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Characterizations of those $P_n(S)$ which are Relatively Stone Nearlattice

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Abstract

For a fixed element *n* of a nearlattice *S*, a convex subnearlattice of *S* containing *n* is called an *n*-ideal of *S*. An *n*-ideal generated by a single element *a* is called a principal *n*-ideal, denoted by $<a>_n$. The set of principal *n*-ideals is denoted by $P_n(S)$. A distributive nearlattice *S* is called *relatively Stone nearlattice* if each closed interval $[x,y]$ with $x < y$ $(x, y \in S)$ is a Stone lattice. In this paper, we give several characterizations of those $P_n(S)$ which are relatively Stone in terms of *n*-ideals and relative *n*-annihilators.

Keywords: Principal *n*-ideal; Central element; Relatively Stone nearlattice.

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1. Introduction

Relatively Stone lattices have studied by many authors including Ali [1], Cornish [2] and mandelker [3]. In this paper we work on relatively Stone nearlattice. A nearlattice *S* is a meet semilattice with the property that any two elements possessing a common upper bound, have a supremum. Nearlattice *S* is distributive if for all $x, y, z \in S$, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ provided $y \vee z$ exists. An element s of a nearlattice *S* is called *standard* if for all $t, x, y \in S$,

$$t \wedge [(x \wedge y) \vee (x \wedge s)] = (t \wedge x \wedge y) \vee (t \wedge x \wedge s).$$

The element s is called neutral if

(i) s is standard and

(ii) for all $x, y, z \in S$, $s \wedge [(x \wedge y) \vee (x \wedge z)] = (s \wedge x \wedge y) \vee (s \wedge x \wedge z)$.

In a distributive nearlattice every element is neutral and hence standard.

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An element n of a nearlattice S is called *medial* if

$m(x,n,y)=(x\wedge y)\vee(x\wedge n)\vee(y\wedge n)$ exists in S for all $x,y\in S$.

A nearlattice S is called a *medial nearlattice* if $m(x,y,z)$ exists for all $x,y,z\in S$. An element n in a nearlattice S is called *sesquimedial* if for all $x,y,z\in S$ $([(x\wedge n)\vee(y\wedge n)]\wedge[(y\wedge n)\vee(z\wedge n)])\vee(x\wedge y)\vee(y\wedge z)$ exists in S. An element n of a nearlattice S is called an *upper element* if $x\vee n$ exists for all $x\in S$. Every upper element is of course a sesquimedial element. An element n is called a *central element* of S if it is neutral, upper and complemented in each interval containing it.

For $a,b\in S$, $<a,b>$ denotes the relative annihilator. That is, $<a,b>=\{x\in S: x\wedge a\le b\}$. Also note that $<a,b>=<a,a\wedge b>$.

Again for $a,b\in L$, where L is a lattice, $<a,b>_d=\{x\in L: x\vee a\ge b\}$ is a dual relative annihilator.

In case of a nearlattice it is not possible to define a dual relative annihilator ideal for any a and b. But if n is an upper element of S, then $x\vee n$ exists for all $x\in S$.

Then for any $a\in(n]$, $a\vee x$ exists for all $x\in S$ by the upper bound property of S.

Thus for any $a\in(n]$, we can talk about dual relative annihilator ideal of the form $<a,b>_d$ for any $b\in S$. That is, for any $a\le n$ in S,

$<a,b>_d=\{x\in S: x\vee a\ge b\}$.

For $a,b\in S$ and an upper element $n\in S$,

We define $<a,b>^n=\{x\in S: m(a,n,x)\in<b>_n\}$

$$=\{x\in S: b\wedge n\le m(a,n,x)\le b\vee n\}.$$

We call $<a,b>^n$ the *annihilator of* a *relative to* b *around the element* n or simply a *relative* n*- annihilator*. For two n-ideals A and B of a nearlattice S, $<A,B>$ denotes $\{x\in S: m(a,n,x)\in B,$ for all $a\in A\}$ when n is a medial element.

