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On Construction of Mean Graphs

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Abstract

A graph $G=(p,q)$ with p vertices and q edges is called a mean graph if there is an injective function f that maps *V(G)* to $\{0,1,2,3,...,q\}$ such that for each edge *uv*, is labeled with $\frac{f(u)+f(v)}{2}$ if $f(u)+f(v)$ is even and $\frac{f(u)+f(v)+1}{2}$ if $f(u)+f(v)$ is odd. Then the resulting edge labels are distinct. In this paper, we prove some general theorems on mean graphs and show that the graphs $G=P_m(+)\overline{K_n}$, Jewel graph J_n, Jelly fish graph $(JF)_n$ and $K_n^c+2P_3$ are mean graphs.

Keywords: Mean labeling; Mean graph.

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1. Introduction

By a graph we mean a finite, simple and undirected one. The vertex set and the edge set of a graph G are denoted by $V(G)$ and $E(G)$ respectively. The disjoint union of m copies of the graph G is denoted by mG. The union of two graphs G_1 and G_2 is the graph $G_1 \cup G_2$ with $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. A vertex of degree one is called a pendant vertex. Let $G=(p,q)$ be a mean graph with p vertices and q edges and let v be a vertex with label q and let one of the mean labelings of G satisfy the following: If q is odd (even) and all the labels of the vertices which are adjacent to v are even (odd), then we call this mean labeling as extra mean labeling [4] and the graph G as extra mean graph.

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The Jewel graph J_n is a graph with vertex set $V(J_n)=\{u,x,v,y,u_i : 1\le i\le n\}$ and edge set $E(J_n)=\{ux,vx,uy,vy,xy,uu_i,vu_i : 1\le i\le n\}$. The graph Jelly fish $(JF)_n$ has $2n$ vertices and $2n+1$ edges with vertex set $V((JF)_n)=\{u,v,u_i,v_j : 1\le i\le n, 1\le j\le n-2\}$ and edge set $E((JF)_n)=\{uu_i : 1\le i\le n\}\cup\{vv_i : 1\le i\le n-2\}\cup\{u_1u_n, vu_1vu_n\}$. Terms and notations not defined here are used in the sense of Harary [1].

The concept of mean labeling was introduced by Somasundaram and Ponraj [2] and further studied by the same authors in [3]. Motivated by the work of the above authors, we have established the mean labeling of some standard graphs in [4,5]. In this paper we extend our study to establish the mean labeling some more graphs like Jewel graph J_n and Jelly fish graph $(JF)_n$.

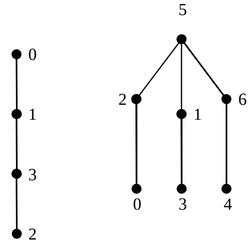
2. Mean Graphs

Remark 2.1: For any mean graph G, $0, q-1$ and q must be the vertex labels. Either 1 or 2 must be a vertex labeling, a vertex of label $q-1$ is adjacent with a vertex of label q and a vertex of label 0 is adjacent with a vertex of label 1 or 2.

Theorem 2. 2: Let $G_1=(p_1,q_1)$ be a mean graph with mean labeling f and let $e=xu$ be an edge with $f(x)=q_1-1$ and $f(u)=q_1$. Let $G_2=(p_2,q_2)$ be a mean graph with mean labeling g and let $e'=yv$ be an edge with $g(y)=0$ and $g(v)=1$ (or 2). If G is a graph obtained by joining the vertex x with y and u with v by an edge, then G is a mean graph.

Proof: Add the number q_1+2 to all the vertex labels of the graph G_2.Then the vertex labels of G_2 remain distinct and the edge labels of G_2 are increased by q_1+2.That is the edge labels of G_2 are $q_1+3, q_1+4, \dots, q_1+q_2+2$. Now the label of the edge xy is $\left\lceil \frac{q_1-1+q_1+2}{2}\right\rceil=\left\lceil\frac{2q_1+1}{2}\right\rceil=q_1+1$. Also the label of the edge uv is $\left\lceil\frac{q_1+q_1+3}{2}\right\rceil=q_1+2$ if $g(v)=1$ and the label of the edge uv is $\left\lceil\frac{q_1+q_1+4}{2}\right\rceil=q_1+2$ if $g(v)=2$. Hence the edge labels of the graph G are 1, 2, 3,…, q_1+q_2+2 and the vertex labels of G are also distinct. This completes the proof.

