Commutativity in Prime $\Gamma$–Near-Rings with Permuting Tri-derivations

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Abstract

The object of this paper is to introduce a permuting tri-derivation in a $\Gamma$-near-ring. We obtain the conditions for a prime $\Gamma$-near-ring to be a commutative $\Gamma$-ring.

Keywords: $\Gamma$-near-ring; Prime $\Gamma$-near-ring; Commutative $\Gamma$-ring; Permuting tri-derivation.

1. Introduction

The derivations in near-rings have been introduced by Bell and Mason [1]. They investigated some basic properties of derivations in near-rings. Then Asci [2] obtained some commutativity conditions for a $\Gamma$-near-ring with derivations. Some characterizations of $\Gamma$-near-rings and some regularity conditions were obtained by Cho [3]. Kazaz and Alkan [4] introduced the notion of two-sided $\Gamma$-$\alpha$-derivation of a $\Gamma$-near-ring and investigated the commutativity of prime and semiprime $\Gamma$-near-rings. Uckun et al. [5] worked on prime $\Gamma$-near-rings with derivations and they investigated the conditions for a $\Gamma$-near-ring to be commutative.

In this paper, the notion of a permuting tri-derivation in a $\Gamma$-near-ring is introduced. We investigate the conditions for a prime $\Gamma$-near-ring to be a commutative $\Gamma$-ring.

2. Preliminaries

A $\Gamma$-near-ring is a triple $(R, +, \Gamma)$ where

(i) $(R, +)$ is a group (not necessarily abelian),
(ii) $\Gamma$ is a non-empty set of binary operations on $R$ such that for each $\alpha \in \Gamma$, $(R, +, \alpha)$ is a left near-ring,
(iii) $x\alpha(y\beta z) = (x\alpha y)\beta z$, for all $x, y, z \in R$ and $\alpha, \beta \in \Gamma$.

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Exactly speaking, it is a left $\Gamma$-near-ring because it satisfies the left distributive law. We will use the word $\Gamma$-near-ring to mean left $\Gamma$-near-ring. For a $\Gamma$-near-ring $R$, the set $R_0 = \{x \in R : 0 \alpha x = 0, \alpha \in \Gamma\}$ is called the zero-symmetric part of $R$. A $\Gamma$-near-ring $R$ is said to be zero-symmetric if $R = R_0$. Throughout this note, $R$ will be a zero-symmetric $\Gamma$-near-ring and $R$ is called prime if $x \Gamma R \Gamma y = \{0\}$ implies $x = 0$ or $y = 0$. Recall that $R$ is called n-torsion-free, where $n$ is a positive integer, if $nx = 0$ implies $x = 0$ for all $x \in R$. The symbol $C(R)$ will represent the multiplicative center of $R$, that is, $C(x) = \{x \in R : xay = yax$ for all $y \in R, \alpha \in \Gamma\}$. For $x \in R$, the symbol $C(x)$ will denote the centralizer of $x$ in $R$. As usual, for $x, y \in R, \alpha \in \Gamma$, $[x, y]_\alpha$ will denote the commutator $xay - yax$, while $(x, y)$ will indicate the additive-group commutator $x + y - x - y$. An additive map $d : R \to R$ is called a derivation if the Leibniz rule $d(xay) = d(x)ay + xad(y)$ holds for all $x, y \in R, \alpha \in \Gamma$. By a bi-derivation we mean a bi-additive map $D : R \times R \to R$ (i.e., $D$ is additive in both arguments) which satisfies the relations $D(xay, z) = D(x, z)ay + xad(y, z)$ and $D(x, yaz) = D(x, y)az + yaD(x, z)$ for all $x, y, z \in R, \alpha \in \Gamma$. Let $D$ be symmetric, that is, $D(x, y) = D(y, x)$ for all $x, y \in R$. The map $d : R \to R$ defined by $d(x) = D(x, x)$ for all $x \in R$ is called the trace of $D$. A map $F : R \times R \times R \to R$ is said to be permuting if the equation $F(x_1, x_2, x_3) = F(x_{\alpha(1)}, x_{\alpha(2)}, x_{\alpha(3)})$ holds for all $x_1, x_2, x_3 \in R$ and for every permutation $(\pi(1), \pi(2), \pi(3))$.

