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Some Theorems for Generalized (U, M)-Derivations in Semiprime Γ-Rings

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Abstract

The objective of this paper is to establish some results for generalized (U, M)-derivations in semiprime Γ-rings, where U is a Lie ideal of a semiprime Γ-ring M. Let d be a (U,M)-derivation and f be a generalized (U,M)-derivation on M then we proved that

- $f(u\alpha v) = f(u)\alpha v + u\alpha d(v)$ for all $u,v \in U$ and $\alpha \in \Gamma$, when U is an admissible Lie ideal of M;
- $f(u\alpha m) = f(u)\alpha m + u\alpha d(m)$ for all $u \in U, m \in M$ and $\alpha \in \Gamma$, when U is a square closed Lie ideal of M.

Keywords: Semiprime Γ-ring; Lie ideal; Square closed Lie ideal; Admissible Lie ideal; (U,M)-derivation; Generalized (U,M)-derivation.

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1. Introduction

(U,R)-derivations in rings have been introduced by A. K. Faraj, C. Haetinger and A. H. Majeed [1] as a generalization of Jordan derivations on a Lie ideal of a ring. We introduced (U,M)-derivations in Γ-rings as a generalization of Jordan derivations on Lie ideals of a Γ-ring in [2] and proved that, $d(u\alpha v) = d(u)\alpha v + u\alpha d(v)$ for all $u,v \in U, \alpha \in \Gamma$, where U is an admissible Lie ideal of M and d is a (U,M)-derivation of M. We also proved that, if $u\alpha u \in U$ for all $u \in U$ and $\alpha \in \Gamma$ then $d(u\alpha m) = d(u)\alpha m + u\alpha d(m)$ for all $u \in U, m \in M$ and $\alpha \in \Gamma$. Following the notion of (U,M)-derivations we then introduced the concept of generalized (U,M)-derivations in [3] and proved the analogous results considering generalized (U,M)-derivations of prime Γ-rings corresponding to the results of (U,M)-derivations. We

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refer the reader to R. Awtar [4], M. Ashraf and N. U. Rehman [5], W. E. Baarnes [6], Y. Ceven [7], I. N. Herstein [8], and A. K. Halder and A. C. Paul [9] where we can find further references and more detailed explanations concerning the motivations and the background of these researches. The notion of a Γ-ring has been developed by N. Nobusawa [10], as a generalization of a ring. Following W. E. Barnes [11] generalized the concept of Nobusawa's Γ-ring as a more general nature in the following way.

Let M and Γ be additive abelian groups. If there is a mapping $M \times \Gamma \times M \to M$ (sending (x, α, y) into $x\alpha y)$ such that

(i) $(x+y)\alpha z = x\alpha z + y\alpha z, x(\alpha+\beta)y = x\alpha y + x\beta y, x\alpha(y+z) = x\alpha y + x\alpha z$,

(ii) $(x\alpha y)\beta z = x\alpha(y\beta z)$ for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$,

then M is called a Γ-ring. A Γ-ring M is semiprime if $a\Gamma M\Gamma a = 0$ (with $a \in M$) implies $a = 0$. We denote the commutator $u\alpha v - v\alpha u$ by $[u,v]_\alpha$ for all $u, v \in M$ and $\alpha \in \Gamma$. An additive subgroup U of a Γ-ring M is a Lie ideal of M if for all $u \in U, m \in M$ and $\alpha \in \Gamma$, implies $[u,m]_\alpha \in U$. A Lie ideal U is a square closed Lie ideal of a Γ-ring M if $u\alpha u \in U$, for all $u \in U, \alpha \in \Gamma$ and if the Lie ideal U is square closed and $U \not\subseteq Z(M)$, where $Z(M)$ denotes the center of M then U is an admissible Lie ideal of M. In this article, we generalize some results of [3] for square closed and admissible Lie ideal of semiprime Γ-rings by the new concept of (U,M)-derivation.

2. Generalized (U,M)-Derivations in Semiprime Γ-Rings

Following the notions of (U,M)-derivation of a Γ-ring in [9], we then introduced the concepts of generalized (U,M)-derivations of Γ-rings in [3] in the following way.

Definition 1. Let U be a Lie ideal of a Γ-ring M. An additive mapping $f: M \to M$ is a generalized (U,M)-derivation of M if there exists a (U,M)-derivation d of M such that $f(u\alpha m + s\alpha u) = f(u)\alpha m + u\alpha d(m) + f(s)\alpha u + s\alpha d(u)$ is satisfied for all $u \in U; m, s \in M$ and $\alpha \in \Gamma$.

The following are examples of (U,M)-derivation and generalized (U,M)-derivation of a Γ-ring *M*.

Example 1. Let R be an associative ring with 1, and let U be a Lie ideal of R. Let $M = M_{1,2}(R)$ and $\Gamma = \left\{ \begin{pmatrix} n.1 \\ 0 \end{pmatrix} : n \in Z \right\}$, then M is a Γ-ring. If $N = \{(x,x) : x \in R\} \subseteq M$, then N is a sub Γ-ring of M. Let $U_1 = \{(u,u) : u \in U\}$, then U_1 is a Lie ideal of N. If $f: R \to R$ is a generalized (U,R)-derivation, then there exists a (U,R)-derivation $d: R \to R$ such that

$f(ux + su) = f(u)x + ud(x) + f(s)u + sd(u)$ for all $u \in U$ and $x, s \in R$. If we define a mapping $D: N \to N$ by $D((x,x)) = (d(x), d(x))$, then we have

$$D((u,u)\begin{pmatrix} n \\ 0 \end{pmatrix}(x,x)+(y,y)\begin{pmatrix} n \\ 0 \end{pmatrix}(u,u)) = D((unx,unx)+(ynu,ynu))$$
$$= D((unx+ynu,unx+ynu))$$
$$= (d(unx+ynu),d(unx+ynu)).$$