Example 2.3: Let $G_1=P_4$ and $G_2=S(K_{1,3})$.The mean labeling of G_1 and G_2 are given below.



The mean graph obtained by the above construction is given in Fig. 1.

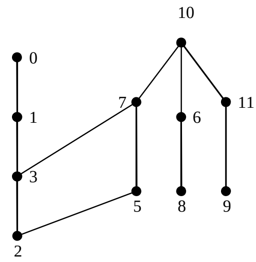


Fig. 1

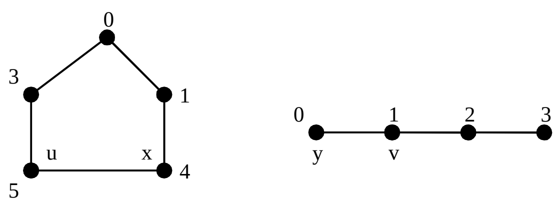
Theorem 2.4: Let $G_1 = (p_1, q_1)$ be a mean graph with mean labeling f and let $e = ux$ be an edge with $f(x) = q_1 - 1$ and $f(u) = q_1$ and let $G_2 = (p_2, q_2)$ be a mean graph with mean labeling g and let $e' = vy$ be an edge with $g(y) = 0$ and $g(v) = 1$. If G is a graph obtained by identifying the edge e' with the edge e (that is identifying u with v and x with y), then G is a mean graph.

Proof: Let $V(G_1) = \{u, x, u_i : 1 \le i \le p_1 - 2\}$ and $V(G_2) = \{v, y, v_i : 1 \le i \le p_2 - 2\}$. Then $V(G) = \{u = v, x = y, u_i, v_j : 1 \le i \le p_1 - 2, 1 \le j \le p_2 - 2\}$. Clearly G has $p_1 + p_2 - 2$ vertices and $q_1 + q_2 - 1$ edges.

Define $h : V(G) \to \{0,1,2,3,...,q_1 + q_2 - 1\}$ by $h(w) = \begin{cases} f(w) & \text{if } w \in V(G_1) \\ g(w) + q_1 - 1 & \text{if } w \in V(G_2) \end{cases}$.

Here $h(u) = h(v) = q_1$ and $h(x) = h(y) = q_1 - 1$. Since G_1 and G_2 are mean graphs and the vertex labels of G_2 are increased by $q_1 - 1$, the vertex labels of G are distinct. The edge labels of the graph G_1 under h are $1,2,3,...,q_1$ and the edge labels of G_2 (except e') under h are $q_1 + 1, q_1 + 2, ..., q_1 + q_2 - 1$. Hence G is a mean graph.

Example 2.5: Let $G_1 = C_5$ and $G_2 = P_4$.The mean labeling of G_1 and G_2 are given below.



The mean graph obtained by the above construction is given Fig. 2.

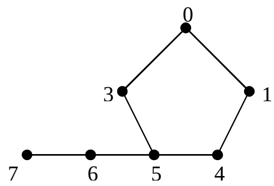


Fig. 2

Theorem 2.6: Let $G_1 = (p_1, q_1)$ be an extra mean graph with an extra mean labeling *f* and let $e = xu$ be an edge with $f(x) = q_1 - 1$ and $f(u) = q_1$. Let $G_2 = (p_2, q_2)$ be a mean graph with mean labeling *g* and let $e' = yv$ be an edge with $g(y) = 0$ and $g(v) = 2$. The graph *G* obtained by identifying the edge e' with the edge e (that is identifying *x* with *y* and *u* with *v*), then *G* is a mean graph.

Proof: Let $V(G_1) = \{u, x, u_i : 1 \le i \le p_1 - 2\}$ and $V(G_2) = \{v, y, v_i : 1 \le i \le p_2 - 2\}$. Then $V(G) = \{u = v, x = y, u_i, v_j : 1 \le i \le p_1 - 2, 1 \le j \le p_2 - 2\}$. Clearly *G* has $p_1 + p_2 - 2$ vertices and $q_1 + q_2 - 1$ edges.