3. Permuting Tri-derivations and Commutativity

A map $f : R \to R$ defined by $f(x) = F(x, x, x)$ for all $x \in R$, where $F : R \times R \times R \to R$ is a permuting map, is called the trace of $F$. It is obvious that, in the case $F : R \times R \times R \to R$ is a permuting map which is also tri-additive (i.e., additive in each argument), the trace $f$ of $F$ satisfies the relation $f(x + y) = f(x) + 2F(x, x, y) + F(x, y, y) + 2F(x, y, y) + f(y)$ for all $x, y \in R$. Since we have $F(0, y, z) = F(0 + 0, y, z) = F(0, y, z) + F(0, y, z)$ for all $y, z \in R$, we obtain $F(0, y, z) = 0$ for all $y, z \in R$. Hence we get $0 = F(0, y, z) = F(x + x, y, z) = F(x, y, z) + F(x, y, z)$ and so we see that $F(x, y, z) = \text{opp}(F(x, y, z))$ for all $x, y, z \in R$. This tells us that $f$ is an odd function.

A tri-additive map $F : R \times R \times R \to R$ will be called a tri-derivation if the relations $D(x_1x_2y_3) = D(x_1, y_3)ax_2 + x_1aD(x_2, y_3)$, $D(x, y_2y_3) = D(x, y_2, y_3)ay_2 + y_1aD(x, y_2, y_3)$ and $D(x, y, z) = D(x, y, z)az + z_1aD(x, y, z)$ are fulfilled for all $x, y, z \in R$, $\alpha \in \Gamma$.

We need the following lemmas to obtain our main results.

**Lemma 3.1** [6, Lemma 2.3] Let $R$ be a prime $\Gamma$-near-ring. If $C(R) - \{0\}$ contains an element $z$ for which $z + \alpha z \in C(R)$, then $(R, +)$ is abelian.

**Lemma 3.2** [7, Lemma 2.2] Let $R$ be a 3!-torsion free $\Gamma$-near-ring. Suppose that there exists a permuting tri-additive map $f : R \times R \times R \to R$ such that $f(x) = 0$ for all $x \in R$, where $f$ is the trace of $F$. Then we have $F = 0$.

**Lemma 3.3.** Let $R$ be a 3!-torsion free prime $\Gamma$-near-ring and let $x \in R$. Suppose that there exists a nonzero permuting tri-derivation $D : R \times R \times R \to R$ such that $xad(y) = 0$ for all $y \in R$, $\alpha \in \Gamma$, where $d$ is the trace of $D$. Then we have $x = 0$. 

**Proof.** Since we have \( d(y+z) = d(y) + 2D(y, y, z) + D(y, z, z) + D(y, y, z) + 2D(y, z, z) + d(z) \) for all \( y, z \in R, \alpha \in \Gamma \), the hypothesis gives

\[
2\alpha D(y, y, z) + \alpha D(y, z, z) + x\alpha D(y, y, z) + 2\alpha D(y, y, z) = 0 \quad \text{for all} \quad y \in R, \quad \alpha \in \Gamma. \tag{1}
\]

Setting \( y = -y \) in (1), it follows that

\[
2\alpha D(y, y, z) - \alpha D(y, z, z) + x\alpha D(y, y, z) - 2\alpha D(y, y, z) = 0 \quad \text{for all} \quad y \in R, \quad \alpha \in \Gamma. \tag{2}
\]

On the other hand, for any \( y \in R, \) \( d(z+y) = d(z) + 2D(z, z, y) + D(z, y, y) + D(z, z, y) + 2D(z, y, y) + d(y) \) and so, by the hypothesis, we have

\[
2\alpha D(y, z, z) + x\alpha D(y, y, z) + x\alpha D(y, z, z) + 2\alpha D(y, y, z) = 0 \quad \text{for all} \quad x, y \in R, \quad \alpha \in \Gamma. \tag{3}
\]

Since \( D \) is permuting. Comparing (1) with (2), we get

\[
2\alpha D(y, z, z) + x\alpha D(y, y, z) + x\alpha D(y, z, z) = x\alpha D(y, y, z) - 3\alpha xD(y, y, z) \quad \text{which means that} \quad 2\alpha xD(y, z, z) + x\alpha D(y, y, z) + x\alpha D(y, z, z) + 2\alpha xD(y, y, z) \quad \text{for all} \quad x, y \in R, \quad \alpha \in \Gamma.
\]

Now, from (3), we obtain

\[
\alpha xD(y, y, z) - 3\alpha xD(y, z, z) + 2\alpha xD(y, y, z) = 0 \quad \text{for all} \quad x, y \in R, \quad \alpha \in \Gamma. \tag{4}
\]

Taking \( y = -y \) in (4) leads to

\[
\alpha xD(y, y, z) + 3\alpha xD(y, z, z) + 2\alpha xD(y, y, z) = 0 \quad \text{for all} \quad x, y \in R, \quad \alpha \in \Gamma. \tag{5}
\]

Combining (4) and (5), we obtain

\[
\alpha xD(y, z, z) = 0 \quad \text{for all} \quad x, y \in R, \quad \alpha \in \Gamma, \tag{6}
\]

since \( R \) is 6-torsion free.

