Convex sublattices of a lattice have been studied by many authors including Koh [1-2]. Set of all convex sublattices of a lattice $L$ is denoted by $CS(L)$. By K. M. Koh [2] $CS(L)$ with the empty set is a lattice. On the other hand standard convex sublattices of a lattice $L$ have been studied by Fried and Schmidt [3]. Recently Lavanya and Bhatta [4] have introduced a new partial ordering relation on $CS(L)$, under which $CS(L)$ is a lattice. Moreover $L$ and $CS(L)$ are in the same equational class. On $CS(L)$, they defined the partial order “$\leq$” as follows:

For $A, B \in CS(L)$, $A \leq B$ if and only if “for every $a \in A$ there exists a $b \in B$, such that $a \leq b$ and for every $b \in B$ there exists an $a \in A$, Such that $b \geq a$.” It is easy to see that ‘$\leq$’ is clearly a partial order and $(CS(L); \leq)$ forms a lattice, where for $A, B \in CS(L)$,

$\text{Inf } \{A, B\} = A \wedge B$

$= \langle\{a \wedge b|a \in A, b \in B\}\rangle$

$= \{x \in L|a \wedge b \leq x \leq a_1 \wedge b_1 \text{ for some } a, a_1 \in A \text{ and } b, b_1 \in B\}$

$\text{Sup } \{A, B\} = A \vee B$

$= \langle\{a \vee b|a \in A, b \in B\}\rangle$

$= \{x \in L|a \vee b \leq x \leq a_1 \vee b_1 \text{ for some } a, a_1 \in A \text{ and } b, b_1 \in B\}$
and for any non-empty subset \( H \) of \( L \), \( \langle H \rangle \) denotes the convex sublattice generated by \( H \). Note that \( A \land B \) and \( A \lor B \) have also been studied by J. Nieminen [5], where the author studied the distributive and neutral sublattices.

In this paper we studied the structure of \( \text{CS}(L) \) with this new approach and then include some properties of \((\text{CS}(L); \leq)\). We have also given a nice characterization of a standard element of \( \text{CS}(L) \).

We start with the construction of \((\text{CS}(L); \leq)\) of a lattice \( L \) of Fig. 1.

![Fig. 1.](image1)

![Fig. 2.](image2)

It is easy to check that Fig. 2 represents the lattice “\((\text{CS}(L); \leq)\)”. Now we include some properties of “\((\text{CS}(L); \leq)\)”. We know that for any congruence of a lattice \( L \), each congruence class is an element of \( \text{CS}(L) \). We have the following results:

**Theorem 1.** For any Congruence \( \Theta \) of a lattice \( L \), \([a] \Theta \leq [b] \Theta \) in \( \frac{L}{\Theta} \) if and only if \([a] \Theta \leq [b] \Theta \) in \( \text{CS}(L) \). In other words, the quotient lattice \( \frac{L}{\Theta} \) is a subposet of \((\text{CS}(L); \leq)\) but \( \frac{L}{\Theta} \) is not necessarily a sublattice of \( \text{CS}(L) \).

**Proof:** Suppose \([a] \Theta \leq [b] \Theta \) in \( \frac{L}{\Theta} \). Let \( s \in [a] \Theta \) then \([s] \Theta = [a] \Theta \leq [b] \Theta \) in \( \frac{L}{\Theta} \). Thus \([b] \Theta \leq [b] \Theta \odot [s] \Theta = [b \lor s] \Theta \), this implies that \( b \lor s e \in [b] \Theta \) and \( s \leq b \lor s \). On the other hand, let \( t \in [b] \Theta \). Then \([a] \Theta \leq [b] \Theta \leq [t] \Theta \) in \( \frac{L}{\Theta} \). Thus \([a] \Theta = [a] \Theta \land [t] \Theta = [a \land t] \Theta \), which implies that \( a \land t \in [a] \Theta \) and \( t \geq a \land t \). Therefore, by the definition of ‘\( \leq \)’ in \( \text{CS}(L) \), \([a] \Theta \leq [b] \Theta \) in \( \text{CS}(L) \).