Define $h : V(G) \to \{0,1,2,3...,q_1+q_2 - 1\}$ by $h(u) = q_1 + 1$; $h(x) = q_1 - 1$; $h(u_i) = f(u_i)$ for $1 \le i \le p_1 - 2$ and $h(v_j) = g(v_j) + q_1 - 1$ for $1 \le j \le p_2 - 2$.

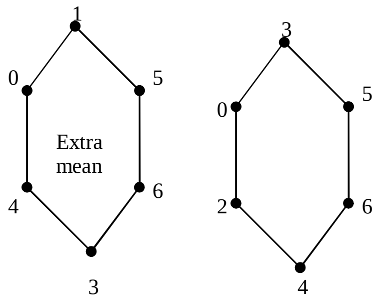
Since G_1 is a mean graph, the vertex labels of G_1 under *h* are remain distinct and $h(V(G_1)) \subseteq \{0,1,2,...,q_1 - 1, q_1 + 1\}$. Since the label of the vertices of $V(G_2) - \{y, v\}$ are increased by $q_1 - 1$ and G_2 is a mean graph, the labels of the vertices of $V(G_2) - \{y, v\}$ are distinct. Also $h(V(G_2) - \{y, v\}) \subseteq \{q_1, q_1 + 2, ..., q_1 + q_2 - 1\}$. The edge labels of the graph G_1, except the edges incident with *u*, under *h* remain distinct. Since G_1 is an extra mean graph with mean labeling *f*, for each vertex *w* incident with *u* in G_1, $f(u)$ and $f(w)$ are of opposite parity. Therefore the induced edge label under *f* is

$$f^*(uw) = \left\lceil \frac{f(u) + f(w)}{2} \right\rceil = \frac{q_1 + f(w) + 1}{2} = k, \text{ an integer. Also,}$$

$$h^*(uw) = \left\lceil \frac{h(u) + h(w)}{2} \right\rceil = \frac{q_1 + 1 + f(w)}{2} = k.$$

Hence, the induced edge labels of G_1 under *h* are $1,2,3,...,q_1$ and the edge labels of G_2 (except e') under *h* are $q_1 + 1, q_1 + 2, ..., q_1 + q_2 - 1$. Hence *G* is a mean graph.

Example 2.7: Let $G_1 = C_6$ and $G_2 = C_6$. The extra mean labeling of G_1 and a mean labeling of G_2 are given below.



The mean graph obtained by the above construction is given in Fig. 3.

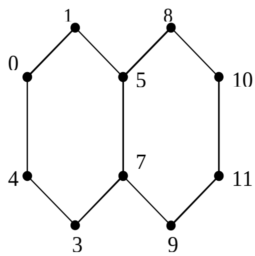


Fig. 3

Theorem 2.8: The Jewel graph J_n is an extra mean graph.

Proof: Let $V(J_n) = \{u, x, v, y, u_i : 1 \le i \le n\}$ and $E(J_n) = \{ux, vx, uy, vy, xy, uu_i, vu_i : 1 \le i \le n\}$. Then J_n has $n+4$ vertices and $2n+5$ edges. Define $f : V(J_n) \to \{0,1,2,...,2n+5\}$ as follows:

$f(u) = 0$; $f(v) = 2n+5$; $f(x) = 2$; $f(y) = 2n+4$; $f(u_i) = 2i+2$ for $1 \le i \le n$.

For each vertex label f, the induced edge label f^* is defined as follows:

$f^*(uu_i) = i+1$ for $1 \le i \le n$; $f^*(vu_i) = n+i+4$ for $1 \le i \le n$;
$f^*(ux) = 1$; $f^*(uy) = n+2$; $f^*(xv) = n+4$; $f^*(vy) = 2n+5$;
$f^*(xy) = n+3$.

Clearly f is a mean labeling of *G*. Moreover *q* is odd and all the vertices which are adjacent to the vertex labeled *q* are even. Thus, *G* is an extra mean graph.

Example 2.9: The mean labeling of J_5 is given in Fig. 4.