A distributive lattice L with 0 and 1 is called a *Stone lattice* if it is pseudocomplemented and for each $a\in L$, $a^*\vee a^{**}=1$. We also know that a distributive pseudocomplemented lattice is a Stone lattice if and only if for each $a,b\in L$, $(a\wedge b)^*=a^*\vee b^*$. A nearlattice S is *relatively pseudocomplemented* if the interval $[a,b]$ for each $a,b\in S$, $a<b$ is pseudocomplemented. A distributive nearlattice S is called *relatively Stone nearlattice* if each closed interval $[x,y]$ with $x<y$ $(x,y\in S)$ is a Stone lattice.

For a fixed element n of a neartattice S, a convex subnearlattice of S containing n is called an n-ideal of S. An n-ideal generated by a single element a is called a principal n-ideal, denoted by $<a>_n$. The set of principal n-ideals is denoted by $P_n(S)$. When $n\in S$ is standard and medial then for any $a\in S$

$$<a>_n=\{y\in S: a\wedge n\le y=(y\wedge a)\vee(y\wedge n)\}$$
$$=\{y\in S: y=(y\wedge a)\vee(y\wedge n)\vee(a\wedge n)\}$$

When n is an upper element, $<a>_n$ is the closed interval $[a \wedge n, a \vee n]$. For detailed literature on n-ideals and principal n-ideals see Akhter et. al. [4].

When n is a sesquimedial element of a distributive nearlattice *S*, then $P_n(S)$ is also a distributive nearlattice. $P_n(S)$ is *relatively pseudocomplemented* if the interval $[<a>_n, <b>_n]$ in $P_n(S)$ for each

$<a>_n, <b>_n \in P_n(S)$, $<a>_n \subset <b>_n$ is pseudocomplemented.

Moreover, $P_n(S)$ is a relatively Stone nearlattice if each closed interval

$[<a>_n, <b>_n]$ with $<a>_n < <b>_n$ $(<a>_n, <b>_n \in P_n(S))$ is a Stone lattice.

Theorem 1.1. *Let S be a distributive nearlattice with an upper element n. Then the following conditions hold :*

(i) $<<x>_n \vee <y>_n, <x>_n> = <<y>_n, <x>_n>$;

(ii) $<<x>_n, J> = \vee_{y \in J} <<x>_n, <y>_n>$, the supremum of n- ideals $<<x>_n, <y>_n>$ in the lattice of n- ideals of S, for any $x \in S$ and any n-ideal J.

Proof. (i). Obviously L.*H*.*S*.$\subseteq$ R. H. S.
To prove the reverse inclusion, let , $t \in R.H.S.$,

then $t \in <<y>_n, <x>_n>$. This implies $m(y,n,t) \in <x>_n$.

That is, $<m(y,n,t)>_n \subseteq <x>_n$ and

so $(<y>_n \cap <t>_n) \vee (<x>_n \cap <t>_n) \subseteq <x>_n$.

That is, $<t>_n \cap [<x>_n \vee <y>_n] \subseteq <x>_n$

which implies $t \in <<x>_n \vee <y>_n, <x>_n>$.

Thus, $t \in L.H.S.$ and so $R.H.S. \subseteq L.H.S.$

Hence L.H.S.=R.H.S..

(ii). Obviously $R.H.S. \subseteq L.H.S.$

To prove the reverse inclusion, let $t \in L.H.S.$, then $m(x,n,t) \in J$ that is $m(x,n,t) = j$ for some $j \in J$.

This implies $t \in <<x>_n, <j>_n>$.

Thus $t \in R.H.S.$ and so (ii) holds. □

Following lemma will be needed for further development of this paper. This is in fact, the dual of Cornish [2, Lemma 3.6] and very easy to prove. So we prefer to omit the proof.

Lemma 1.2. *Let L be a distributive lattice . Then the following conditions hold :*

(i) $<x \wedge y, x>_d = <y, x>_d$;

(ii) $<[x),F>_d = \vee_{y\in F} <x,y>_d$, where F is a filter of L;

(iii) $\{<x,a>_d \vee <y,a>_d\} \cap [a,b] = \{<x,a>_d \cap [a,b]\} \vee \{<y,a>_d \cap [a,b]\}$, *where* $[a,b]$ represents any interval in L. □

Lemma 1.3 and 1.4 are essential for the proof of our main result of this paper.

Lemma 1.3 *Let S be a distributive nearlattice with an upper element n. Suppose* $a,b,c \in S$.

