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Exact Solutions to the (2+1)-Dimensional Boussinesq Equation via exp(Φ(η))-Expansion Method

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Abstract

The exp*(Φ(η))*-expansion method is applied to find exact traveling wave solutions to the (2+1)-dimensional Boussinesq equation which is an important equation in mathematical physics. The traveling wave solutions are expressed in terms of the exponential functions, the hyperbolic functions, the trigonometric functions and the rational functions. The procedure is simple, direct and constructive without the help of a computer algebra system. The applied method will be used in further works to establish more new solutions for other kinds of nonlinear evolution equations arising in mathematical physics and engineering.

Keywords: exp*(Φ(η))*-Expansion method; (2+1)-Dimensional Boussinesq equation; Traveling wave solutions; Solitary wave solutions.

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1. Introduction

Nonlinear complex physical phenomena are related to nonlinear partial differential equations (PDEs) which are involved in many fields from physics to biology, chemistry, mechanics, etc. Searching for exact solutions of nonlinear PDEs plays an important role in the study of these physical phenomena and gradually becomes one of the most important and significant tasks.

A great deal of research work has been carried out during the past decades for the study of the nonlinear evolution equation. Powerful methods which make it possible to generate exact traveling wave solutions to nonlinear equations have emerged from the literatures in the past decades. Among them are the tanh-function method [1], the Hirota's bilinear method [2], the auxiliary equation method [3], the inverse scattering transform [4], the

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complex hyperbolic function method [5,6], the rank analysis method [7], the ansatz method [8,9], the *(G′/G)*-expansion method [10-21], the exp-functions method [22], the modified simple equation method [23,24], the Jacobi elliptic function expansion method [25,26], the Adomian decomposition method [27,28], the homogeneous balance method [29-31], the F-expansion method [32,33], the Backlund transformation method [34], the Darboux transformation method [35], the homotopy perturbation method [36,37], the generalized Riccati equation method [38], the tanh-coth method [39], the exp$(-\varphi(\eta))$-expansion method [40-43] and so on.

The objective of this article is to implement the exp$(-\varphi(\eta))$-expansion method to construct exact solutions for the nonlinear evolution equations in mathematical physics via the (2+1)-dimensional Boussinesq equation for the first time. However, the exp$(-\varphi(\eta))$-expansion method have provided some new analytical solutions than the other methods. The (2+1)-dimensional Boussinesq equation is an important class of NLEEs and arises in physics, biophysics, optical fibers, propagation of shallow water waves, plasma physics and quantum mechanics to analyze the basic properties of nonlinear propagation of many physical phenomena. Water flow to subterranean drains is described by the (2+1)-dimensional Boussinesq equation. The (2+1)-dimensional Boussinesq equation describes the propagation of long waves in shallow water under gravity propagating in both directions. It also arises in other physical applications such as nonlinear lattice waves, iron sound waves in plasma and in vibrations in a nonlinear string. It is used in many physical applications such as the percolation of water in porous subsurface of a horizontal layer of material.

The rest of the paper is organized as follows: In Section 2, we give a description of the exp$(-\varphi(\eta))$-expansion method. In Section 3, we apply this method to the (2+1)-dimensional Boussinesq equation and graphical representations of the solutions. In Section 4, we have compared the obtained solutions with Zheng's [21] solutions and Conclusions are given in the last section.

2. Materials and Methods

In this section, we briefly highlight the main features of the exp$(\Phi(\eta))$-expansion method. We refer to [40-43] for more details. The nonlinear wave and evolution equations we want to investigate are commonly written as

$$u_t = F(u, u_x, u_{xx}, u_{yy} \ldots) \text{ or } u_{tt} = F(u, u_x, u_{xx}, u_{yy} \ldots) \tag{1}$$

Where $u=u(x,t)$ is an unknown function, F is a polynomial in $u(x,t)$ and its derivatives in which highest order derivatives and nonlinear terms are involved and the subscripts stand for the partial derivatives. In order to solve eq. (1) by this method, one has to resort the following steps:

Step 1: To find the traveling wave solution of (1), introduce the wave variable $\eta=x+y\pm Vt$, so that $u=u(x,t)=u(\eta)$. Based on this

$$\frac{\delta}{\delta t}=-V\frac{\delta}{\delta\eta},\ \frac{\delta^2}{\delta t^2}=V^2\frac{\delta^2}{\delta\eta^2},\ \frac{\delta}{\delta x}=\frac{\delta}{\delta\eta},\ \frac{\delta^2}{\delta x^2}=\frac{\delta^2}{\delta\eta^2}, \tag{2}$$

and so on for other derivatives. With the help of (2), the NLEE (1) changes to an ODE as

$$\Re(u,u',u'',u''',\cdots)=0\,, \tag{3}$$

where $\Re$ is a polynomial of u and its derivatives and the superscripts indicate the ordinary derivatives with respect to η.