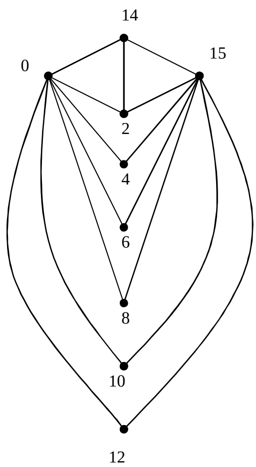


Fig. 4

Theorem 2.10: Let $G = P_m(+)\overline{K_n}$ be the graph with the vertex set $V(G) = \{u_i, v_j : 1 \le i \le m,\ 1 \le j \le n\}$ and the edge set $E(G) = \{u_i u_{i+1}, u_1 v_j, u_m v_j : 1 \le i \le m-1$ and $1 \le j \le n\}$. Then G is a mean graph.

Proof: Let $V(G) = \{u_i, v_j : 1 \le i \le m, 1 \le j \le n\}$.
Define $f : V(G) \to \{0,1,2,...,m+2n-1\}$ as follows:

$f(u_1) = 0;$

$$f(u_i) = \begin{cases} 2n + 2i - 3 & \text{for } 2 \le i \le \left\lceil \frac{m+1}{2} \right\rceil \\ 2n + 2 + 2(m-i) & \text{for } \left\lceil \frac{m+1}{2} \right\rceil + 1 \le i \le m \end{cases} \quad \text{and}$$

$f(v_j) = 2j$ for $1 \le j \le n$. Then $f(V(G)) = \{0,2,4,...,2n,2n+1,2n+2,...,2n+m-1\}$.

For each vertex label f, the induced edge label f^* is defined as follows:

$f^*(u_1 v_j) = j$ for $1 \le j \le n$, $f^*(u_1 u_2) = \left\lceil \frac{2n+1}{2} \right\rceil = n+1$,

$f^*(u_m v_j) = \left\lceil \frac{2n+1+2j}{2} \right\rceil = n+1+j$ for $1 \le j \le n$,

$f^*(u_i u_{i+1}) = \left\lceil \frac{2n+2i-3+2n+2i-1}{2} \right\rceil = 2n+2i-2$ for $2 \le i \le \left\lceil \frac{m+1}{2} \right\rceil$,

$f*(u_iu_{i+1})=\left\lceil \frac{2n+2+2(m-i)+2n+2+2(m-i-1)}{2} \right\rceil = 2n+2(m-i)+1$ for $\left\lceil \frac{m+1}{2} \right\rceil +1 \le i \le m-1$. Now $\{f*(e): e\in E(G)\}=\{1,2,3,...,m+2n-1\}$.

It can be verified that f is a mean labeling of G. Hence G is a mean graph.

Example 2.11: The mean labeling of $P_9(+)\overline{K_5}$ is given in Fig. 5.

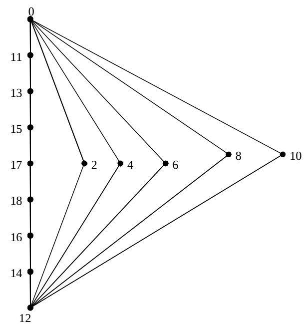


Fig. 5

Theorem 2.12: The graph Jelly fish $(JF)_n$ is a mean graph.

Proof: Let $V((JF)_n)=\{u,v,u_i,v_j : 1\le i\le n, 1\le j\le n-2\}$ and $E((JF)_n)=\{uu_i : 1\le i\le n\}\cup\{vv_j : 1\le j\le n-2\}\cup\{u_1u_n, vu_1 vu_n\}$.

Define $f : V((JF)_n)\to\{0,1,2,...,2n+1\}$ as follows:
$f(u)=0;\ f(u_i)=2i$ for $1\le i\le n$; $f(v)=2n+1$; $f(v_j)=2j+3$ for $1\le j\le n-2$.
For each vertex label f, the induced edge label $f*$ is defined as follows:

$f*(uu_i)=i$ for $1\le i\le n$, $f*(vv_j)=n+j+2$ for $1\le j\le n-2$;
$f*(u_1u_n)=\left\lceil \frac{2n+2}{2} \right\rceil = n+1$, $f*(vu_1)=\left\lceil \frac{2n+3}{2} \right\rceil = n+2$, $f*(vu_n)=\left\lceil \frac{4n+1}{2} \right\rceil = 2n+1$.

Therefore, $\{f*(e): e\in E(G)\}=\{1,2,3,...,n,n+1,n+2,...,2n,2n+1\}$.