Conversely, let \([a] \Theta \leq [b] \Theta \) in \( \text{CS}(L) \). Since \( a \in [a] \Theta \) there exists \( t \in [b] \Theta \) such that \( a \leq t \). Then \( a = a \land t = (a \land b) \Theta \) and so \([a] \Theta = [a \land b] \Theta = [a] \Theta \land [b] \Theta \) in \( \frac{L}{\Theta} \). Thus implies \([a] \Theta \leq [b] \Theta \) in \( \frac{L}{\Theta} \).

To prove the last part, consider the following lattice \( L \) in Fig. 3.
Consider the congruence $\Theta = \{0, a\}, \{b\}, \{c\}, \{1\}$. In $L$, $[b] \cap [c] \in \Theta = [b \cap c] \in \Theta = [a] \in \Theta$. But in $CS(L)$, $[b] \cap [c] \in \Theta \neq \{a\}$. Therefore $L$ is not a sublattice of $CS(L)$.

**Theorem 2.** For any $A, B \in CS(L)$, $A \leq B$ if and only if $(A) \subseteq (B)$ and $(A) \supseteq (B)$.

**Proof:** Suppose $A \leq B$, let $a \in (A)$, then $a \leq a_1$ for some $a_1 \in A$. Since $A \leq B$, so there exists a $b_1 \in B$ such that $a \leq b_1$ and so $a \in (B)$. Hence $(A) \subseteq (B)$. Now let $b \in [B]$, then $b \geq b_1$ for some $b_1 \in B$. Since $A \leq B$, so there exists $a_1 \in A$ such that $b_1 \geq a_1$. Thus $b \geq a_1$, which implies that $b \in (A)$. Hence $(A) \subseteq (B)$.

Conversely, suppose $(A) \subseteq [B]$ and $(A) \supseteq [B]$. Let $a \in A$, then $a \in (A) \subseteq (B)$. This implies that $a \leq b$ for some $b \in B$. Again for any $b \in B$, $b \in [B] \subseteq [A]$ and so $b \geq a$ for some $a \in A$. Hence by definition, $A \leq B$ in $CS(L)$.

For a lattice $L$, $I(L)$ and $D(L)$ are Lattice of ideals and dual ideals respectively. From the above theorem, we have the following corollary.

**Corollary 3.** For $I, J \in I(L)$, $I \leq J$ if and only if $I \subseteq J$ and for $D, K \in D(L)$, $D \leq K$ if and only if $D \supseteq K$.

**Theorem 4.** For any lattice $L$, $I(L)$ is a principal ideal generated by $L$ in $CS(L)$ and $D(L)$ is a principal dual ideal generated by $L$ in $CS(L)$.

**Proof:** By Corollary 3, $I(L)$ is a sublattice of $CS(L)$ with $L$ as its largest element. Now let $I \in I(L)$ and $A \in CS(L)$ with $A \leq I$. We need to show that $A$ has the hereditary property. Suppose, $x \in A$ and $y \leq x$. Since $x \in A$ and $A \leq I$, so by definition there exists $i \in I$, such that $x \leq i$. Since $I$ is an ideal, so $y \leq x \leq i$ implies that $y \in I$. Now $A \leq I$ implies that there exists an element $z \in A$, such that $y \geq z$. Then $z \leq y \leq x$ and so by convexity $y \in A$. Hence $A$ has the hereditary property and thus $A$ is an ideal, that is, $A \in I(L)$. Therefore $I(L)$ is an ideal of $CS(L)$ with $L$ as its largest element and so it is a principal ideal generated by $L$. Similarly, we can show that $D(L)$ is a principal dual ideal generated by $L$. Therefore, we can show that $D(L)$ is a principal dual ideal generated by $L$.

