Dynamics of Boundary Graphs

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Abstract

In a graph G, the distance \( d(u,v) \) between a pair of vertices \( u \) and \( v \) is the length of a shortest path joining them. A vertex \( v \) is a boundary vertex of a vertex \( u \) if \( d(u,w) \leq d(u,v) \) for all \( w \in N(v) \). The boundary graph \( B(G) \) based on a connected graph \( G \) is a simple graph which has the vertex set as in \( G \). Two vertices \( u \) and \( v \) are adjacent in \( B(G) \) if either \( u \) is a boundary of \( v \) or \( v \) is a boundary of \( u \). If \( G \) is disconnected, then each vertex in a component is adjacent to all other vertices in the other components and is adjacent to all of its boundary vertices within the component. Given a positive integer \( m \), the \( m \)th iterated boundary graph of \( G \) is defined as \( B^m(G) = B(B^{m-1}(G)) \). A graph \( G \) is periodic if \( B^m(G) \cong G \) for some \( m \). A graph \( G \) is said to be an eventually periodic graph if there exist positive integers \( m \) and \( k > 0 \) such that \( B^{mk+i}(G) \cong B^i(G) \), \( \forall i \geq k \). We give the necessary and sufficient condition for a graph to be eventually periodic.

Keywords: Boundary graph; Periodic graph.

1. Introduction and Definitions

The graphs considered here are nontrivial and simple. For other graph theoretic notation and terminology, we follow [1]. In a graph \( G \), the distance \( d(u,v) \) between a pair of vertices \( u \) and \( v \) is the length of a shortest path joining them. The eccentricity \( e(u) \) of a vertex \( u \) is the distance to a vertex farthest from \( u \). The radius \( r(G) \) of \( G \) is defined as \( r(G) = \min\{e(u) : u \in V(G)\} \) and the diameter \( d(G) \) of \( G \) is defined as \( d(G) = \max\{e(u) : u \in V(G)\} \). A graph \( G \) for which \( r(G) = d(G) \) is called a self-centered graph of radius \( r(G) \). A vertex \( v \) is called an eccentric vertex of a vertex \( u \) if \( d(u,v) = e(u) \).

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A vertex \( v \) of \( G \) is called an eccentric vertex of \( G \) if it is an eccentric vertex of some vertex of \( G \). The eccentric graph based on \( G \) is denoted by \( G_e \), whose vertex set is \( V(G) \) and two vertices \( u \) and \( v \) are adjacent in \( G_e \) if and only if \( d(u,v) = \min\{e(u),e(v)\} \).

Gimbert et al. [3] studied the iterations of eccentric digraphs. The eccentric digraph of a digraph \( G \), denoted by \( ED(G) \), is the digraph on the same vertex set as \( G \) but with an arc from a vertex \( u \) to a vertex \( v \) in \( ED(G) \) if and only if \( v \) is an eccentric vertex of \( u \) in \( G \).

Given a positive integer \( k \), the \( k^{th} \) iterated eccentric digraph of \( G \) is written as \( ED^k(G) = ED(ED^{k-1}(G)) \), where \( ED^0(G) = G \). For every digraph \( G \), there exists smallest integer \( p \geq 0 \) and \( t \geq 0 \) such that \( ED^p(G) \cong ED^{p+t}(G) \), where \( \cong \) denotes graph isomorphism. We call \( p \), the iso-period of \( G \) and \( t \), the iso-tail of \( G \); these quantities are denoted by \( p(G) \) and \( t(G) \), respectively.

Kathiuren et al. [4] introduced a new type of graph called radial graph. Two vertices of a graph \( G \) are said to be radial to each other if the distance between them is equal to the radius of the graph. Two vertices of graph \( G \) are said to be radial to each other if the distance between them is equal to the radius of the graph. The radial graph of a graph \( G \) denoted by \( R(G) \) has the vertex set as \( G \) and two vertices are adjacent in \( R(G) \) if and only if they are radial in \( G \). If \( G \) is disconnected, then two vertices are adjacent in \( R(G) \) if they belong to different components of \( G \). A graph \( G \) is called a radial graph if \( R(H) \cong G \) for some graph \( H \). In [5] Kathiresan et al. studied the properties of iteration of radial graphs. Given a positive integer \( m \), the \( m^{th} \) iterated radial graph of \( G \) is defined as \( R^m(G) = R(R^{m-1}(G)) \).