It can be verified that f is a mean labeling of $(JF)_n$ and hence $(JF)_n$ is a mean graph.

Example 2.13: The mean labeling of $(JF)_5$ is given in Fig. 6.

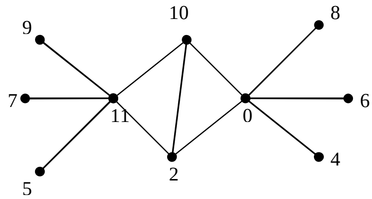


Fig. 6

Theorem 2.14: Let G be a mean tree with $V(G)=\{v_1,v_2,...,v_p\}$ and let G' be a copy of G and with $V(G')=\{v_1',v_2',...,v_p'\}$. Then the graph $G^{(+)}$ obtained by joining the vertex v_i with v_i' by an edge for all $1\le i\le p$, is a mean graph.

Proof: Let f be a mean labeling of G. Clearly $V(G^{(+)})=V(G)\cup V(G')$. Add the number $2p-1$ to the label of the vertices v_i' for $1\le i\le p$. Then the vertex labels of the graph G' remain distinct and the edge labels of G' are increased by $2p-1$. Since G is a tree, $f(V(G))=\{0,1,2,3,...,p-1\}$ and the edge labels of G are $1,2,3,..., p-1$. Also the induced edge labels of G' are $2p, 2p+1, 2p+2,...,3p-2$. For each $i=1$ to n, the label of the edge v_iv_i' is $\left\lceil \frac{f(v_i)+f(v_i)+2p-1}{2}\right\rceil = f(v_i)+p.$ Therefore the induced edge labels of v_iv_i' for $1\le i\le p$ are $p, p+1, p+2,...,2p-1$. Thus $G^{(+)}$ is a mean graph.

Example 2.15: Let G be a Comb obtained from the path P_4. The mean labeling of $G^{(+)}$ is given in Fig. 7.

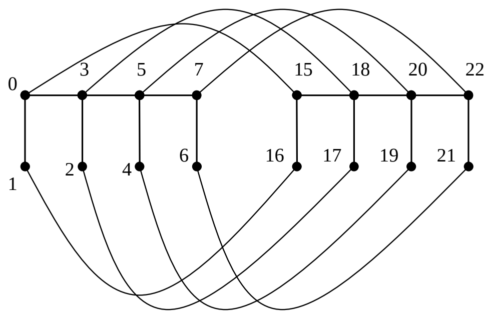


Fig. 7

Theorem 2.16: The graph $K_n^{\,c}+2P_3$ is a mean graph for all n.

Proof: Let $V(K_n)=\{u_1,u_2,u_3,...,u_n\}$. Let $V(2P_3)=\{u,v,w,x,y,z\}$ and $E(2P_3)=\{uv,vw,xy,yz\}$.

Define $f:V(K_n^{\,c}+2P_3)\rightarrow\{0,1,2,...,q=6n+4\}$ as follows:

$f(u)=2,\quad f(v)=0;$

$f(w)=4,\quad f(u_i)=5+6(i-1)$ for $1\le i\le n$,

$f(x)=6n+1,\ f(y)=6n+4,\ f(z)=6n+3.$

For each vertex label f, the induced edge label f^* is defined as follows:

$f^*(uv)=1,\quad f^*(vw)=2,$

$f^*(uu_i)=3i+1$ for $1\le i\le n$,

$f^*(vu_i)=3i$ for $1\le i\le n$,

$f^*(wu_i)=3i+2$ for $1\le i\le n$,

$f^*(xu_i)=3(n+i)$ for $1\le i\le n$,

$f^*(yu_i)=3(n+i)+2$ for $1\le i\le n$,

$f^*(zu_i)=3(n+i)+1$ for $1\le i\le n$,

$f^*(xy)=6n+3,\quad f^*(yz)=6n+4$.

It can be verified that f is a mean labeling and hence $K_n^{\,c}+2P_3$ is a mean graph.

Example 2.17: The mean labeling of $K_3^{\,c}+2P_3$ is given in Fig. 8.

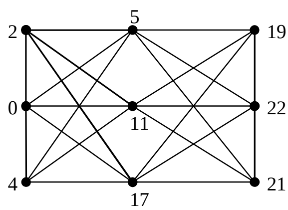


Fig. 8

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