After calculation , we get

$D(u_1\alpha x_1 + y_1\alpha u_1) = D(u_1)\alpha x_1 + u_1\alpha D(x_1) + D(y_1)\alpha u_1 + y_1\alpha D(u_1),$

where $u_1 = (u,u), \alpha = \begin{pmatrix} n \\ 0 \end{pmatrix}, x_1 = (x,x)$ and $y_1 = (y,y)$. Hence D is a $(U_1,N)-$ derivation on N. Let $F : N \to N$ be the additive mapping defined by $F((x,x)) = (f(x), f(x))$, then considering $u_1 = (u,u) \in U_1, \alpha = \begin{pmatrix} n \\ 0 \end{pmatrix} \in \Gamma$ and $x_1 = (x,x), y_1 = (y,y) \in N,$ we have

$$F(u_1\alpha x_1 + y_1\alpha u_1) = F((unx+ynu,unx+ynu))$$
$$= (f(unx+ynu), f(unx+ynu))$$
$$= (f(u)nx+und(x)+f(y)nu+ynd(u), f(u)nx+und(x)+f(y)nu+ynd(u))$$
$$= (f(u)nx+und(x), f(u)nx+und(x)) + (f(y)nu+ynd(u), f(y)nu+ynd(u))$$
$$= (f(u)nx, f(u)nx) + (und(x),und(x)) + (f(y)nu, f(y)nu) + (ynd(u), ynd(u))$$
$$= (f(u), f(u))\begin{pmatrix} n \\ 0 \end{pmatrix}(x,x) + (u,u)\begin{pmatrix} n \\ 0 \end{pmatrix}(d(x),d(x)) + (f(y), f(y))\begin{pmatrix} n \\ 0 \end{pmatrix}(u,u)$$
$$+ (y,y)\begin{pmatrix} n \\ 0 \end{pmatrix}(d(u),d(u))$$
$$= F((u,u))\begin{pmatrix} n \\ 0 \end{pmatrix}(x,x) + (u,u)\begin{pmatrix} n \\ 0 \end{pmatrix}(D((x,x)) + F((y,y))\begin{pmatrix} n \\ 0 \end{pmatrix}(u,u)$$
$$+ (y,y)\begin{pmatrix} n \\ 0 \end{pmatrix}D((u,u)).$$

$\Rightarrow F(u_1\alpha x_1 + y_1\alpha u_1) = F(u_1)\alpha x_1 + u_1\alpha D(x_1) + F(y_1)\alpha u_1 + y_1\alpha D(u_1).$

Hence F is a generalized $(U_1,N)-$derivation on N.

Except otherwise mentioned, throughout this paper, M is a 2-torsion free semiprime Γ-ring which satisfies the condition (*) $a\alpha b\beta c = a\beta b\alpha c$ for all $a,b,c \in M$; $\alpha,\beta \in \Gamma$ and U is a Lie ideal of M.

To generalize some results of [3] in semiprime Γ-rings with generalized (U,M)-derivations, we develop some important results proceeding as follows.

Lemma 2.1 *If f is a generalized $(U,M)-$derivation of M for which d is the associated $(U,M)-$derivation of M. Then for all $u,v \in U; m \in M$ and $\alpha,\beta \in \Gamma$,*

(*i*) $f(u\alpha m\beta u) = f(u)\alpha m\beta u + u\alpha d(m)\beta u + u\alpha m\beta d(u);$

(*ii*) $f(u\alpha m\beta v + v\alpha m\beta u) = f(u)\alpha m\beta v + u\alpha d(m)\beta v + u\alpha m\beta d(v) + f(v)\alpha m\beta u + v\alpha d(m)\beta u + v\alpha m\beta d(u).$

Proof. By the definition of a generalized (U,M)-derivation of M, we have $f(u\alpha m + s\alpha u) = f(u)\alpha m + u\alpha d(m) + f(s)\alpha u + s\alpha d(u)$ for all $u \in U; m,s \in M$ and $\alpha \in \Gamma$. Replacing m and s by $(2u)\beta m + m\beta(2u)$ and let

$w = u\alpha((2u)\beta m + m\beta(2u)) + ((2u)\beta m + m\beta(2u))\alpha u$.

On the one hand

$$f(w) = 2(f(u)\alpha(u\beta m + m\beta u) + u\alpha d(u\beta m + m\beta u) + f(u\beta m + m\beta u)\alpha u + (u\beta m + m\beta u)\alpha d(u))$$
$$= 2(f(u)\alpha u\beta m + f(u)\alpha m\beta u + u\alpha d(u)\beta m + u\alpha u\beta d(m) + u\alpha d(m)\beta u + u\alpha m\beta d(u)$$
$$+ f(u)\beta m\alpha u + u\beta d(m)\alpha u + f(m)\beta u\alpha u + m\beta d(u)\alpha u + u\beta m\alpha d(u) + m\beta u\alpha d(u))$$
$$= 2(f(u)\alpha u\beta m + f(u)\alpha m\beta u + u\alpha d(u)\beta m + u\alpha u\beta d(m) + u\alpha d(m)\beta u + u\alpha m\beta d(u)$$
$$+ f(u)\alpha m\beta u + u\alpha d(m)\beta u + f(m)\alpha u\beta u + m\alpha d(u)\beta u + u\alpha m\beta d(u) + m\alpha u\beta d(u)). \quad (1)$$

On the other hand

$$f(w) = f((2u\alpha u)\beta m + m\beta(2u\alpha u)) + 2f(u\alpha m\beta u) + 2f(u\beta m\alpha u)$$
$$= 2(f(u)\alpha u\beta m + u\alpha d(u)\beta m + u\alpha u\beta d(m) + f(m)\beta u\alpha u$$
$$+ m\beta d(u)\alpha u + m\beta u\alpha d(u)) + 4f(u\alpha m\beta u)$$
$$= 2(f(u)\alpha u\beta m + u\alpha d(u)\beta m + u\alpha u\beta d(m) + f(m)\alpha u\beta u$$
$$+ m\alpha d(u)\beta u + m\alpha u\beta d(u)) + 4f(u\alpha m\beta u) \quad (2)$$

Comparing (1) and (2), and since M is 2-torsion free

$$f(u\alpha m\beta u) = f(u)\alpha m\beta u + u\alpha d(m)\beta u + u\alpha m\beta d(u), \forall\, u \in U; m \in M; \alpha, \beta \in \Gamma.$$

If we linearize (3) on u, then (ii) is obtained.