Note that \( R^0(G) \cong G \). A graph \( G \) is periodic if \( R^m(G) \cong G \) for some \( m \). If \( p \) is the least positive integer with this property, then \( G \) is called a periodic graph with iso-period \( p \). When \( p=1 \), \( G \) is called as a fixed graph. A graph \( G \) is said to be eventually periodic if there exist positive integers \( m \) and \( k \), such that \( R^{m+i}(G) \cong R^k(G), \forall i \geq k \). If \( p \) and \( k \) are the least positive integers with this property, then \( G \) is eventually periodic with iso-period \( p \) and iso-tail \( k \).

Based on the concept of radial graphs, Marimuthu and Sivanandha Saraswathy [6] introduced the concept of boundary graphs. A vertex \( v \) is a boundary vertex of a vertex \( u \) if \( d(u,w) \leq d(u,v) \) for all \( w \in N(v) \). The boundary graph \( B(G) \) based on a connected graph \( G \) is the graph which has the vertex set as \( G \). Two vertices \( u \) and \( v \) are adjacent in \( B(G) \) if either \( u \) is a boundary of \( v \) or \( v \) is a boundary of \( u \). If \( G \) is disconnected, then each vertex in a component is adjacent to all the vertices in the other components and is adjacent to all of its boundary vertices within the component. A graph \( G \) is called a boundary graph if there exists a graph \( H \) such that \( B(H) = G \). We defined the neighborhood \( N_B(u) = \{w \in N(v) / d(u,w) = k\} \).

Motivated by the work of J. Gimbert et al., [2,3] and KM. Kathiresan et al., [5], we study here an iterated version of a distance dependent mapping. Given a positive integer \( m \), the \( m^{th} \) iterated boundary graph of \( G \) is defined as \( B^m(G) = B(B^{m-1}(G)) \). Note that \( B^0(G) \cong G \).
Definition 1.1: A graph $G$ is periodic if $B^m(G) \cong G$ for some $m$. If $p$ is the least positive integer with this property, then $G$ is called a periodic graph with iso-period $p$. When $p = 1$, $G$ is called as a fixed graph.

Definition 1.2: A graph $G$ is said to be eventually periodic if there exist positive integers $m$ and $k > 0$, such that $B^{n+k}(G) \cong B^i(G), \forall i \geq k$. If $p$ and $k$ are the least positive integers with this property, then $G$ is called an eventually periodic graph with iso-period $p$ and iso-tail $k$.

Figs. 1, 2 and 3 illustrate these definitions showing boundary graph of $G$ and its iterated boundary graphs.

![Fig. 1. The graph $G$.](image1)

![Fig. 2. The graph $B(G)$.](image2)

![Fig. 3. The graph $B^2(G)$.](image3)

In the above example $B^2(G) \cong B(G)$. Here $k(G) = 1$ and $p(G) = 2$ where $k$ denotes the iso-tail and $p$ denotes the iso-period of $G$.

Let $F_{11}, F_{12}, F_{22}, F_{23}, F_{24}$ and $F_3$ denote the set of all graphs $G$ such that $r(G) = 1$ and $d(G) = 1$; $r(G) = 1$ and $d(G) = 2$; $r(G) = 2$ and $d(G) = 2$; $r(G) = 2$ and $d(G) = 3$; $r(G) = 2$
and \( d(G) = 4 \) and \( r(G) \geq 3 \) respectively and \( \mathcal{F}_4 \) denote the set of all disconnected graphs. It is well known that \( d(G) \geq 4 \) implies that \( d(\overline{G}) \leq 2 \).

2. Previous results

The following theorems are appeared in [6].

