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Vertex Equitable Labeling of Cyclic Snakes and Bistar Graphs

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Abstract

Let G be a graph with p vertices and q edges and let $A = \{0, 1, 2, \ldots, \lceil \frac{q}{2} \rceil\}$. A vertex labeling $f: V(G) \to A$ induces an edge labeling f^* defined by $f^*(uv) = f(u) + f(v)$ for all edges uv. For $a \in A$, let $v_f(a)$ be the number of vertices v with $f(v) = a$. A graph G is said to be vertex equitable if there exists a vertex labeling f such that for all a and b in A, $|v_f(a) - v_f(b)| \le 1$ and the induced edge labels are 1, 2, 3,…, q. In this paper, we establish vertex equitable labeling of square graph of $B_{n,n}$ and splitting graph of $B_{n,n}$ and kC_4 –snake($k\ge1$) and the generalized kC_n –snake is vertex equitable if $n\equiv0(\mathrm{mod}\,4), n\ge4$.

Keywords: Vertex equitable labeling; Vertex equitable graph.

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1. Introduction

All graphs considered here are simple, finite, connected and undirected .We follow the basic notations and terminologies of graph theory as in [1]. A graph labeling is an assignment of integers to the vertices or edges or both, subject to certain conditions. There are several types of labeling. A detailed survey of graph labeling can be found in ref. [2].

The concept of vertex equitable labeling was introduced by Lourdusamy and Seenivasan [3]. Let G be a graph with p vertices and q edges and $A=\{0,1,2, \ldots, \lceil \frac{q}{2} \rceil\}$. A graph G is said to be vertex equitable if there exists a vertex labeling $f : V \to A$ that induces an edge labeling f^* defined by $f^*(uv) = f(u) + f(v)$ for all edges uv such that for all

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a and *b* in *A*, $\left|v_f(a) - v_f(b)\right| \leq 1$ and the induced edge labels are 1, 2, 3,…, *q*, where $v_f(a)$ be the number of vertices *v* with *f(v) = a* for $a \in A$. The vertex labeling *f* is known as vertex equitable labeling. A graph *G* is said to be a vertex equitable graph if it admits vertex equitable labeling. They proved that the graphs like path, bistar *B(n,n),* combs, cycle C_n if $n \equiv 0$ or 3 (mod 4), $K_{2,n}$, $C_3^{(t)}$ for $t \geq 2$, quadrilateral snake, $K_2 + mK_1$, $K_{1,n} \cup K_{1,n+k}$ iff $1 \leq k \leq 3$, ladder, arbitrary super division of any path and cycle C_n with $n \equiv 0$ or 3 (mod4) are vertex equitable. Also they proved that the graphs $K_{1,n}$ if $n \geq 4$, any Eulerian graph with n edges where $n \equiv 1$ or 2 (mod4), the wheel W_n, the complete graph K_n if $n > 3$ and triangular cactus with $q \equiv 0$ or 6 or 9 (mod12) are not vertex equitable. In addition, they proved that if *G* is a graph with *p* vertices and *q* edges, *q* is even and $p < \left\lceil \frac{q}{2} \right\rceil + 2$ then *G* is not vertex equitable graph.

Jeyanthi and Maheswari [4-7] proved that T_p-tree, $T \,\square\, \overline{K_n}$ where *T* is a T_p-tree with even number of vertices, $T\,\hat{o}P_n$, $T\,\hat{o}2P_n$, $T\hat{o}C_n$ ($n \equiv$ 0,3 (mod4)), $T\tilde{o}C_n$ ($n \equiv$ 0,3 (mod4)), bistar *B(n, n+1),* the caterpillar $S(x_1, x_2, \ldots, x_n)$ and $C_n \,\square\, K_1$, P_n^2 ,tadpoles, $C_m \oplus C_n$, armed crowns, $[P_m; C_n^2]$, $\left\langle P_m \hat{o} K_{1,n} \right\rangle$ and the graphs obtained by duplicating an arbitrary vertex and an arbitrary edge of a cycle C_n, total graph of P_n, splitting graph of P_n and fusion of two edges of a cycle C_n are vertex equitable graphs. In this paper we prove that kC_4 –snakes for all $k \geq 1$ are vertex equitable graph and generalized kC_n –snake is vertex equitable if $n \equiv 0 (\text{mod}4), n \geq 4$.Also we prove that square graph of $B_{n,n}$ and splitting graph of $B_{n,n}$ are vertex equitable.

The following definitions are used in the present study.

