D^q e_4) \\ &= -K_1 e_1^2 - K_2 e_2^2 - K_3 e_3^2 - K_4 e_4^2 \\ &< 0 \end{aligned}$$

i.e.  $V(e(t))$  is a positive definite function with a negative definite derivative. Hence, by Lyapunov Stability Theory we have that error tends to zero, implying desired anti-synchronization has been achieved.

### 5. Conclusion

In this paper four hyper-chaotic fractional order systems have been synchronized in compound difference anti-synchronization manner by designing suitable controllers. This technique will find application in secure communication, control systems etc.

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### References

1. L. M. Pecora and T. L. Carroll, Phys. Rev. Lett. **64**, 821 (1990). <https://link.aps.org/doi/10.1103/PhysRevLett.64.821>
2. A. Khan and P. Trikha, Ind. J. Indust. Appl. Math. **8**, 118 (2017). <https://doi.org/10.5958/1945-919X.2017.00010.X>
3. A. Khan and P. Trikha, SN Appl. Sci. **1**, 757 (2019). <https://doi.org/10.1007/s42452-019-0776-x>
4. J. Sun, Y. Shen, and G. Cui, Adv. Math. Phys. **2015**, ID 921515 (2015). <https://doi.org/10.1155/2015/921515>
5. A. Khan, D. Khattar, and N. Prajapati, Pramana **89**, 90 (2017). <https://doi.org/10.1007/s12043-017-1488-7>