**Theorem 2.1** [6]: \( \mathcal{B}(G) = G \) if and only if \( G \) is complete.

**Theorem 2.2** [6]: For a graph \( G \in \mathcal{F}_4, \mathcal{B}(G) = K_n \) if and only if either 
\[ N(u) - \{v\} \subseteq N(v) - \{u\} \] or 
\[ N(v) - \{u\} \subseteq N(u) - \{v\} \] for any two adjacent vertices \( u \) and \( v \) of \( G \).

**Theorem 2.3** [6]: Let \( G \) be a graph. Then \( \mathcal{B}(G) = \overline{G} \) if and only if the following conditions hold.

(i) \( G \) has no complete vertex.

(ii) neither \( N(u) - \{v\} \subseteq N(v) - \{u\} \) nor \( N(v) - \{u\} \subseteq N(u) - \{v\} \) for any two adjacent vertices \( u \) and \( v \) of \( G \).

(iii) either \( N_k(v) = \emptyset \) or \( N_k(u) = \emptyset \) for any two non-adjacent vertices \( u \) and \( v \) of \( G \), where \( k = d(u, v) + 1 \).

**Theorem 2.4** [6]: If \( G \) has at least one isolated vertex, then \( G \) is not a boundary graph.

**Theorem 2.5** [6]: Let \( G \in \mathcal{F}_4 \) without isolated vertices. If \( \overline{G} \) without complete vertices has the following properties

(i) neither \( N(u) - \{v\} \subseteq N(v) - \{u\} \) nor \( N(v) - \{u\} \subseteq N(u) - \{v\} \) for any two adjacent vertices \( u \) and \( v \) of \( G \).

(ii) either \( N_k(u) = \emptyset \) or \( N_k(v) = \emptyset \) for any two non-adjacent vertices \( u \) and \( v \) of \( G \), where \( k = d(u, v) + 1 \) then, \( G \) is a boundary graph.

3. Main Results

**Proposition 3.1**: Every graph is either periodic or eventually periodic.

**Proof.** Consider the set \( A = \{B^m(G) : m=0,1,2,...\} \) where \( B^0(G) = G \). If \( G \) has \( n \) vertices, then \( B^m(G) \) also has \( n \) vertices. Moreover, the possible number of graphs in \( A \) is at most \( \frac{n(n-1)}{2} \). Thus, there exist non-negative integer \( k \) and positive integer \( m \) such that \( B^{m+k}(G) \cong B^k(G) \) and hence \( B^{m+i}(G) \cong B^k(G), \forall i \geq k \). If \( k = 0 \), then \( G \) is periodic. If \( k > 0 \), then \( G \) is eventually periodic.
Proposition 3.2: Let \( C_n \) be any cycle. Then \( C_n \) is periodic with iso-period 1 if it is odd and eventually periodic with iso-period 1 if it is even.

Proof. Case (i) If \( n \) is odd, \( B(C_n) \cong C_n \). Hence \( C_n \) is periodic with iso-period 1.

Case(ii) If \( n \) is even, \( B(C_n) = \frac{n}{2} K_2 \). a disconnected graph with each component \( K_2 \). By the definition of \( B(G) \), \( B^2(C_n) \) is a complete graph. Hence by Theorem 2.1, \( C_n \) is eventually periodic with iso-period 1.

Let us find some graphs of order \( n \) which is either periodic or eventually periodic.

Observation 3.3: \( C_n+C_n \) is a periodic graph for odd values of \( n \), \( \forall n \geq 3 \) whose \( k(G)=0 \) and \( p(G)=2 \) where + denotes the usual addition of graphs.

Observation 3.4: We also observed that \( p(C_m+C_n) = p(C_m) + p(C_n) \) where \( m = 2n+1 \), \( \forall n \geq 2 \).

Observation 3.5: \( C_{2m+1} \times C_{2m+1} \) is a fixed graph whose \( k(G)=0 \) and \( p(G)=1 \), \( \forall m \geq 1 \) where \( \times \) denotes the Cartesian product of graphs.

Observation 3.6: \( C_{2m} \times C_{2m} \) is eventually periodic with \( k(G)=2 \), \( p(G)=1 \).