(i) *If* $a,b,c \geq n$, then $<<m(a,n,b)>_n, <c>_n> =$ $<<a>_n, <c>_n> \vee <<b>_n, <c>_n>$ is equivalent to $<a\wedge b, c> = <a,c> \vee <b,c>$.

(ii) *If* $a,b,c \leq n$, then $<<m(a,n,b)>_n, <c>_n> =$ $<<a>_n, <c>_n> \vee <<b>_n, <c>_n>$ is equivalent to $<a\vee b, c>_d = <a,c>_d \vee <b,c>_d$.

Proof. (i). Suppose $a,b,c \geq n$, and

$$<<m(a,n,b)>_n, <c>_n> = <<a>_n, <c>_n> \vee <<b>_n, <c>_n>.$$

That is, $<<a>_n \cap <b>_n, <c>_n> = <<a>_n, <c>_n> \vee <<b>_n, <c>_n>$.

Let $x \in <a\wedge b, c>$. Then $x\wedge a\wedge b \leq c$,

$$\begin{aligned} <x>_n \cap <a\wedge b>_n &= <x>_n \cap [n, a\wedge b] \\ &= [n, (x\vee n)\wedge(a\wedge b)] \\ &= [n, (x\wedge a\wedge b)\vee n] \\ &\subseteq [n,c]. \end{aligned}$$

Hence $$\begin{aligned} x &\in <<a\wedge b>_n, <c>_n> \\ &= <<m(a,n,b)>_n, <c>_n> \\ &= <<a>_n, <c>_n>> \vee <<b>_n, <c>_n> \end{aligned}$$

Thus $x \leq p\vee q$, where $p \in <<a>_n, <c>_n>$ and $q \in <<b>_n, <c>_n>$.

Then $<p>_n \cap <a>_n \subseteq <c>_n$. That is, $[p\wedge n, p\vee n] \cap [n,a] \subseteq [n,c]$.

Thus, $[n, (p\vee n)\wedge a] \subseteq [n,c]$ which implies $p\wedge a \leq c$ and so $p \in <a,c>$.

Similarly, $q \in <b,c>$ and so $x \in <a,c> \vee <b,c>$.

Hence $<a\wedge b, c> \subseteq <a,c> \vee <b,c>$.

But $<a,c> \vee <b,c> \subseteq <a\wedge b, c>$ is obvious.

Therefore, $<a\wedge b, c> = <a,c> \vee <b,c>$.

Conversely, suppose $<a\wedge b, c> = <a,c> \vee <b,c>$.

Let $x \in <<m(a,n,b)>_n, <c>_n>$.

Then $<x>_n \cap <m(a,n,b)>_n = [x\wedge n, x\vee n] \cap [n, a\wedge b] \subseteq [n,c]$.

That is, $[n,(x\vee n)\wedge(a\wedge b)]\subseteq[n,c]$.
Thus $[n,(x\wedge a\wedge b)\vee n]\subseteq[n,c]$ which implies
$x\wedge a\wedge b\le c$ and so $x\in\, <a\wedge b,c> = <a,c>\vee<b,c>$.
This implies $x=r\vee s$, where $r\in\, <a,c>$ and $s\in\, <b,c>$.
Then $r\wedge a\le c$ and $s\wedge b\le c$.
Now, $<r>_n\cap<a>_n=[r\wedge n,r\vee n]\cap[n,a]$
$=[n,(r\vee n)\wedge a]$
$=[n,(r\wedge a)\vee n]$
$\subseteq[n,c]=\,<c>_n$
Hence $r\in\, <<a>_n,<c>_n>$. Similarly $s\in\, <<b>_n,<c>_n>$
Thus $x\in\, <<a>_n,<c>_n>\vee<<b>_n,<c>_n>$ and so
$<<m(a,n,b)>_n,<c>_n>\subseteq\, <<a>_n,<c>_n>\vee<<b>_n,<c>_n>$.
Since $<<a>_n,<c>_n>\vee<<b>_n,<c>_n>\subseteq\, <<m(a,n,b)>_n,<c>_n>$
is obvious, so
$<<m(a,n,b)>_n,<c>_n>\, = \, <<a>_n,<c>_n>\vee<<b>_n,<c>_n>$.
A dual calculation of above proof proves (ii). □

Lemma 1.4. *Let S be a distributive nearlattice with an upper element n. Suppose* $a,b,c\in S$.