Definition 2. Let f be a generalized (U,M)-derivation with the associated (U,M)-derivation d of M. We define $\Psi_\alpha(u,m) = f(u\alpha m) - f(u)\alpha m - u\alpha d(m)$ and $\Phi_\alpha(u,m) = d(u\alpha m) - d(u)\alpha m - u\alpha d(m)$ for all $u \in U; m \in M$ and $\alpha \in \Gamma$.
Directly from the definition, the following properties follow at once.

Lemma 2.2 If f is a generalized (U,M)-derivation of M, then for all $u,v \in U; m,n \in M$ and $\alpha, \beta \in \Gamma$,
(i) $\Psi_\alpha(u,m) = -\Psi_\alpha(m,u)$; (ii) $\Psi_\alpha(u+v,m) = \Psi_\alpha(u,m) + \Psi_\alpha(v,m)$;
(iii) $\Psi_\alpha(u,m+n) = \Psi_\alpha(u,m) + \Psi_\alpha(u,n)$; (iv) $\Psi_{\alpha+\beta}(u,m) = \Psi_\alpha(u,m) + \Psi_\beta(u,m)$.

Proof. (i) By the definition of $\Psi_\alpha(u,m)$, we have
$\Psi_\alpha(u,m) = f(u\alpha m) - f(u)\alpha m - u\alpha d(m)$. Using Definition 1, we get

$$\Psi_\alpha(u,m) + \Psi_\alpha(m,u) = f(u\alpha m) - f(u)\alpha m - u\alpha d(m) + f(m\alpha u) - f(m)\alpha a - m\alpha d(u)$$
$$= f(u\alpha m + m\alpha u) - f(u)\alpha m - u\alpha d(m) - f(m)\alpha u - m\alpha d(u)$$
$$= f(u)\alpha m + f(m)\alpha a + u\alpha d(m) + m\alpha d(u) - f(u)\alpha m - u\alpha d(m)$$
$$- f(m)\alpha u - m\alpha d(u) = 0.$$

$\Rightarrow \Psi_\alpha(u,m) = -\Psi_\alpha(m,u)$.

(ii) By the definition of $\Psi_\alpha(u,m)$, we get

$$\Psi_\alpha(u+v,m) = f((u+v)\alpha m) - f(u+v)\alpha m - (u+v)\alpha d(m)$$
$$= f(u\alpha m + v\alpha m) - f(u)\alpha m - f(v)\alpha m - u\alpha d(m) - v\alpha d(m)$$
$$= f(u\alpha m) - f(u)\alpha m - u\alpha d(m) + f(v\alpha m) - f(v)\alpha m - v\alpha d(m)$$
$$= \Psi_\alpha(u,m) + \Psi_\alpha(v,m).$$

(iii)- (iv): These are too easy to prove.

Lemma 2.3 *With our notations as above, for any* $u,v \in U; m \in M$ *and* $\alpha, \beta \in \Gamma$, *the following are true:* (i) $\Phi_\alpha(u,m) = -\Phi_\alpha(m,u)$; (ii) $\Phi_\alpha(u+v,m) = \Phi_\alpha(u,m) + \Phi_\alpha(v,m)$;

(iii) $\Phi_\alpha(u,m+n)=\Phi_\alpha(u,m)+\Phi_\alpha(u,n)$;

(iv) $\Phi_{\alpha+\beta}(u,m)=\Phi_\alpha(u,m)+\Phi_\beta(u,m)$.

Proof. Proceeding in the same way of the proof of above lemma.

Lemma 2.4 *Let* U be a Lie ideal of a 2-torsion free Γ-ring M satisfying the condition(*) then $T(U)=\{x\in M:[x,M]_\Gamma\subseteq U\}$ is both a subring and a Lie ideal of M such that $U\subseteq T(U)$.

Proof. We have U is a Lie ideal of M, so $[U,M]_\Gamma\subseteq U$. Thus $U\subseteq T(U)$. Also we have $[T(U),M]_\Gamma\subseteq U\subseteq T(U)$. Hence $T(U)$ is a Lie ideal of M. Now suppose that $x,y\in T(U)$ then $[x,m]_\alpha\in U$ and $[y,m]_\alpha\in U$ for all $m\in M$ and $\alpha\in\Gamma$.

Now $[x\alpha y,m]_\beta=x\alpha[y,m]_\beta+[x,m]_\beta\alpha y\in U$. Therefore, $[x\alpha y,m]_\beta\in U$ for all $x,y\in T(U),m\in M$ and $\alpha,\beta\in\Gamma$. Hence $x\alpha y\in T(U)$.

Lemma 2.5 Let $U\not\subseteq Z(M)$ be a Lie ideal of a 2-torsion free semiprime Γ-ring M satisfying the condition (*) then there exists a nonzero ideal $K=M\Gamma[U,U]_\Gamma\Gamma M$ of M generated by $[U,U]_\Gamma$ such that $[K,M]_\Gamma\subseteq U$.