Let us say that a class is periodic if every graph in the class is periodic. As we observed earlier \( C_n+C_n \), complete graph \( C_{2m+1} \times C_{2m+1} \) are periodic graphs.

Observation 3.7: Every complete n-partite graph with \( |V| \geq 2 \) for each \( i^{th} \) partition is eventually periodic with iso-period 1.

Proof. Let \( G \) be a complete n-partite graph with \( |V| \geq 2 \) for each \( i^{th} \) partition. Any two vertices \( v_i \) and \( v_j \) in \( G \) are adjacent in \( B(G) \) if and only if they are in the same partition. Therefore \( B(G) \) is a disconnected graph with each component complete. By the definition of boundary graph, \( B^2(G) \) is complete. By Theorem 2.1, \( G \) is eventually periodic with iso-period 1.

Proposition 3.8: Every path \( P_n \), \( n \geq 3 \) is eventually periodic with iso-period 1.

Proof. Let \( v_1, v_2 \ldots v_n \) be a path on \( n \) vertices. Since the end vertices are complete, \( B(P_n) \in F_{12} \). Further \( v_2, v_3 \ldots v_{n-1} \) are non-adjacent vertices of eccentricity 2 in \( B(P_n) \) and \( B^2(P_n) = K_n \). Hence by Theorem2.1, \( P_n \) is eventually periodic with iso-period 1.

Lemma 3.9: A graph \( G \in F_{12} \) is eventually periodic with iso-period 1 if and only if either \( N(u) - \{v\} \subseteq N(v) - \{u\} \) or \( N(v) - \{u\} \subseteq N(u) - \{v\} \) for any two adjacent vertices \( u \) and \( v \) of \( G \).
Let \( G \in F_{12} \). Assume for any two adjacent vertices \( u \) and \( v \) of \( G \), either 
\[ N(u) - \{v\} \subseteq N(v) - \{u\} \] 
or 
\[ N(v) - \{u\} \subseteq N(u) - \{v\} \] 
. Then by Theorem 2.2, 
\[ B(G) = K_n \] 
. Therefore \( B^2(G) = B(B(G)) = B(K_n) \cong K_n \) implies \( G \) is eventually periodic with iso-period 1.

Conversely, assume \( G \in F_{12} \) is eventually periodic. Suppose for any two adjacent vertices neither \( N(u) - \{v\} \subseteq N(v) - \{u\} \) nor \( N(v) - \{u\} \subseteq N(u) - \{v\} \). This implies \( uv \notin B(G) \). Therefore non-adjacent vertices in \( G \) are adjacent in \( B(G) \) together with the full degree vertices in \( G \) continue to have the same degree in \( B(G) \). Hence \( B(G) \in F_{12} \).

With the assumption of the condition mentioned for adjacent vertices, \( B^2(G) \equiv G \), implies \( G \) is periodic which is a contradiction.

**Lemma 3.10:** If \( G \) is not a boundary graph, then \( G \) is eventually periodic.

**Proof.** Since \( G \) is not a boundary graph, there is no graph \( H \) such that \( B(H) \cong G \). Therefore for any \( m \), \( B^n(H) \neq G, m \geq 1 \) and thus \( G \) is not a periodic graph. Hence by proposition 3.1, \( G \) is eventually periodic.

**Lemma 3.11:** Let \( G \) be a disconnected graph. If \( G \) has at least one isolated vertex, then \( G \) is eventually periodic.

**Proof.** Suppose that \( G \) has at least one isolated vertex, then by Theorem 2.4, \( G \) is not a boundary graph. Hence by Lemma 3.10, \( G \) is eventually periodic.

**Lemma 3.12:** Let \( G \in F_4 \) without isolated vertices. If \( \overline{G} \) without complete vertices has the following properties

(i) neither \( N(u) - \{v\} \subseteq N(v) - \{u\} \) nor \( N(v) - \{u\} \subseteq N(u) - \{v\} \) for any two adjacent vertices \( u \) and \( v \) of \( G \).

(ii) either \( N_k(u) = \phi \) or \( N_k(v) = \phi \) for any two non-adjacent vertices \( u \) and \( v \) of \( G \), where \( k = d(u,v)+1 \), then \( G \) is periodic.