Replacing \( z = z + w \) to linearize (6) and using the conditions show that

\[
\alpha xD(w, y, z) = 0 \quad \text{for all} \quad w, x, y, z \in R, \quad \alpha \in \Gamma. \tag{7}
\]

Substituting \( w\beta v \) for \( w \) in (7), we get \( \alpha xw\beta D(v, y, z) = 0 \) for all \( v, w, x, y, z \in R, \beta \in \Gamma \). Since \( R \) is prime and \( D \neq 0 \), we arrive at \( x = 0 \). This completes the proof of the theorem.

**Lemma 3.4.** Let \( R \) be a \( \Gamma \)-near-ring and let \( D : R \times R \times R \to R \) be a permuting triderivation. Then we have \( [D(x, z, w) \alpha y + x\alpha D(y, z, w)] \beta v = D(x, z, w) \alpha y \beta v + x\alpha D(y, z, w) \beta v \) for all \( v, w, x, y, z \in R, \alpha, \beta \in \Gamma \).

**Proof.** Since we have \( D(x\alpha y, z, w) = D(x, z, w) \alpha y + x\alpha D(y, z, w) \) for all \( w, x, y, z \in R, \alpha \in \Gamma \), the associative law gives

\[
D((x\alpha y) \beta v, z, w) = D(x\alpha y, z, w) \beta v + x\alpha D(y, z, w)
\]

\[= [D(x, z, w) \alpha y + x\alpha D(y, z, w)] \beta v + x\alpha y \beta D(v, z, w) \tag{8}\]

and

\[
D(x\alpha (y \beta v), z, w) = D(x, z, w) \alpha y \beta v + x\alpha D(y, \beta v, z, w)
\]

\[= D(x, z, w) \alpha y \beta v + x\alpha [D(y, z, w) \beta v + y \beta D(v, z, w)]
\]

\[= D(x, z, w) \alpha y \beta v + x\alpha D(y, z, w) \beta v + x\alpha y \beta D(v, z, w) \tag{9}\]

for all \( v, w, x, y, z \in R, \alpha, \beta \in \Gamma \).
Comparing (8) and (9), we see that 
\[ [D(x, z, w)α + xaD(y, z, w)]βv = D(x, z, w)αβv + \]
\[ xαD(y, z, w)βv \] 
for all \( v, w, x, y, z \in R \), \( α, β \in \Gamma \).

The proof of the lemma is complete.

Now we are ready to prove our main results in this section.

**Theorem 3.5.** Let \( R \) be a 31-torsion free prime \( \Gamma \)-near-ring. Suppose that there exists a nonzero permuting tri-derivation \( D : R \times R \times R \to R \) such that \( D(x, y, z) \in C(R) \) for all \( x, y, z \in R \). Then \( R \) is a commutative \( \Gamma \)-ring.

**Proof.** Assume that \( D(x, y, z) \in C(R) \) for all \( x, y, z \in R \). Since \( D \) is nonzero, there exist \( x_0, y_0, z_0 \in R \) such that \( D(x_0, y_0, z_0) \in C(R) \setminus \{0\} \) and \( D(x_0, y_0, z_0) + D(x_0, y_0, z_0) = D(x_0, y_0, z_0 + z_0) \in C(R) \).

So \( (R, +) \) is abelian by Lemma 3.1.

Since the hypothesis implies that
\[ wβD(x, y, z) = D(x, y, z)βw \] 
for all \( w, x, y, z \in R \), \( β \in \Gamma \),
we replace \( x \) by \( xaβ \) in (10) to get \( wβ[D(x, y, z)α + xaD(v, y, z)] = [D(x, y, z)α + xaD(v, y, z)]βw \) and thus, from Lemma 3.4 and the hypothesis, it follows that \( D(x, y, z)βwαv + D(v, y, z)αwβx = D(x, y, z)αβw + D(v, y, z)βxαw \) which means that
\[ D(x, y, z)[βw α]_a = D(v, y, z)[βx α]_a \] 
for all \( v, w, x, y, z \in R \), \( α, β \in \Gamma \).