Proof. First we prove that if $[U,U]_\Gamma=0$ then $U\subseteq Z(M)$, so let $[U,U]_\Gamma=0$ for $u\in U$ and $\alpha\in\Gamma$, we have $[u,[u,x]_\alpha]_\alpha=0$ for all $x\in M$. For all $z\in M$ and $\beta\in\Gamma$, we replace x by $x\beta z$ in $[u,[u,x]_\alpha]_\alpha=0$ and obtain

$$\begin{aligned}0&=[u,[u,x\beta z]_\alpha]_\alpha\\&=[u,x\beta[u,z]_\alpha+[u,x]_\alpha\beta z]_\alpha\\&=[u,x\beta[u,z]_\alpha]_\alpha+[u,[u,x]_\alpha\beta z]_\alpha\\&=x\beta[u,[u,z]_\alpha]_\alpha+[u,x]_\alpha\beta[u,z]_\alpha+[u,[u,x]_\alpha]_\alpha\beta z+[u,x]_\alpha\beta[u,z]_\alpha\\&=2[u,x]_\alpha\beta[u,z]_\alpha\end{aligned}$$

By the 2-torsion freeness of M, we obtain $[u,x]_\alpha\beta[u,z]_\alpha=0$. Now replacing z by $z\gamma x$, we obtain

$$\begin{aligned}0&=[u,x]_\alpha\beta[u,z\gamma x]_\alpha\\&=[u,x]_\alpha\beta z\gamma[u,x]_\alpha+[u,x]_\alpha\beta[u,z]_\alpha\gamma x\\&=[u,x]_\alpha\beta z\gamma[u,x]_\alpha\end{aligned}$$

That is, $[u,x]_\alpha\beta M\gamma[u,x]_\alpha=0$. Since M is semiprime, $[u,x]_\alpha=0$. This implies that $u\in Z(M)$ and therefore, $U\subseteq Z(M)$ is a contradiction. So let $[U,U]_\Gamma\neq 0$. Then $K=M\Gamma[U,U]_\Gamma\Gamma M$ is a nonzero ideal of M generated by $[U,U]_\Gamma$. Let $x,y\in U,m\in M$ and $\alpha,\beta\in\Gamma$, we have $[x,y\beta m]_\alpha,y,[x,m]_\alpha\in U\subseteq T(U)$. Hence $[x,y]_\alpha\beta m=[x,y\beta m]_\alpha-y\beta[x,m]_\alpha\in T(U)$.

Also we can show that, $m\beta[x,y]_\alpha\in T(U)$ and therefore, we obtain $[[U,U]_\Gamma,M]_\Gamma\subseteq U$. That is, $[[[[x,y]_\alpha,m]_\alpha,s]_\alpha,t]_\alpha\in U$ for all $m,s,t\in M$ and $\alpha\in\Gamma$.

Hence $[x,y]_\alpha\alpha m\alpha s-m\alpha[x,y]_\alpha\alpha s+[s,m]_\alpha\alpha[x,y]_\alpha-[s\alpha[x,y]_\alpha,m]_\alpha,m]_\alpha t]_\alpha\in T(U)$.

Since $[x,y]_\alpha \alpha m \alpha s, s\alpha[x,y]_\alpha, [s,m]_\alpha \alpha[x,y]_\alpha \in T(U)$. Thus we have, $[m\alpha[x,y]_\alpha \alpha s, t]_\alpha \in U$ for all $m,s,t \in M$ and $\alpha \in \Gamma$. Hence $[K,M]_\Gamma \subseteq U$.

Lemma 2.6 Let $U \not\subseteq Z(M)$ be a Lie ideal of a 2-torsion free semiprime Γ-ring M satisfying the condition (*) then $a\alpha a = 0$ and there exists a nonzero ideal $K = M\Gamma[U,U]_\Gamma \Gamma M$ of M generated by $[U,U]_\Gamma$ such that $[K,M]_\Gamma \subseteq U$ and $K\Gamma a = a\Gamma K = \{0\}$.

Proof. If $a\alpha U\beta a = \{0\}$ for all $\alpha,\beta \in \Gamma$, then $a\alpha[a,a\delta m]_\alpha \beta a = 0$ for all $m \in M$ and $\delta \in \Gamma$. Therefore, by our assumption

$$\begin{aligned} 0 &= a\alpha(a\alpha a\delta m - a\delta m\alpha a)\beta a \\ &= a\alpha a\alpha a\delta m\beta a - a\alpha a\delta m\alpha a\beta a \\ &= a\alpha a\delta a\alpha m\beta a - a\alpha a\delta m\beta a\alpha a. \end{aligned}$$

Since $a\alpha a\delta a = 0$, we have $(a\alpha a)\delta m\beta(a\alpha a) = 0$. Since M is semiprime, $a\alpha a = 0$. Now we obtain $a\alpha[k\gamma a,m]_\mu \alpha u\beta a = 0$ for all $k \in K, m \in M, u \in U$ and $\alpha,\beta,\mu, \in \Gamma$. Again using our assumption and $a\alpha U\beta a = \{0\}$.

$$\begin{aligned} 0 &= a\alpha(k\gamma a\mu m - m\mu k\gamma a)\alpha u\beta a \\ &= a\alpha k\gamma a\mu m\alpha u\beta a - a\alpha m\mu k\gamma a\alpha u\beta a \\ &= a\alpha k\gamma a\mu m\beta u\alpha a. \end{aligned}$$

So, we obtain $a\alpha k\gamma a\mu m\beta[k,a]_\gamma \alpha a = 0$. This implies that $a\alpha k\gamma a\mu m\beta(k\gamma a - a\gamma k)\alpha a = 0$ and hence $a\alpha k\gamma a\mu m\beta k\gamma a\alpha a - a\alpha k\gamma a\mu m\beta a\gamma k\alpha a = 0$. By using assumption and $a\alpha a = 0$, we obtain $(a\alpha k\gamma a)\mu m\beta(a\alpha k\gamma a) = 0$. Since M is semiprime, $a\alpha k\gamma a = 0$. Thus we find that $(a\alpha k)\Gamma M\beta(a\alpha k) = 0$. Hence $a\alpha k = 0$ for all $k \in K$, that is $a\alpha K = \{0\}$. Similarly we obtain $K\alpha a = \{0\}$.