**Proof.** With the above assumption by Theorem 2.5, \( G \) is a boundary graph. Then there exists a graph \( H \) such that \( B(H) \cong G \). Since \( G \) has \( n \) vertices, we can find a graph in the set of all graphs with \( n \) vertices such that \( B^m(G) \cong G \) for some least positive integer \( m \). Therefore \( G \) is periodic.

**Lemma 3.13:** Let \( G \) be a disconnected graph with at least one complete component. If for any two adjacent vertices \( u \) and \( v \) in \( B(G) \), either \( N(u) - \{v\} \subseteq N(v) - \{u\} \) or \( N(v) - \{u\} \subseteq N(u) - \{v\} \) then, \( G \) is eventually periodic with iso-period 1.

**Proof.** Let \( G \) be a disconnected graph with at least one complete component. Then \( B(G) \in F_{12} \). Since for any two adjacent vertices \( u \) and \( v \) in \( B(G) \), either
\[ N(u) - \{v\} \subseteq N(v) - \{u\} \text{ or } N(v) - \{u\} \subseteq N(u) - \{v\}. \]

Now, consider \( B^2(G) = B(B^2(G)) = K_n \cong B^3(G). \) Hence \( G \) is eventually periodic with iso-period 1.

**Open Problem 3.14:** Characterize all disconnected periodic graphs in which each component is non-complete.

**Theorem 3.15:** Let \( G \) be a connected graph. If the following conditions hold in two successive iterations in \( B^k(G), K \geq 1 \)

(i) No complete vertex

(ii) neither \( N(u) - \{v\} \subseteq N(v) - \{u\} \) nor \( N(v) - \{u\} \subseteq N(u) - \{v\} \) for any two adjacent vertices \( u \) and \( v \).

(iii) either \( N_k(u) = \emptyset \) or \( N_k(v) = \emptyset \) for any two non-adjacent vertices \( u \) and \( v \)

where \( k = d(u,v) + 1 \), then \( G \) is eventually periodic with iso-period 2.

**Proof.** Suppose two successive iterations in \( B^k(G), K \geq 1 \) satisfies (i), (ii) and (iii), then by Theorem 2.3, \( B^k(G) \cong (B^{k+1}(G)) \) and \( B^{k+1}(G) \cong (B^k(G)). \) Consider, \( B^{k+1}(G) \cong (B^k(G)) \cong (B^{k-1}(G)) = B^{k-1}(G). \) This proves that \( G \) is eventually periodic with iso-period 2.

From the above theorem it is clear that, If \( G \) and \( B(G) \) holds the conditions in Theorem 3.15 then \( G \) is periodic with iso-period 2.

**Remark 3.16:** There are some graphs in \( F_{22} \) which does not satisfies the condition mentioned in Theorem 3.15, but they are eventually periodic with iso-period 2.

The following example illustrates the above remark.

![Graph G](image)

Fig. 4. The graph \( G. \)

The graph mentioned in Fig. 4 does not satisfy the condition in Theorem 3.15 but it is eventually periodic with iso-period 2.

**Conjecture 3.17:** We have observed, but not proven that a self centered graph of radius two is eventually periodic with iso-period 2.

**Lemma 3.18:** Let \( G \) be a connected graph. If \( \overline{G} \) has the following properties
(i) $\overline{G}$ has no complete vertex.
(ii) neither $N(u) - \{v\} \subseteq N(v) - \{u\}$ nor $N(v) - \{u\} \subseteq N(u) - \{v\}$ for any two adjacent vertices $u$ and $v$ of $\overline{G}$.
(iii) either $N_k(u) = \phi$ or $N_k(v) = \phi$ for any two non-adjacent vertices $u$ and $v$ of $\overline{G}$, where $k = d(u,v)+1$, with $B(G) \cong \overline{G}$, then $G$ is periodic with iso-period 2.

Proof. Since $\overline{G}$ has the properties (i), (ii) and (iii) by Theorem 2.3, $B(\overline{G}) \cong G$. $B^2(G) = B(B(G)) \cong B(\overline{G}) \cong G$ implies that $G$ is periodic with iso-period 2.