(i) *If* $a,b,c\ge n$ and $a\vee b$ exists then $<<c>_n,<a>_n\vee<b>_n> =$
$<<c>_n,<a>_n>\vee<<c>_n,<b>_n>$ is equivalent to
$<c,a\vee b> = <c,a>\vee<c,b>$.

(ii) *If* $a,b,c\le n$, then $<<c>_n,<a>_n\vee<b>_n> =$
$<<c>_n,<a>_n>\vee<<c>_n,<b>_n>$
is equivalent to $<c,a\wedge b>_d = <c,a>_d\vee<c,b>_d$.

Proof. (i). Suppose $a,b,c\ge n$ and $a\vee b$ exists and
$<<c>_n,<a>_n\vee<b>_n> = <<c>_n,<a>_n>\vee<<c>_n,<b>_n>$.
Let $x\in\, <c,a\vee b>$. Then $x\wedge c\le a\vee b$.
Then $<x>_n\cap<c>_n=[x\wedge n,x\vee n]\cap[n,c]$
$=[n,(x\vee n)\wedge c]$
$=[n,(x\wedge c)\vee n]$
$\subseteq[n,a\vee b]$
$=\,<a>_n\vee<b>_n$
That is $<x>_n\cap<c>_n\subseteq\, <a>_n\vee<b>_n$.
Thus $x\in\, <<c>_n,<a>_n\vee<b>_n> = <<c>_n,<a>_n>\vee<<c>_n,<b>_n>$.

So, $x \le p \vee q$ where $p \in \ll c >_n, < a >_n >$ and $q \in \ll c >_n, < b >_n >$

Since $p \in \ll c >_n, < a >_n >$ so $< p >_n \cap < c >_n \subseteq < a >_n$.

That is $[p \wedge n, p \vee n] \cap [n, c] \subseteq [n, a]$.

Thus $[n, (p \vee n) \wedge c] \subseteq [n, a]$.

That is $[n, (p \wedge c) \vee n] \subseteq [n, a]$.

This implies $p \wedge c \le a$ and so $p \in < c, a >$.

Similarly, $q \in < c, b >$.

Hence $x \in < c, a > \vee < c, b >$ and so $< c, a \vee b > \subseteq < c, a > \vee < c, b >$.

Since the reverse inequality is trivial,

so $< c, a \vee b > = < c, a > \vee < c, b >$.

Conversely, suppose $< c, a \vee b > = < c, a > \vee < c, b >$.

Let $x \in \ll c >_n, < a >_n \vee < b >_n >$.

Then $< x >_n \cap < c >_n \subseteq < a >_n \vee < b >_n$.

That is $[x \wedge n, x \vee n] \cap [n, c] \subseteq [n, a \vee b]$

and so $[n, (x \vee n) \wedge c] \subseteq [n, a \vee b]$.

That is $[n, (x \wedge c) \vee n] \subseteq [n, a \vee b]$.

This implies $x \wedge c \le a \vee b$ and so $x \in < c, a \vee b > = < c, a > \vee < c, b >$.

Thus $x = r \vee t$, where $r \in < c, a >$ and $t \in < c, b >$.

Now, $< r >_n \cap < c >_n = [r \wedge n, r \vee n] \cap [n, c]$

$= [n, (r \wedge c) \vee n]$

$\subseteq [n, a] = < a >_n$

(Here $r \in < c, a >$ implies $r \wedge c \le a$)

So $r \in \ll c >_n, < a >_n >$. Similarly $t \in \ll c >_n, < b >_n >$.

Hence $x \in \ll c >_n, < a >_n > \vee \ll c >_n, < b >_n >$, and so

$\ll c >_n, < a >_n \vee < b >_n > \subseteq \ll c >_n, < a >_n > \vee \ll c >_n, < b >_n >$.

Since the reverse inequality is trivial, so

$\ll c >_n, < a >_n \vee < b >_n > = \ll c >_n, < a >_n > \vee \ll c >_n, < b >_n >$.