Observe that in Fig. 2, both $I(L)$ and $D(L)$ are principal ideal and principal dual ideal respectively, in $CS(L)$ generated by $L$. 

**Fig. 3.**
Since $I(L)$ is a sub lattice of $CS(L)$, we have the following result.

**Theorem 5.** The mapping $f: L \rightarrow CS(L)$ defined by $f(a)=(a)$ is an embedding. Moreover, an element $a$ is join irreducible in $L$ if and only if $f(a)$ is join irreducible in $CS(L)$.

**Proof:** The mapping $f$ is obviously an embedding of $L$ into $CS(L)$. Now suppose $a$ is join irreducible in $L$. Let for $A, B \in CS(L)$, $A \lor B=f(a)=(a)$, implies $A \subseteq (a)$ and $B \subseteq (a)$ in $CS(L)$. Then each $x \in A$ implies $x \leq a$, so $x \in (a)$ and hence $A \subseteq (a)$. Similarly $B \subseteq (a)$. Since $a \in A \lor B$, so by definition $a_1 \lor b_1 \leq a_2 \lor b_2$ for some $a_1,a_2 \in A$ and $b_1,b_2 \in B$. Now $A, B \subseteq (a)$ so $a_2,b_2 \leq a$. Without loss of generality, suppose $a=a_2$, then $a \in A$. Now we prove that $A=(a)$. If not, then there exist an element $t \in (a)$ such that $t \not\in A$. Since $t \in (a)=(a_2 \lor b_2)$, so there exist $p_1,p_2 \in A$, $q_1,q_2 \in B$, such that $p_1 \lor q_1 \leq t \leq p_2 \lor q_2$ this implies $p_1 \leq t \leq a$ and so by convexity $t \in A$, which is a contradiction. Therefore $A=(a)$. Similarly, by considering $a=b_2$ we can show that $B=(a)$, therefore $f(a)=(a)$ is join irreducible in $CS(L)$.

Conversely, suppose $f(a)$ is join irreducible in $CS(L)$. Let $a=b \lor c$ in $L$, then $(a)=(b) \lor (c) \subseteq (a)$ in $CS(L)$. Since $f(a)=(a)$ is join irreducible in $CS(L)$, so either $(b)=(a)$ or $(c)=(a)$, that is, either $b=a$ or $c=a$. Therefore $a$ is join irreducible in $L$.

Since $D(L)$ is also a sub lattice of $CS(L)$ a dual proof of above gives the following result.

**Theorem 6.** The mapping $f : L \rightarrow CS(L)$ defined by $f(a)=[a)$ is an embedding. Moreover, an element $a$ is meet irreducible in $L$ if and only if $f(a)$ is meet irreducible in $CS(L)$.

The following theorem is due to S. Lavanya and S. P. Bhatta [4]. This gives a clear idea on the structure of $(CS(L);\leq)$.

**Theorem 7.** For any lattice $L$ the map $f: CS(L) \rightarrow I(L) \times D(L)$ defined by for any $X \in CS(L)$, $f(x)=((X],[X))$ is an imbedding. In fact, $CS(L)$ is isomorphic to the sublattice $\{(I,D) \mid I \in I(L), D \in D(L), I \cap D \neq \emptyset\}$ of $I(L) \times D(L)$.

We know from Grätzer [6] that the identities of lattices are preserved under the function of sublattices, homomorphic images, direct products, ideal lattices and dual ideal lattices. Also it is easily seen that $L$ can be embedded in $CS(L)$. Therefore, by above theorem we have the following result, which is also mentioned by Lavanya and Bhatta [4].

**Corollary 8.** $CS(L)$ satisfies all the identities satisfied by $L$ and conversely.

Thus in particular, a lattice $L$ is distributive (modular) if and only if $CS(L)$ is distributive (modular).

According to Grätzer [6] an element $n$ of a lattice $L$ is called a standard element if for all $x, y \in L$, $x \land (y \lor n) = (x \land y) \lor (x \land n)$ Element $n$ is called a neutral element if $(i)$ $n$ is standard, and
(ii) \( n \land (x \lor y) = (n \land x) \lor (n \land y) \) for all \( x, y \in L \).