Step 2: Suppose the traveling wave solution of Eq. (3) can be expressed as follows:

$$u(\eta)=A_0+\sum_{i=1}^{N}A_i(\exp(-\Phi(\eta)))^i\,, \tag{4}$$

where $A_i (0 \le i \le N)$ are constants to be determined, such that $A_N \neq 0$ and $\Phi = \Phi(\eta)$ satisfies the following ordinary differential equation:

$$\Phi'(\eta)=\exp(-\Phi(\eta))+\mu\exp(\Phi(\eta))+\lambda, \tag{5}$$

Eq. (5) gives the following solutions:

Cluster 1: When $\mu \neq 0$, $\lambda^2 - 4\mu > 0$,

$$\Phi(\eta)=\ln\left(\frac{-\sqrt{(\lambda^2-4\mu)}\tanh\left(\frac{\sqrt{(\lambda^2-4\mu)}}{2}(\eta+E)\right)-\lambda}{2\mu}\right) \tag{6}$$

Cluster 2: When $\mu \neq 0$, $\lambda^2 - 4\mu < 0$,

$$\Phi(\eta)=\ln\left(\frac{\sqrt{(4\mu-\lambda^2)}\tan\left(\frac{\sqrt{(4\mu-\lambda^2)}}{2}(\eta+E)\right)-\lambda}{2\mu}\right) \tag{7}$$

Cluster 3: When $\mu = 0$, $\lambda \neq 0$, and $\lambda^2 - 4\mu > 0$,

$$\Phi(\eta)=-\ln\left(\frac{\lambda}{\exp(\lambda(\eta+E))-1}\right) \tag{8}$$

Cluster 4: When $\mu \neq 0$, $\lambda \neq 0$, and $\lambda^2 - 4\mu = 0$,

$$\Phi(\eta)=\ln\left(-\frac{2(\lambda(\eta+E)+2)}{\lambda^2(\eta+E)}\right) \tag{9}$$

Cluster 5: When $\mu = 0$, $\lambda = 0$, and $\lambda^2 - 4\mu = 0$,

$$\Phi(\eta)=\ln(\eta+E) \tag{10}$$

A_N,----,V, λ, μ are constants to be determined latter, $An \neq 0$, the positive integer N can be determined by considering the homogeneous balance between the highest order derivatives and the nonlinear terms appearing in Eq. (3).

Step 3: We substitute Eq. (4) into Eq. (3) and then we account the function exp$(\Phi(\eta))$. As a result of this substitution, we get a polynomial of exp$(\Phi(\eta))$. We equate all the coefficients of same power of exp$(\Phi(\eta))$ to zero. This procedure yields a system of algebraic equations whichever can be solved to find A_N,---,V, λ, μ. Substituting the values of A_N,---,V, λ, μ into Eq. (4) along with general solutions of Eq. (5) completes the determination of the solution of Eq. (1).

3. Exact solution for the (2+1)-dimensional Boussinesq equation and the graphical representions to the obtained solutions

3.1. *Exact solution for the (2+1)-dimensional Boussinesq equation*

In this sub-section we will solve the (2+1)-dimensional Boussinesq equation which contains the second order partial derivative u_{tt} in addition to other partial derivatives. This family of nonlinear equations gained its importance because it appears in many scientific applications and physical phenomena. The new family is of the form $u_{tt} - u_{xx} - u_{yy} + p(u) = 0$, where $u(x, y, t)$ is a function of space x, y and time variable t and the nonlinear term $P(u) = -u_{xxx}-(u^2)_{xx}$. Therefore, the equation becomes u_{tt}-u_{xx}-u_{yy}-u_{xxxx}-$(u^2)_{xx}$ = 0, with $u(x,y,t)$ is a sufficiently often differentiable function. This is called the (2+1)-dimensional Boussinesq equation. The (2+1)-dimensional Boussinesq equation was introduced by Boussinesq to describe the propagation of long waves in shallow water under gravity propagating in both directions. The (2+1)-dimensional Boussinesq equation describes motions of long waves in shallow water under gravity and in a two-dimensional nonlinear lattice. This particular form the (2+1)-dimensional Boussinesq equation is of special interest because it is completely integrable and admits inverse scattering formalism. However, the good Boussinesq equation or the well-posed equation can be handled in a like manner.