Setting \( d(u) \) in place of \( v \) in (11) and using \( d(x) \in C(R) \) for all \( x \in R \), by the hypothesis, we obtain
\[ D(d(u), y, z)[βx α]_a = 0 \] 
for all \( u, w, x, y, z \in R \), \( α, β \in \Gamma \).

The substitution \( ναx \) for \( x \) in (12) yields that \( D(d(u), y, z)[βνα]_a = 0 \) for all \( u, v, w, x, y, z \in R \), \( α, β \in \Gamma \). Since \( R \) is prime, we obtain either \( D(d(u), y, z) = 0 \) or \( [x, w]_a = 0 \) for all \( u, w, x, y, z \in R \), \( α \in \Gamma \).

Assume that
\[ D(d(u), y, z) = 0 \] 
for all \( u, y, z \in R \).

Let us take \( u + x \) instead of \( u \) in (13). Then we obtain
\[ 0 = D(d(u + x), y, z) = D(d(u) + d(x) + 3D(u, u, x), y, z) + 3D(D(u, u, x), y, z), \]
that is,
\[ D(D(u, u, x), y, z) = 0 \] 
for all \( v, w, x, y \in R \).

Setting \( u = -u \) in (14) and then comparing the result with (14), we see that
\[ D(D(u, u, x), y, z) = 0 \] 
for all \( u, x, y, z \in R \).

Substituting \( uλx \) for \( x \) in (15) and employing (13) give the relation \( d(u)λD(x, y, z) + D(u, y, z)λD(u, u, x) = 0 \) and so it follows from the hypothesis that
\[ d(u)λD(x, y, z) + D(u, u, x)λD(u, y, z) = 0 \] 
for all \( u, x, y, z \in R \), \( λ \in \Gamma \).

We put \( u = y = x \) in (16) to obtain,
\[d(x)\lambda D(x, x, w) = 0\] for all \(w, x \in \mathbb{R}, \lambda \in \Gamma.\] (17)

Taking \(w\lambda x\) in substitute for \(w\) in (17) yields \(d(x)\lambda w\lambda d(x) = 0\) for all \(\lambda \in \Gamma\), and so the primeness of \(R\) implies that \(d(x) = 0\) for all \(x \in \mathbb{R}\). Hence, by Lemma 3.2, we have \(D = 0\) which is a contradiction. So \(R\) is a commutative \(\Gamma\)-ring. This proves the theorem.

**Theorem 3.6.** Let \(R\) be a 3\,!-torsion free prime \(\Gamma\)-near-ring. Suppose that there exists a nonzero permuting tri-derivation \(D: \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}\) such that \(d(x), d(x) + d(x) \in C(D(u, v, w))\) for all \(u, v, w, x \in \mathbb{R}\), where \(d\) is the trace of \(D\). Then \(R\) is a commutative \(\Gamma\)-ring.

**Proof.** Assume that
\[d(x), d(x) + d(x) \in C(D(u, v, w))\] for all \(u, v, w, x \in \mathbb{R}.\) (18)

From (18), we get
\[
D(u + t, v, w)\alpha d(x) + d(x))
= (d(x) + d(x))\alpha D(u + t, v, w)
= (d(x) + d(x))\alpha [D(u, v, w) + D(t, v, w)]
= d(x)\alpha D(u, v, w) + d(x)\alpha D(t, v, w) + d(x)\alpha D(t, v, w)
= d(x)\alpha [D(u, v, w) + D(t, t, v, w) + D(t, t, v, w) + d(x)\alpha d(x)]
= [D(u, v, w) + D(t, t, v, w) + D(t, t, v, w) + d(x)\alpha d(x)]
\]
and
\[
D(u + t, v, w)\alpha d(x) + d(x))
= D(u + t, v, w)\alpha d(x) + D(u + t, v, w)\alpha d(x)
= [D(u, v, w) + D(t, t, v, w)]\alpha d(x) + [D(u, v, w) + D(t, t, v, w)]\alpha d(x)
= [D(u, v, w) + D(t, t, v, w) + D(u, v, w) + D(t, t, v, w)]\alpha d(x)
\]
Comparing (19) and (20), we obtain \(D((u, t), v, w)\alpha d(x) = 0\) for all \(t, u, v, w, x \in \mathbb{R}, \alpha \in \Gamma.\) Hence it follows from Lemma 3.3 that
\[D((u, t), v, w) = 0\] for all \(t, u, v, w \in \mathbb{R}.\) (21)