Definition 1.1 A kC_n-snake is defined as a connected graph in which all the *k*-blocks are isomorphic to the cycle C_n and the block-cut point graph is a path. Let P be the path of minimum length that contains all the cut vertices of a kC_n –snake. Barrientos [8] has proved that any kC_n-snake is represented by a string $s_1, s_2, \ldots, s_{k-2}$ of integers of length *k*-2 where the i^{th} integer, s_i on the string is the distance between i^{th} and $(i+1)^{th}$ cut vertices on the path *P* from one extreme and is taken from $S_n = \{1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor\}$.The strings obtained for both extremes are assumed to be the same. Then there are at most $\left\lfloor \frac{n}{2} \right\rfloor^{k-2}$ non isomorphic kC_n –snakes. For example, the string of a $10C_4$-snake is shown in Fig.1 is 2,2,1,2,1,1,2,1, , , .A kC_n –snake is said to be linear if each integer of its string is $\left\lfloor \frac{n}{2} \right\rfloor$.

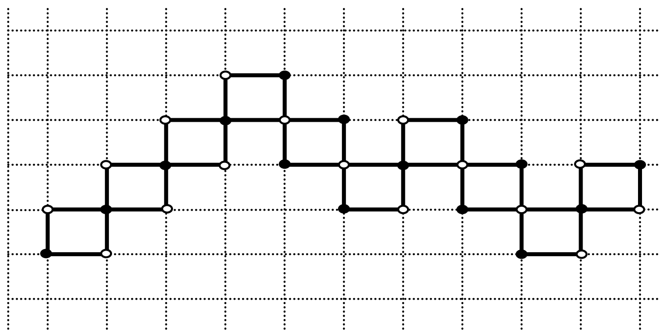

Fig. 1.

Definition 1.2 A generalized kC_n-snake is defined as a connected graph in which each block is isomorphic to a cycle C_n for some n and the block-cut point graph is a path. It is denoted by $CS(n_1,n_2,\ldots,n_k)$ where $B_1, B_2,\ldots,B_k$ are the consecutive blocks and B_i is isomorphic to C_{n_i}. By applying the same methods used to obtain the strings of a kC_n-snake, we can show that any generalized kC_n-snake can also be represented by a string of integers $s_1,s_2,\ldots,s_{k-2}$ of length k-2 where $s_{i-1} \in S_{n_i}$.

Definition 1.3 The square of a graph G denoted by G^2 has the same vertex set $V(G)$ and the two vertices are adjacent in G^2 if they are at a distance of 1 or 2 in G.

Definition 1.4 Let G be a graph. For each point v of a graph G, take a new point v' and join v' to the vertices of G which are adjacent to v. The graph thus obtained is called the splitting graph of G and is denoted by $S'(G)$.

2. Main Results

Theorem 2.1: The kC_4 –snake is a vertex equitable graph.

Proof: Let G be a kC_4 –snake with k blocks. As G is bipartite, let one partite set has black vertices and the other white vertices. Therefore, G can be embedded on a square grid as shown in Figure 1.

Consider the assignment of numbers to the vertices of C_4 as shown in Fig. 2 where x is an even integer. Then the induced edge labels are $2x$+1, $2x$+2, $2x$+3, $2x$+4 four successive integers starting from $2x$+1.

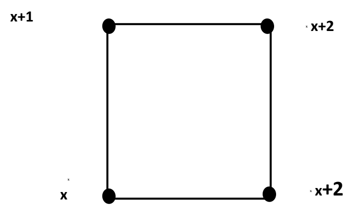

Fig. 2.

The vertices of G are labeled as follows: The black vertices in the diagonals, except the first vertex of each diagonal, ordered from left to right and inside the diagonals from bottom to top are assigned the numbers from an arithmetic progression of common difference and first term both equal to 2.

The first vertex in the first diagonal is labeled with 0 and the first vertex in each of the remaining diagonals is labeled with the integer which is one more than the label assigned with the last vertex of the previous diagonal.

The white vertices in the diagonals, except the first vertex of each diagonal are ordered from left to right and inside the diagonals from top to bottom are assigned the numbers from an arithmetic progression of common difference 2 and first term 2.

The first vertex in the first diagonal is labeled with 1 and the first vertex in each of the remaining diagonals is labeled with the integer which is one more than the label assigned for the last vertex of the previous diagonal.

The labeling of each copy of C_4 is the same as in Figure 2 and the black and white vertices are labeled respectively with integers from *0, 1, 2,…,2k*. It can be easily verified that the set of induced edge labels defined by the sum of the labels on the end vertices is the set *1,2,…,4k* and $|v_f(a) - v_f(b)| \le 1$ for all $a, b \in 0, 1, 2, \dots, 2k$ .Then G is a vertex equitable graph.