Lemma 2.7 Let $U \not\subseteq Z(M)$ be a Lie ideal of a 2-torsion free semiprime Γ-ring M satisfying the condition (*) (i) if $a\alpha U\beta a = \{0\}$, then $a = 0$; (ii) If $a\alpha U = \{0\}$ (or $U\alpha a = \{0\}$), then $a = 0$; (iii) if $u\alpha u \in U$ for all $u \in U$ and $a\alpha U\beta b = \{0\}$ then $a\alpha b = 0$ and $b\alpha a = 0$ for all $\alpha \in \Gamma$.

Proof. (i) By Lemma 2.5, we have $K\alpha a = M\Gamma[U,U]_\Gamma \Gamma M\alpha a = \{0\}$ and $a\alpha a = 0$ for all $\alpha \in \Gamma$. Therefore, for all $x,y \in M$ and $\alpha,\beta \in \Gamma$, we obtain

$$\begin{aligned} 0 &= [[a,x]_\alpha, a]_\gamma]\beta y\alpha a \\ &= [a\alpha x - x\alpha a, a]_\gamma \beta y\alpha a \\ &= a\alpha[x,a]_\gamma \beta y\alpha a - [x,a]_\gamma \alpha a\beta y\alpha a \\ &= a\alpha x\gamma a\beta y\alpha a - a\alpha a\gamma x\beta y\alpha a - x\gamma a\alpha a\beta y\alpha a + a\gamma x\alpha a\beta y\alpha a \\ &= a\alpha x\gamma a\beta y\alpha a + a\gamma x\alpha a\beta y\alpha a \\ &= 2a\alpha x\gamma a\beta y\alpha a \end{aligned}$$

By the 2-torsion freeness of M, we have $a\alpha x\gamma a\beta y\alpha a = 0$. Thus we obtain, $a\alpha x\gamma a\beta y\alpha a\delta x\gamma a = 0$. By using $a\alpha b\beta c = a\beta b\alpha c$ for all $a,b,c \in M$ and $\alpha,\beta \in \Gamma$, we have $(a\alpha x\gamma a)\beta y\delta(a\alpha x\gamma a) = 0$. This implies that $(a\alpha x\gamma a)\beta M\delta(a\alpha x\gamma a) = 0$. Since M is

semiprime $a\alpha x\gamma a = 0,$ for all $x \in M$ and $\alpha, \gamma \in \Gamma$. Again using the semiprimeness of M ,we obtain $a = 0.$
(ii) If $a\alpha U = \{0\}$, then $a\alpha U\beta a = \{0\})$ for all $\beta \in \Gamma$, therefore by (i), we obtain $a = 0$. Similarly, if $U\alpha a = \{0\}$, then $a = 0$.
(iii) If $a\alpha U\beta b = \{0\}$, then we have $(b\gamma a)\alpha U\beta(b\gamma a) = \{0\}$ and hence by (i), $b\gamma a = 0$ for all $\gamma \in \Gamma$. Also $(a\gamma b)\alpha U\beta(a\gamma b) = \{0\}$ if $a\alpha U\beta b = \{0\}$ and hence $a\gamma b = 0.$
In obtaining our main results the following lemma plays an important role.

Lemma 2.8 If U is an admissible Lie ideal of a 2-torsion free semiprime Γ-ring M satisfying the condition (*) and f is a generalized $(U,M)-$derivation of M for which d is the associated $(U,M)-$derivation of M , then for all $u,v,w \in U$ and $\alpha,\beta,\gamma \in \Gamma$, (i) $\Psi_\alpha(u,v)\beta w\gamma[u,v]_\alpha = 0;$ (ii) $\Psi_\alpha(u,v)\alpha w\alpha[u,v]_\alpha = 0;$ (iii) $\Psi_\alpha(u,v)\beta w\beta[u,v]_\alpha = 0.$

Proof. (i) Let $x = 4(u\alpha v\beta w\gamma v\alpha u + v\alpha u\beta w\gamma u\alpha v)$. Using Lemma 2.1(ii), we have

$$f(x) = f((2u\alpha v)\beta w\gamma(2v\alpha u) + (2v\alpha u)\beta w\gamma(2u\alpha v))$$
$$= 4f(u\alpha v)\beta w\gamma v\alpha u + 4u\alpha v\beta d(w)\gamma v\alpha u + 4u\alpha v\beta w\gamma d(v\alpha u) + 4f(v\alpha u)\beta w\gamma u\alpha v$$
$$+ 4v\alpha u\beta d(w)\gamma u\alpha v + 4v\alpha u\beta w\gamma d(u\alpha v).$$