We substitute \(u\beta z\) for \(u\) and \(u\beta t\) for \(t\) in (21) to get
\[0 = D(u\beta z, t, v, w) = D(u, v, w)\beta(z, t) + u\beta D((z, t), v, w) = D(u, v, w)\beta(z, t), \beta \in \Gamma.\]

That is,
\[D(u, v, w)\beta(z, t) = 0\] for all \(t, u, v, w, z \in \mathbb{R}, \beta \in \Gamma.\) (22)

Letting \(z = s\delta z\) in (22) and comparing the results (22) we obtain,
\[D(u, v, w)\beta s\delta(z, t) = 0\] for all \(s, t, u, v, w, z \in \mathbb{R}, \beta, \delta \in \Gamma.\) (23)

Since \(D \neq 0,\) we conclude, from (23) and the primeness of \(R,\) that \((z, t) = 0\) is fulfilled for all \(t, z \in \mathbb{R.}\) Therefore \((R, +)\) is abelian.

By the hypothesis, we know that
Hence if we let $x = x + y$ in (24) and since $d(x + y) = d(x) + 2D(x, y, y) + D(x, x, y) + 2D(x, y, y) + d(y)$, then we deduce from (24) that $3[D(x, y, y), D(u, v, w)]_a + 3[D(x, y, y), D(u, v, w)]_a = 0$ for all $u, v, w, x, y \in R, \alpha \in \Gamma$.

Since $R$ is 3-torsion-free, we obtain,

$$[D(x, x, y), D(u, v, w)]_a + [D(x, y, y), D(u, v, w)]_a = 0 \text{ for all } u, v, w, x, y \in R, \alpha \in \Gamma. \quad (25)$$

Setting $y = -y$ in (25) and comparing the result with (25), we obtain

$$[D(x, y, y), D(u, v, w)]_a = 0 \text{ for all } u, v, w, x, y \in R, \alpha \in \Gamma. \quad (26)$$

Replacing $y$ by $y + z$ in (26) and using (26), we have $[D(x, y, z), D(u, v, w)]_a = 0, \alpha \in \Gamma$, since $D$ is permuting, i.e.,

$$D(x, y, z)\beta \alpha D(u, v, w) = D(u, v, w)\alpha D(x, y, z) \text{ for all } u, v, w, x, y, z \in R, \alpha \in \Gamma. \quad (27)$$

Taking $u \beta \alpha$ instead of $u$ in (27), we obtain,

$$D(u, v, w)\beta \alpha D(x, y, z) - D(x, y, z)\alpha D(u, v, w)\beta \alpha + u \beta \alpha D(t, v, w)\alpha D(x, y, z)
- D(x, y, z)\beta \alpha u \alpha D(t, v, w) = 0 \text{ for all } t, u, v, w, x, y, z \in R, \alpha, \beta \in \Gamma. \quad (28)$$

Substituting $d(u)$ for $u$ in (28) and then utilizing the hypothesis and (27), we get

$$D(d(u), v, w)\beta \alpha [t, D(x, y, z)]_a = 0 \text{ for all } t, u, v, w, x, y, z \in R, \alpha, \beta \in \Gamma. \quad (29)$$

Let us write in (29) $\delta \alpha \beta$ instead of $w$. Then we have $D(d(u), v, w)\delta \alpha \beta [t, D(x, y, z)]_a = 0 \text{ for all } s, t, u, v, w, x, y, z \in R, \alpha, \beta, \delta \in \Gamma$. Since $R$ is prime, we arrive at either $D(d(u), v, w) = 0$ or $[t, D(x, y, z)]_a = 0 \text{ for all } t, u, v, w, x, y, z \in R, \alpha \in \Gamma$. As in the proof of Theorem 3.5, the case when $D(d(u), v, w) = 0$ holds for all $u, v, w \in R$ leads to the contradiction. Consequently, we arrive at $[t, D(x, y, z)]_a = 0 \text{ for all } t, x, y, z \in R, \alpha \in \Gamma$, i.e. $D(x, y, z) \in C(R)$ for all $x, y, z \in R$. Therefore, Theorem 3.5 yields that $R$ is a commutative $\Gamma$-ring which completes the proof.

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