Since \( L \) is the largest element and the smallest element of \((I(L); \subseteq)\) and \((D(L); \supseteq)\) respectively, so it is a neutral element of both \( I(L) \) and \( D(L) \). Therefore, by Theorem 7, we have the following result.

**Corollary 9.** \( L \) is a neutral element of \( CS(L) \).

We conclude the paper with the following characterization of standard elements of \( CS(L) \)

**Theorem 10.** For a lattice \( L \), a convex sublattice \( S \) is a standard element of \( CS(L) \) if and only if for any \( a, b \in L \), \( \{a\} \bigwedge (S \bigvee \{b\}) = (\{a\} \bigwedge S) \bigvee (\{a\} \bigwedge \{b\}) \).

**Proof:** Suppose, \( S \) is standard in \((CS(L); \leq)\). Then of course the given condition holds. Conversely, suppose the given condition holds for any \( a, b \in S \). We have to show that

\[ A \bigwedge (S \bigvee B) = (A \bigwedge S) \bigvee (A \bigwedge B) \]

for any \( A, B \in CS(L) \). Since \((CS(L); \bigwedge, \bigvee)\) is a lattice, so clearly \((A \bigwedge S) \bigvee (A \bigwedge B) \leq A \bigwedge (S \bigvee B) \). For the reverse inequality, let \( x \in A \bigwedge (S \bigvee B) \). Then \( x \leq a_1 \land t_1 \) for some \( a_1 \in A \) and \( 7t_1 \in S \bigvee B \). Now \( t_1 \in S \bigvee B \) implies that \( t_1 \leq s_1 \lor b_1 \) for some \( s_1 \in S \) and \( b_1 \in B \). Then \( x \leq a_1 \land (s_1 \lor b_1) = y \) (say). But \( y = a_1 \land (s_1 \lor b_1) \in \{a_1\} \bigwedge (S \bigvee \{b\}) = (\{a_1\} \bigwedge S) \bigvee (\{a_1\} \bigwedge \{b\}) \) (using the given condition) \( \subseteq (A \bigwedge S) \bigvee (A \bigwedge B) \). In other words, there exists an element \( y \in (A \bigwedge S) \bigvee (A \bigwedge B) \) with \( x \leq y \). Now let \( p \in (A \bigwedge S) \bigvee (A \bigwedge B) \). Then \( p \geq c_1 \lor d_1 \) for some \( c_1 \in A \bigwedge S \) and \( d_1 \in A \bigwedge B \). Now \( c_1 \in A \bigwedge S \) implies \( c_1 \geq a_2 \land s_2 \) and \( d_1 \in A \bigwedge B \) implies \( d_1 \geq a_3 \land b_3 \) for some \( a_2, a_3 \in A \), \( s_2 \in S \) and \( b_3 \in B \). Thus, \( p \geq (a_2 \land a_3 \land s_3) \lor (a_2 \land a_3 \land b_3) \in (a' \land s_3) \lor (a' \land b_3) \) where \( a' = a_2 \land a_3 \). But \( (a' \land s_3) \lor (a' \land b_3) \in \{a'\} \bigwedge S = (\{a'\} \bigwedge S) \bigvee (\{a'\} \bigwedge B) \) (by the given condition) \( \subseteq A \bigwedge (S \bigvee B) \). That is, for \( p \in (A \bigwedge S) \bigvee (A \bigwedge B) \), there exists \( q = (a' \land s_3) \lor (a' \land b_3) \in A \bigwedge (S \bigvee B) \) with \( p \geq q \).

Therefore, \( A \bigwedge (S \bigvee B) \leq (A \bigwedge S) \bigvee (A \bigwedge B) \) and so \( A \bigwedge (S \bigvee B) = (A \bigwedge S) \bigvee (A \bigwedge B) \).

**References**