Let us consider the (2+1)-dimensional Boussinesq equation

$$u_{tt} - u_{xx} - u_{yy} - u_{xxxx} - (u^2)_{xx} = 0. \qquad (11)$$

We use the traveling wave variable $u(\eta)=u(x,t)$, $\eta=x+y-Vt$, Eq. (11) is carried to an ODE

$$V^2u'' - 2u'' - u^{iv} - (u^2)'' = 0. \qquad (12)$$

Eq. (12) is integrable, therefore, integrating twice with respect to η once yields:

$$(V^2 - 2)u - u'' - u^2 + C = 0, \qquad (13)$$

where C is an integration constant which is to be determined.
Taking the homogeneous balance between highest order nonlinear term u^2 and linear term of the highest order u'' in Eq. (13), we obtain N=2. Therefore, the solution of Eq. (13) is of the form:

$$u(\eta) = A_0 + A_1(\exp(-\Phi(\eta))) + A_2(\exp(-\Phi(\eta)))^2, \tag{14}$$

where A_0, A_1, A_2 are constants to be determined such that $A_N \neq 0$, while λ, μ are arbitrary constants.
Substituting Eq. (14) into Eq. (13) and then equating the coefficients of exp$(\Phi(\eta))$ to zero, we obtain

$$-6A_2 - A_2^{\,2} = 0, \tag{15}$$

$$-2A_1 - 10A_2\lambda - 2A_1A_2 = 0, \tag{16}$$

$$-2A_2 + V^2A_2 - 8A_2\mu - A_1^{\,2} - 3A_1\lambda - 2A_0A_2 - 4A_2\lambda^2 = 0, \tag{17}$$

$$-2A_0A_1 + V^2A_1 - A_1\lambda^2 - 2A_1 - 6A_2\mu\lambda - 2A_1\mu = 0, \tag{18}$$

$$C + V^2A_0 - 2A_0 - 2A_2\mu^2 - A_0^{\,2} - A_1\lambda\mu = 0, \tag{19}$$

Solving the Eqs. (15)-(19) yields

$$C = -12\mu^2 - A_0^{\,2} - 6\lambda^2\mu - \lambda^2A_0 - 8A_0\mu,\ V = \pm\sqrt{\lambda^2 + 2 + 8\mu + 2A_0},\ A_0 = A_0,\ \ A_1 = -6\lambda,\ \ A_1 = -6\,.$$

where λ, μ are arbitrary constants.
Now substituting the values of V, A_o, A_1, A_2 into Eq. (14) yields

$$u(\eta) = A_0 - 6\lambda(\exp(-\Phi(\eta))) - 6(\exp(-\Phi(\eta)))^2, \tag{20}$$

where $\eta = x - (\pm\sqrt{\lambda^2 + 2 + 8\mu + 2A_0})t\,.$

Now substituting Eqs. (6)-(10) into Eq. (20) respectively, we get the following five traveling wave solutions of the (2+1)-dimensional Boussinesq equation.
When $\mu \neq 0$, $\lambda^2 - 4\mu > 0$,

$$u_1(\eta) = A_0 + \left(\frac{12\lambda\mu}{\sqrt{\lambda^2 - 4\mu}\tanh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}(\eta + E)\right) + \lambda}\right) - 6\left(\frac{2\mu}{\sqrt{\lambda^2 - 4\mu}\tanh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}(\eta + E)\right) + \lambda}\right)^2.$$

where $\eta = x - (\pm\sqrt{\lambda^2 + 2 + 8\mu + 2A_0})t$ and E is an arbitrary constant.
When $\mu \neq 0$, $\lambda^2 - 4\mu < 0$,

$$u_2(\eta)=A_0-(\frac{12\lambda\mu}{\sqrt{4\mu-\lambda^2}\tan(\frac{\sqrt{4\mu-\lambda^2}}{2}(\eta+E))-\lambda})$$
$$-6(\frac{2\mu}{\sqrt{4\mu-\lambda^2}\tan(\frac{\sqrt{4\mu-\lambda^2}}{2}(\eta+E))-\lambda})^2.$$