An example for the **v**ertex equitable labeling of $10C_4$ –snake is shown in Fig. 3.

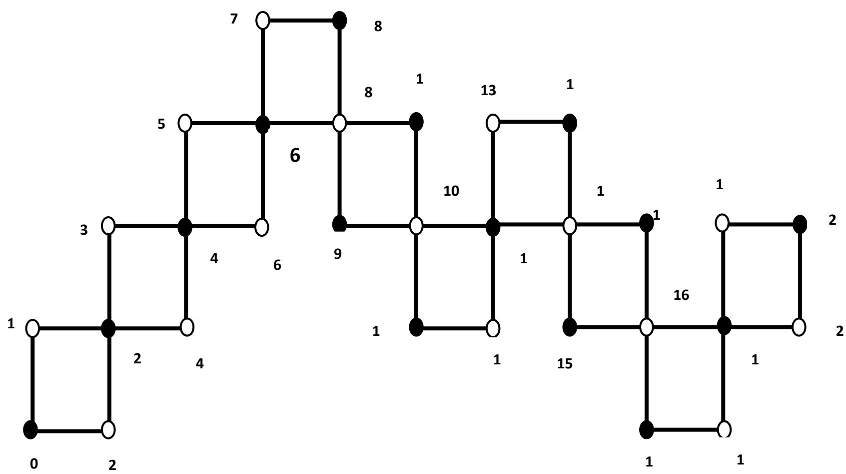

Fig. 3

Theorem 2.2: Let $G = CS(n_1,n_2,\dots,n_k)$, $n_i \equiv 0(\text{mod}4), n_i \geq 4$, be a generalized kC_n –snake with its strings $s_1,s_2,\dots,s_{k-2}$ where $s_i \in 1, 2$, $1 \le i \le k$. Then G is a vertex equitable graph.

Proof: Let $B_1, B_2, \ldots, B_k$ be the consecutive blocks of G. Here each block B_i is isomorphic to C_{ni}. Let $V(C_{ni}) = v_{ij} : 1 \le j \le n_i$ be the vertex set of the cycle C_{ni} $(1 \le i \le k)$. First we define a vertex labeling f_i for the cycle C_{ni} $(1 \le i \le k)$, as $f_i : V(C_{ni}) \rightarrow \left\{ x, x+1, x+2, \ldots, x+\frac{n_i}{2} \right\}$ as follows with $x = 0$ if $i = 1$ and $x = \frac{n_1 + n_2 + \ldots + n_{i-1}}{2}$ *if* $i = 2, 3, \ldots, k.$

$$f(v_{ij}) = \begin{cases} x + \left\lfloor \frac{j}{2} \right\rfloor & \text{if } 1 \le j \le \frac{n_i}{2} \\ x + \left\lceil \frac{j}{2} \right\rceil & \text{if } \frac{n_i}{2} + 1 \le j \le n_i \end{cases}.$$

Then the edge labels induced on the blocks B_i $(1 \le i \le k)$ are $\sum_{r=1}^{i-1} n_r + 1$, $\sum_{r=1}^{i-1} n_r + 2$, …, $\sum_{r=1}^{i} n_r$. Therefore, the edge labels induced on G are 1, 2, 3,…, $\sum_{r=1}^{k} n_r$. To get the vertex equitable labeling of G, identify the vertex v_{21} of C_{n2} with either v_{n1} or v_{n_1-1} of C_{n1} and identify the vertex $v_{i+1\,1}$ of C_{n_i+1} with v_{in_i} of C_{ni} if s_i=1 or with v_{in_i-1} if s_i=2 for $2 \le i \le k.$

An example for the vertex equitable labeling of CS (8,4,12,8,16) is shown in Fig. 4.

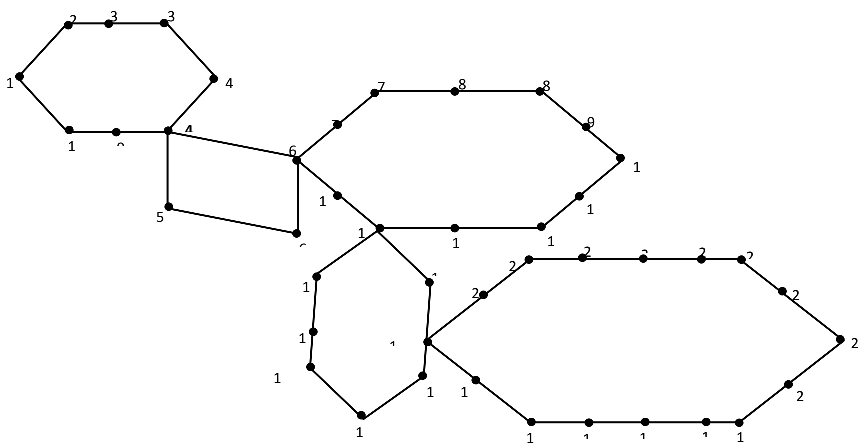

Fig. 4

Theorem 2.3: The square graph $B^2_{n,n}$ is a vertex equitable graph.