On the other hand, using Lemma 2.1(i), we have

$$f(x) = f(u\alpha(4v\beta w\gamma v)\alpha u + v\alpha(4u\beta w\gamma u)\alpha v)$$
$$= f(u)\alpha 4v\beta w\gamma v\alpha u + u\alpha d(4v\beta w\gamma v)\alpha u + u\alpha 4v\beta w\gamma v\alpha d(u) + f(v)\alpha 4u\beta w\gamma u\alpha v$$
$$+ v\alpha d(4u\beta w\gamma u)\alpha v + v\alpha 4u\beta w\gamma u\alpha d(v)$$
$$= 4f(u)\alpha v\beta w\gamma v\alpha u + 4u\alpha d(v)\beta w\gamma v\alpha u + 4u\alpha v\beta d(w)\gamma v\alpha u + 4u\alpha v\beta w\gamma d(v)\alpha u$$
$$+ 4u\alpha v\beta w\gamma v\alpha d(u) + 4f(v)\alpha u\beta w\gamma u\alpha v + 4v\alpha d(u)\beta w\gamma u\alpha v + 4v\alpha u\beta d(w)\gamma u\alpha v$$
$$+ 4v\alpha u\beta w\gamma d(u)\alpha v + 4v\alpha u\beta w\gamma u\alpha d(v).$$

Comparing the right side of $f(x)$ and using the 2-torsion freeness of M

$$f(u\alpha v)\beta w\gamma v\alpha u + u\alpha v\beta w\gamma d(v\alpha u) + f(v\alpha u)\beta w\gamma u\alpha v + v\alpha u\beta w\gamma d(u\alpha v)$$
$$= f(u)\alpha v\beta w\gamma v\alpha u + u\alpha d(v)\beta w\gamma v\alpha u + u\alpha v\beta w\gamma d(v)\alpha u + u\alpha v\beta w\gamma v\alpha d(u)$$
$$+ f(v)\alpha u\beta w\gamma u\alpha v + v\alpha d(u)\beta w\gamma u\alpha v + v\alpha u\beta w\gamma d(u)\alpha v + v\alpha u\beta w\gamma u\alpha d(v).$$

Therefore,

$$(f(u\alpha v) - f(u)\alpha v - u\alpha d(v))\beta w\gamma v\alpha u + (f(v\alpha u) - f(v)\alpha u - v\alpha d(u))\beta w\gamma u\alpha v$$
$$+ u\alpha v\beta w\gamma(d(v\alpha u) - d(v)\alpha u - v\alpha d(u)) + v\alpha u\beta w\gamma(d(u\alpha v) - d(u)\alpha v - u\alpha d(v)) = 0.$$

Using Definition 2, we obtain

$$\Psi_\alpha(u,v)\beta w\gamma v\alpha u + \Psi_\alpha(v,u)\beta w\gamma u\alpha v + u\alpha v\beta w\gamma\Phi_\alpha(v,u) + v\alpha u\beta w\gamma\Phi_\alpha(u,v) = 0.$$

Now, using Lemma 2.2(i) and 2.3(i), we have

$$\Psi_\alpha(u,v)\beta w\gamma[u,v]_\alpha + [u,v]_\alpha\beta w\gamma\Phi_\alpha(u,v) = 0, \forall\, u,v,w \in U; \alpha,\beta,\gamma \in \Gamma.$$

Since d is a (U,M)-derivation, we have $\Phi_\alpha(u,v) = 0$ for all $u,v \in U$ and $\alpha \in \Gamma$, by [9].
Using this we obtain the desired result. All other results are proved similarly.

Lemma 2.9 Let U be an admissible Lie ideal of a 2-torsion free semiprime Γ-ring M and let $a,b \in U$. If $a\alpha u\beta b + b\alpha u\beta a = 0$ for all $u \in U$ and $\alpha,\beta \in \Gamma$ then $a\alpha u\beta b = 0 = b\alpha u\beta a$.

Proof. Let $x \in U$ and $\gamma \in \Gamma$ be any elements. Using the relation $a\alpha u\beta b + b\alpha u\beta a = 0$ for all $u \in U$ and $\alpha,\beta \in \Gamma$ repeatedly, we get

$$\begin{aligned} 4(a\alpha u\beta b)\gamma x\gamma(a\alpha u\beta b) &= -4(b\alpha u\beta a)\gamma x\gamma(a\alpha u\beta b) \\ &= -(b\alpha(4u\beta a\gamma x)\gamma a)\alpha u\beta b \\ &= (a\alpha(4u\beta a\gamma x)\gamma b)\alpha u\beta b \\ &= a\alpha u\beta(4a\gamma x\gamma b)\alpha u\beta b \\ &= -a\alpha u\beta(4b\gamma x\gamma a)\alpha u\beta b \\ &= -4(a\alpha u\beta b)\gamma x\gamma(a\alpha u\beta b). \end{aligned}$$

This implies, $8((a\alpha u\beta b)\gamma x\gamma(a\alpha u\beta b)) = 0$. Since M is 2-torsion free, $(a\alpha u\beta b)\gamma x\gamma(a\alpha u\beta b) = 0.$ Therefore, $(a\alpha u\beta b)\gamma U\gamma(a\alpha u\beta b) = 0.$ Thus by Lemma 2.7 (i), we get $a\alpha u\beta b = 0.$ Similarly, it can be shown that $b\alpha u\beta a = 0.$

Lemma 2.10 Let M be a 2-torsion free semiprime Γ-ring satisfying the condition (*) and U be an admissible Lie ideal of M. Let f be a Jordan generalized derivation on U of M. Then for all $u,v,w \in U$ and $\alpha,\beta,\gamma \in \Gamma$,

(i) $[u,v]_\alpha\beta w\gamma\Psi_\alpha(u,v) = 0$; (ii) $[u,v]_\alpha\alpha w\alpha\Psi_\alpha(u,v) = 0$; (iii) $[u,v]_\alpha\beta w\beta\Psi_\alpha(u,v) = 0$.

Proof. (iii) We have $[u,v]_\alpha\beta w\beta\Psi_\alpha(u,v)\beta w\beta[u,v]_\alpha\beta w\beta\Psi_\alpha(u,v) = 0,$ for all $v \in U$.
By Lemma 2.7(i), $[u,v]_\alpha\beta w\beta\Psi_\alpha(u,v) = 0.$ All other results are proved similarly.