where $\eta=x-(\pm\sqrt{\lambda^2+2+8\mu+2A_0})t$ and E is an arbitrary constant.
When μ=0, $\lambda\neq 0$, and $\lambda^2-4\mu>0$

$$u_3(\eta)=A_0-(\frac{6\lambda^2}{\exp(\lambda(\eta+E))-1})-6(\frac{\lambda}{\exp(\lambda(\eta+E))-1})^2.$$

where $\eta=x-(\pm\sqrt{\lambda^2+2+8\mu+2A_0})t$ and E is an arbitrary constant.
When $\mu\neq 0$, $\lambda\neq 0$, and $\lambda^2-4\mu=0$,

$$u_4(\eta)=A_0+(\frac{3\lambda^3(\eta+E)}{(\lambda(\eta+E))+2)})-\frac{3}{2}(\frac{\lambda^2(\eta+E)}{(\lambda(\eta+E))+2)})^2.$$

where $\eta=x-(\pm\sqrt{\lambda^2+2+8\mu+2A_0})t$ and E is an arbitrary constant.
When $\mu=0$, $\lambda=0$, and $\lambda^2-4\mu=0$,

$$u_5(\eta)=A_0-\frac{6\lambda}{(\eta+E)}-6(\frac{1}{(\eta+E)})^2.$$

where $\eta=x-(\pm\sqrt{\lambda^2+2+8\mu+2A_0})t$ and E is an arbitrary constant.

3.2. *Graphical representations of the obtained solutions*

The graphical illustrations of the solutions are given below in the Figs. with the aid of Maple.

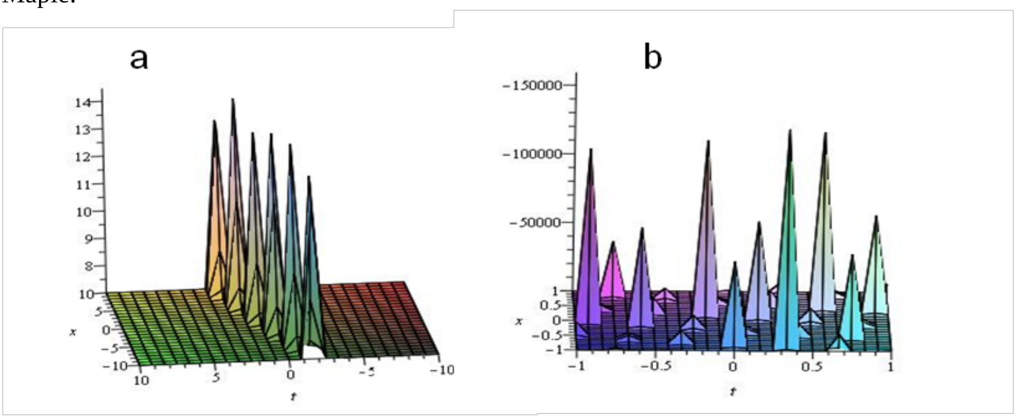

Fig. 1. a) Traveling wave solution $u_1(\eta)$ when μ =1, λ=3, y=0, E=1, A_0=1 and -10 ≤ x,t ≤ 10; b) Traveling wave solution $u_2(\eta)$ when μ =3, λ=1, y=0, E=1, A_0=1 and -1 ≤ x,t ≤ 1.