Proof: Let $V(B_{n,n})$ = $u, v, u_i, v_i, 1 \le i \le n$ where u_i, v_i are the pendant vertices adjacent to u and v respectively. Let G = $B^2_{n,n}$. Then $|V(G)| = 2n+2$ and $|E(G)| = 4n+1$. Let A = {0, 1, 2,..., $\left\lceil \frac{4n+1}{2} \right\rceil$}. Define f: $V(B^2_{n,n}) \rightarrow A$ as follows: $f(u)=0$, $f(v)=\left\lceil \frac{4n+1}{2} \right\rceil$, $f(u_i) = i$,$1 \le i \le n$ and $f(v_i)=n+i$,$1 \le i \le n$. It can be verified that the induced edge labels of $B^2_{n,n}$ are 1, 2,…,4n+1 and $|v_f(i) - v_f(j)| \le 1$ for $i, j \in A$. Therefore, $B^2_{n,n}$ is a vertex equitable graph.

An example for the vertex equitable labeling of $B^2_{4,4}$ is given in Fig.5.

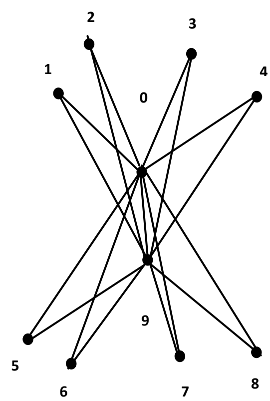

Fig. 5

Theorem 2.4: The splitting graph $S'(B_{n,n})$ is a vertex equitable graph.

Proof: Let $V(B_{n,n})$ = $u, v, u_i, v_i, 1 \le i \le n$ where u_i, v_i are the pendant vertices adjacent to u and v respectively. Let u', v', u_i', v_i' be the vertices corresponding to u, v, u_i, v_i ($1 \le i \le n$) in $S'(B_{n,n})$ and let G = $S'(B_{n,n})$. Then $|V(G)| = 4(n+1)$ and $|E(G)| = 6n+3$. Let A = {0, 1, 2,..., $\left\lceil \frac{6n+3}{2} \right\rceil$}. Define f: $V(S'(B_{n,n})) \rightarrow A$ as follows: $f(u)$ =0, $f(v)=\left\lceil \frac{6n+3}{2} \right\rceil$, $f(u') = n+1$, $f(v')$ = 2n+1. For $1 \le i \le n$ $f(u_i) = n+i$, $f(u_i') = i$, $f(v_i)$ = n+1+i and $f(v_i')$ =2n+1+i . It can be verified that the induced edge labels of $S'(B_{n,n})$

are 1, 2,…,6n+3 and $|v_f(i) - v_f(j)| \le 1$ for i, $j \in A$. Therefore, $S'(B_{n,n})$ is a vertex equitable graph.

An example for the vertex equitable labeling of $S'(B_{4,4})$ is given in Fig. 6.

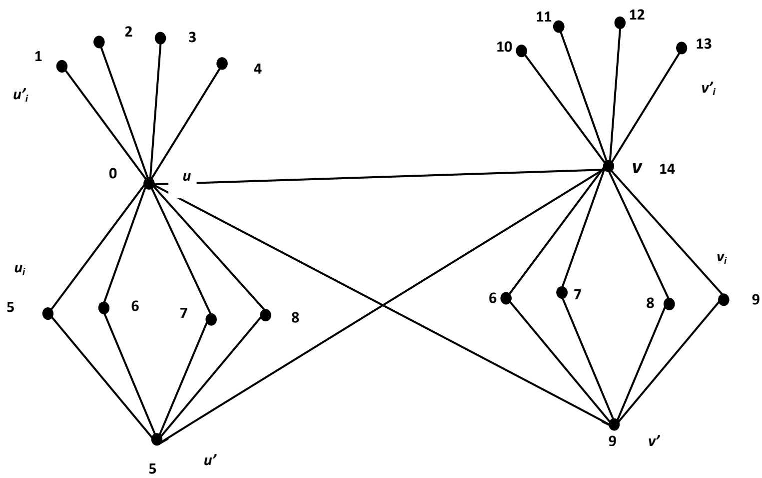

Fig. 6

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