Lemma 2.11 Let M be a 2-torsion free semiprime Γ-ring satisfying the condition (*) and U be an admissible Lie ideal of M. If f is a Jordan generalized derivation *on* U *of* M*, then for all* $u,v,x,y,w \in U$ *and* $\alpha,\beta,\gamma \in \Gamma$,

(i) $\Psi_\alpha(u,v)\beta w\beta[x,y]_\alpha = 0$; (ii) $[x,y]_\alpha\beta w\beta\Psi_\alpha(u,v) = 0$;

(iii) $\Psi_\alpha(u,v)\beta w\beta[x,y]_\gamma = 0$; (iv) $[x,y]_\gamma\beta w\beta\Psi_\alpha(u,v) = 0$.

Proof. (i) If we substitute $u + x$ for u in the Lemma 2.8 (iii), we get

$$\Psi_\alpha(u+x,v)\beta w\beta[u+x,v]_\alpha = 0.$$

This implies

$$\Psi_\alpha(u,v)\beta w\beta[u,v]_\alpha + \Psi_\alpha(u,v)\beta w\beta[x,v]_\alpha + \Psi_\alpha(x,v)\beta w\beta[u,v]_\alpha + \Psi_\alpha(x,v)\beta w\beta[x,v]_\alpha = 0.$$

Which gives

$$\Psi_\alpha(u,v)\beta w\beta[x,v]_\alpha + \Psi_\alpha(x,v)\beta w\beta[u,v]_\alpha = 0.$$

Now by using Lemma 2.10 (iii), we obtain

$$\begin{aligned} (\Psi_\alpha(u,v)\beta w\beta[x,v]_\alpha)\beta u\beta(\Psi_\alpha(u,v)\beta w\beta[x,v]_\alpha) &= -\Psi_\alpha(u,v)\beta w\beta[x,v]_\alpha\beta u\beta\Psi_\alpha(x,v)\beta w\beta[u,v]_\alpha \\ &= 0. \end{aligned}$$

Hence, by Lemma 2.7(i), we get $\Psi_\alpha(u,v)\beta w\beta[x,v]_\alpha = 0.$

Similarly, by replacing $v + y$ for v in this result, we get $\Psi_\alpha(u,v)\beta w\beta[x,y]_\alpha = 0.$

(ii) Proceeding in the same way as described above by the similar replacements successively in Lemma 2.10 (iii), we obtain

$$[x,y]_\gamma \beta w \beta \Psi_\alpha(u,v)=0, \forall u,v,x,y,w \in U, \alpha,\beta \in \Gamma.$$

(iii) Replacing $\alpha+\gamma$ for α in (i), we get

$$\Psi_{\alpha+\gamma}(u,v)\beta w\beta[x,y]_{\alpha+\gamma}=0.$$

This implies

$$(\Psi_\alpha(u,v)+\Psi_\gamma(u,v))\beta w\beta([x,y]_\alpha+[x,y]_\gamma)=0.$$

Therefore

$$\Psi_\alpha(u,v)\beta w\beta[x,y]_\alpha+\Psi_\alpha(u,v)\beta w\beta[x,y]_\gamma+\Psi_\gamma(u,v)\beta w\beta[x,y]_\alpha+\Psi_\gamma(u,v)\beta w\beta[x,y]_\gamma=0.$$

Thus by using Lemma 2.10 (iii), we get

$$\Psi_\alpha(u,v)\beta w\beta[x,y]_\gamma+\Psi_\gamma(u,v)\beta w\beta[x,y]_\alpha=0.$$

Thus, we obtain

$$\begin{aligned}(\Psi_\alpha(u,v)\beta w\beta[x,y]_\gamma)\beta u\beta(\Psi_\alpha(u,v)\beta w\beta[x,y]_\gamma) &= -\Psi_\alpha(u,v)\beta w\beta[x,y]_\gamma\beta u\beta\Psi_\gamma(u,v)\beta w\beta[x,y]_\alpha \\ &= 0.\end{aligned}$$

Hence, by Lemma 2.7 (i), we obtain $\Psi_\alpha(u,v)\beta w\beta[x,y]_\gamma=0.$

(iv) As in the proof of (iii), the similar replacement in (ii) produces (iv).

Now, we prove the following two theorems with generalized (U,M)-derivation of a semiprime Γ-ring M.

Theorem 2.1 Assume that U is an admissible Lie ideal of a 2-torsion free semiprime Γ-ring M satisfying the condition (*) and f is a generalized (U,M)-derivation of M, then $\Psi_\alpha(u,v)=0$ for all $u,v\in U$ and $\alpha\in\Gamma.$

Proof. By Lemma 2.8 (iii), we have

$$\Psi_\alpha(u,v)\beta w\beta[u,v]_\alpha=0, \forall u,v,w\in U;\alpha,\beta\in\Gamma.$$

By Lemma 2.11 (iii), we have

$$\Psi_\alpha(u,v)\beta w\beta[x,y]_\gamma=0, \forall u,v,w,x,y\in U;\alpha,\beta,\gamma\in\Gamma.$$

Since U is not contained in $Z(M)$, so $[x,y]_\gamma\neq 0.$ Thus, by Lemma 2.7, we get $\Psi_\alpha(u,v)=0$ for all $u,v\in U$ and $\alpha\in\Gamma$.

Remark 2.1 If we replace U by a square closed Lie ideal in Theorem 2.1, then the theorem is also true.

Theorem 2.2 Let U be a square closed Lie ideal of a 2-torsion free semiprime Γ-ring M satisfying the condition (*) then $f(u\alpha m)=f(u)\alpha m+u\alpha d(m)$ for all $u\in U; m\in M$ and $\alpha\in\Gamma$.