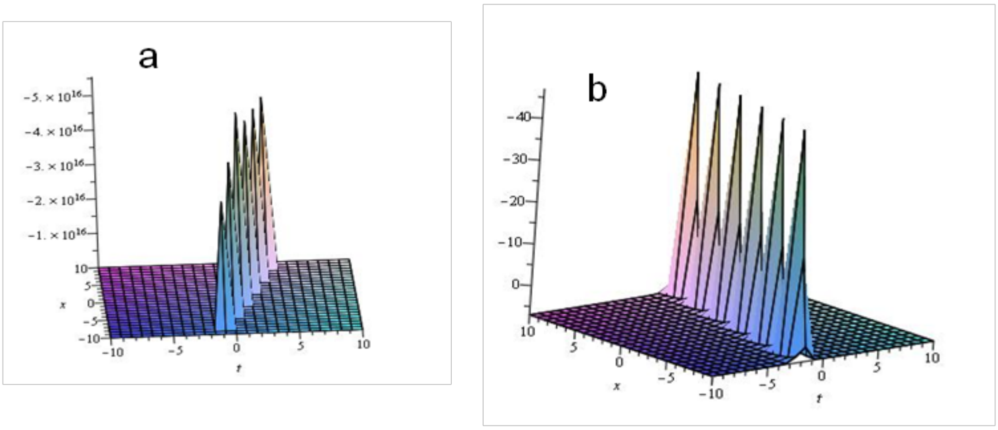

Fig. 2. a) Traveling wave solution $u_3(\eta)$ when $\mu=0$, $\lambda=2$, y=0, E=5, $A_0=5$ and $-10 \leq x,t \leq 10$); b) Traveling wave solution $u_4(\eta)$ when $\mu=1$, $\lambda=2$, y=0, E=1, $A_0=1$ and $-10 \leq x,t \leq 10$).

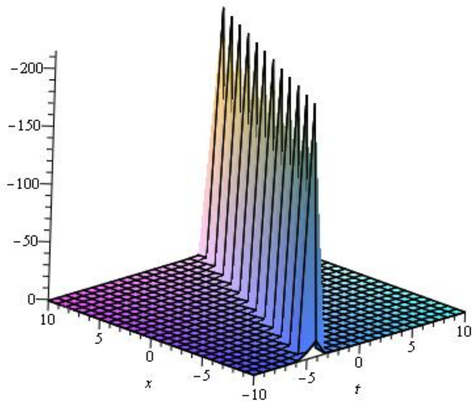

Fig. 3. Traveling wave solution $u_5(\eta)$ when $\mu=0$, $\lambda=0$, y=0, E=1, y=0, $A_0=1$ and $-10 \leq x,t \leq 10$).

4. Comparisons between the *(G′/G)*-expansion method and the exp*(-Φ(ξ))*-expansion method of the (2+1)-dimensional Boussinesq equation

Zheng [21] examined exact traveling wave solutions of the (2+1)-dimensional Boussinesq equation by using the *(G′/G)*-expansion method and obtained only three solutions (A1)-(A3) (see appendix). On the contrary by using the exp*(-Φ(ξ))*-expansion method we have obtained five solutions (See section 3) at least two of which are different from Zheng [21] solutions. On the other hand, the auxiliary equation used in this paper is different.

5. Conclusion

In this article, the exp*(-Φ(η))*-expansion method has successfully been applied to find the exact solutions for nonlinear partial differential equations, such as the (2+1)-dimensional Boussinesq equation. The exp*(-Φ(η))*-expansion method is used to find a new traveling wave solution. The results show that the exp*(-Φ(η))*-expansion method is a powerful mathematical tool to solve the (2+1)-dimensional Boussinesq equation; it is also a promising method to solve other nonlinear equations.

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Appendix: Zheng solutions [21]

Zheng [21] studied the exact traveling wave solutions of the (2+1)-dimensional Boussinesq equation by using the *(G′/G)*-expansion method and achieved the following three exact solutions:

$$u_1(\xi)=\frac{3}{2}k^2\lambda^2-\frac{3}{2}k^2(\lambda^2-4\mu)\times\frac{C_1\sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)+C_2\cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)}{C_1\cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)+C_2\sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)} -\frac{1}{2}\frac{k^2+k^4\lambda^2+8k^4\mu-m^2+l^2}{k^2} \qquad \text{(A.1)}$$

$$u_2(\xi)=\frac{3}{2}k^2\lambda^2-\frac{3}{2}k^2(4\mu-\lambda^2)\times\left(\frac{-C_1\sin(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi)+C_2\cos(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi)}{C_1\cos(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi)+C_2\sin(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi)}\right)^2 -\frac{1}{2}\frac{k^2+k^4\lambda^2+8k^4\mu-m^2+l^2}{k^2} \tag{A.2}$$

$$u_3(\xi)=\frac{3}{2}k^2\lambda^2-\frac{6k^2{C_2}^2}{(C_1+C_2\xi)^2}-\frac{1}{2}\frac{k^2+k^4\lambda^2+8k^4\mu-m^2+l^2}{k^2} \tag{A.3}$$

where $\xi=kx+ly+mt+d$, k, l, m, d, C_1, C_2 are arbitrary constants.