Lemma 3.19: If $G$ is a periodic graph with iso-period $m>1$ and if $\overline{G}$ has the following properties

(i) $\overline{G}$ has no complete vertex.
(ii) neither $N(u) - \{v\} \subseteq N(v) - \{u\}$ nor $N(v) - \{u\} \subseteq N(u) - \{v\}$ for any two adjacent vertices $u$ and $v$ of $\overline{G}$.
(iii) either $N_k(u) = \phi$ or $N_k(v) = \phi$ for any two non-adjacent vertices $u$ and $v$ of $\overline{G}$, where $k = d(u,v)+1$, then $\overline{G}$ is eventually periodic with iso-period $m$.

Proof. By hypothesis $B^m(G) \cong G$. Then there exists a graph $H = B^{m-1}(G)$ such that $B(H) \cong B(B^{m-1}(G)) \cong B^m(G) \cong G$. This implies $G$ is a boundary graph. By Theorem 2.3 $B(\overline{G}) \cong G$. Consider $B^m(G) \cong G$ implies $B^m(B(\overline{G})) \cong B(\overline{G})$. Therefore $B^{m+1}(\overline{G}) \cong B(\overline{G})$. Hence $\overline{G}$ is eventually periodic with iso-period $m$.

Theorem 3.20: A graph $G$ is eventually periodic if and only if one of the following holds

(i) $G$ is a complete $n$-partite graph with $|V_i| \geq 2$ for each $i^{th}$ partition.
(ii) $G \in F_{12}$ and for any two adjacent vertices $u$ and $v$ in $G$ either

$N(u) - \{v\} \subseteq N(v) - \{u\}$ or $N(v) - \{u\} \subseteq N(u) - \{v\}$.

(iii) Any two successive iterations in $B^k(G), k \geq 1$ holds the following conditions

(a) No complete vertex

(b) neither $N(u) - \{v\} \subseteq N(v) - \{u\}$ nor $N(v) - \{u\} \subseteq N(u) - \{v\}$ for any two adjacent vertices $u$ and $v$.

(c) either $N_k(u) = \phi$ or $N_k(v) = \phi$ for any two non-adjacent vertices $u$ and $v$ where $k = d(u,v)+1$.

(iv) $G$ be a disconnected graph with at least one isolated vertex.

(v) Let $G$ be a disconnected graph with at least one component complete. If for any two adjacent vertices $u$ and $v$ in $B(G)$, either $N(u) - \{v\} \subseteq N(v) - \{u\}$ or $N(v) - \{u\} \subseteq N(u) - \{v\}$.

Proof. If (i) holds, then by Observation 3.7, $G$ is eventually periodic. If (ii) holds, then by Lemma 3.9, $G$ is eventually periodic. If (iii) holds true, by Theorem 3.15 $G$ is eventually
periodic. If (iv) holds, by Lemma 3.11 $G$ is eventually periodic. If (v) holds, then by Lemma 3.13 $G$ is eventually periodic.

Conversely, Suppose $G$ is eventually periodic. Assume that (i), (iii), (iv) and (v) do not hold. Now we have to prove that (ii) definitely holds. Suppose this is not. Let $G \in F_{12}$ and for any two adjacent vertices $u$ and $v$ in $G$ neither $N(u) - \{v\} \subseteq N(v) - \{u\}$ nor $N(v) - \{u\} \subseteq N(u) - \{v\}$. This implies $uv \notin B(G)$. Therefore non-adjacent vertices in $G$ are adjacent in $B(G)$ together with the full degree vertices in $G$ continue to have the same degree in $B(G)$. Hence $B(G) \in F_{12}$. With the assumption of the condition mentioned for adjacent vertices, $B^2(G) \cong G$. implies $G$ is periodic which is a contradiction.

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References

1. F. Buckley and F. Harary, Distance in Graphs (Addison-Wesley Reading, 1990).
   http://web.udl.es/usuaris/p4088280/research/abs/GLMR08.html