Proof. From Theorem 2.1 and Remark 2.1, we have

$$\Psi_\alpha(u,v)=0, \forall u,v\in U;\alpha\in\Gamma \qquad (4)$$

Replacing v by $u\beta m-m\beta u$ in (4), we get $\Psi_\alpha(u,u\beta m-m\beta u)=0.$ Since $u\beta m-m\beta u\in U$ for all $u\in U$, $m\in M$ and $\alpha,\beta\in\Gamma$. Therefore,

$$\begin{aligned}
0 &= \Psi_\alpha(u, u\beta m - m\beta u)\\
&= f(u\alpha(u\beta m - m\beta u)) - f(u)\alpha(u\beta m - m\beta u) - u\alpha d(u\beta m - m\beta u)\\
&= f(u\alpha u\beta m) - f(u\alpha m\beta u) - f(u)\alpha u\beta m + f(u)\alpha m\beta u - u\alpha d(u)\beta m\\
&\quad - u\alpha u\beta d(m) + u\alpha d(m)\beta u + u\alpha m\beta d(u)\\
&= f(u\alpha u\beta m) - f(u)\alpha m\beta u - u\alpha d(m)\beta u - u\alpha m\beta d(u) - f(u)\alpha u\beta m\\
&\quad + f(u)\alpha m\beta u - u\alpha d(u)\beta m - u\alpha u\beta d(m) + u\alpha d(m)\beta u + u\alpha m\beta d(u)\\
&= f(u\alpha u\beta m) - f(u)\alpha u\beta m - u\alpha d(u)\beta m - u\alpha u\beta d(m).
\end{aligned}$$

This implies,

$$\begin{aligned}
& f(u\alpha u\beta m) = f(u)\alpha u\beta m + u\alpha d(u)\beta m + u\alpha u\beta d(m)\\
&\Rightarrow f((u\alpha u)\beta m) - f(u\alpha u)\beta m - (u\alpha u)\beta d(m) = 0.\\
&\Rightarrow \Psi_\beta(u\alpha u, m) = 0, \forall\, u \in U; m \in M; \alpha, \beta \in \Gamma.
\end{aligned} \tag{5}$$

Now, let $x = u\alpha u\beta m + u\beta m\alpha u$. Then by the definition of generalized (U, M)-derivation, we have

$$\begin{aligned}
f(x) &= f(u)\alpha u\beta m + u\alpha d(u\beta m) + f(u\beta m)\alpha u + u\beta m\alpha d(u)\\
&= f(u)\alpha u\beta m + u\alpha d(u)\beta m + u\alpha u\beta d(m) + f(u\beta m)\alpha u + u\beta m\alpha d(u).
\end{aligned} \tag{6}$$

On the other hand, using (5) and Lemma 2.1(i)

$$\begin{aligned}
f(x) &= f(u\alpha u\beta m) + f(u\beta m\alpha u)\\
&= f(u)\alpha u\beta m + u\alpha d(u)\beta m + u\alpha u\beta d(m) + f(u)\beta m\alpha u + u\beta d(m)\alpha u + u\beta m\alpha d(u).
\end{aligned} \tag{7}$$

Comparing (6) and (7), we get

$$(f(u\beta m) - f(u)\beta m - u\beta d(m))\alpha u = 0.$$

This yields,

$$\Psi_\beta(u, m)\alpha u = 0, \forall\, u \in U; m \in M; \alpha, \beta \in \Gamma. \tag{8}$$

Linearize (8) on u and using equation (8), we get

$$\Psi_\beta(u, m)\alpha v + \Psi_\beta(v, m)\alpha u = 0. \tag{9}$$

Replacing v by $v\gamma v$ in equation (9), we obtain

$$\Psi_\beta(u, m)\alpha v\gamma v + \Psi_\beta(v\gamma v, m)\alpha u = 0.$$

Since $\Psi_\beta(v\gamma v, m) = 0$ for all $v \in U, m \in M$ and $\beta, \gamma \in \Gamma$. This is seen in the equation (5) for $v\gamma v$ in place of $u\alpha u$. Therefore, we have

$$\Psi_\beta(u, m)\alpha v\gamma v = 0, \forall\, u, v \in U; m \in M; \alpha, \beta, \gamma \in \Gamma. \tag{10}$$

Replacing v by $u + v$ in (10) and using (5), we obtain

$$\begin{aligned}
& \Psi_\beta(u, m)\alpha(u + v)\gamma(u + v) = 0\\
&\Rightarrow \Psi_\beta(u, m)\alpha(u\gamma u + u\gamma v + v\gamma u + v\gamma v) = 0.\\
&\Rightarrow \Psi_\beta(u, m)\alpha u\gamma v + \Psi_\beta(u, m)\alpha v\gamma u = 0.
\end{aligned}$$

Now using (8), this implies $\Psi_\beta(u, m)\alpha v\gamma u = 0$ for all $u, v \in U; m \in M$ and $\alpha, \beta, \gamma \in \Gamma$. Since U is noncentral, by Lemma 2.7, $\Psi_\beta(u, m) = 0$ for all $u \in U; m \in M$ and $\beta \in \Gamma$. Consequently, $f(u\alpha m) = f(u)\alpha m + u\alpha d(m)$ *for all* $u \in U; m \in M$ *and* $\alpha \in \Gamma$.

3. Conclusion

If the Lie ideal U is square closed and $U \not\subseteq Z(M)$ then U is an admissible Lie ideal of M so, for both the cases $f(u\alpha v) = f(u)\alpha v + u\alpha d(v)$ for all $u, v \in U$ and $\alpha \in \Gamma$ but for only square closed case $f(u\alpha m) = f(u)\alpha m + u\alpha d(m)$ for all $u \in U, m \in M$ and $\alpha \in \Gamma$.

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