By the dual calculation of above we can easily prove (ii). □

Following result on Stone lattices is well known due to Cornish [2] and Katrinak [5, 6].

Theorem 1.5. *Let L be a pseudocomplemented distributive lattice. Then the following conditions are equivalent :*

(i) L is Stone ;

(ii) For each $x, y \in L$, $(x \wedge y)^* = x^* \vee y^*$;

(iii) If $x \wedge y = 0$, $x, y \in L$ then $x^* \vee y^* = 1$. □

Similarly we can prove the following result which is dual to above Theorem.

Theorem 1.6. *Let L be a dual pseudocomplemented distributive lattice. Then the following conditions are equivalent :*

(i) L is dual Stone ;

(ii) For each $x, y \in L$, $(x \vee y)^{*d} = x^{*d} \wedge y^{*d}$;

(iii) If $x \vee y = 1$, $x, y \in L$ then $x^{*d} \wedge y^{*d} = 0$, where x^{*d} denotes the dual pseudocomplement of x. □

Ali [1] in his Theorem 3.2.7 has given a nice characterization of relatively dual Stone lattices in terms of dual relative annihilators, which is in fact the dual of Cornish [2, Theorem 3.7]. As we have mentioned earlier that in nearlattices the idea of dual relative annihilators is not always possible. But when n is an upper element in S then $x \vee n$ exists for all $x \in S$. Thus for any $a \in (n]$, $x \vee a$ exists for all $x \in S$. Hence we can define $< a, b >_d$ for all $a \in (n]$ and $b \in S$.

Theorem 1.7. *Let* n *be an upper element of a distributive nearlattice S such that* (*n*] *is relatively dual pseudocomplemented. Let* $a, b, c \in (n]$ *be arbitrary elements and A, B be arbitrary filters of* (*n*]. *Then the following conditions are equivalent :*

(i) (*n*] is relatively dual Stone ;

(ii) $< a, b >_d \vee < b, a >_d = (n]$;

(iii) $< c, a \wedge b >_d = < c, a >_d \vee < c, b >_d$;

(iv) $< [c), A \vee B >_d = < [c), A >_d \vee < [c), B >_d$;

(v) $< a \vee b, c >_d = < a, c >_d \vee < b, c >_d$;

Proof. (i) $\Rightarrow$ (ii). Suppose (i) holds.

Let $z \in (n]$ be arbitrary. Consider the interval $I = [z, a \vee b \vee z]$.

Then $a \vee b \vee z$ is the largest element of I.

Since by (i), I is dual Stone, then by Theorem 1.5(iii), there exists $r, s \in I$ such that $a \vee s = a \vee b \vee z = b \vee z \vee r$ and $z = s \wedge r$.

Now, $a \vee s \geq b$ implies $s \in < a, b >_d$ and $b \vee r = b \vee z \vee r = a \vee b \vee z \geq a$ implies $r \in < b, a >_d$.

Hence (ii) holds.

(ii) $\Rightarrow$ (iii). Suppose (ii) holds.

In (iii), $R.H.S \subseteq L.H.S$ is obvious.

Let $z \in < c, a \wedge b >_d$, then $z \vee c \geq a \wedge b$.

Since (ii) holds, so $z = x \wedge y$ where $x \in < a, b >_d$ and $y \in < b, a >_d$.

Then $x \vee a \geq b$ and $y \vee b \geq a$.

Thus, $x \vee c = x \vee z \vee c$

$\geq x \vee (a \wedge b)$

$= (x \vee a) \wedge (x \vee b) \geq b$,

which implies $x \in < c, b >_d$. Similarly, $y \in < c, a >_d$.

Hence $z = x \wedge y \in < c, a >_d \vee < c, b >_d$ and so

$< c, a \wedge b >_d \subseteq < c, a >_d \vee < c, b >_d$.

Thus (iii) holds.

(iii) $\Rightarrow$ (iv) follows from Lemma 1.2(ii).

(iv) $\Rightarrow$ (iii) is trivial.

(iii) $\Rightarrow$ (ii) follows from Lemma 1.2(i) by putting $c = a \wedge b$.

(ii) $\Rightarrow$ (v). Suppose (ii) holds.

Let $z \in < a \vee b, c >_d$. Then by (ii), $z = x \wedge y$, where $x \vee a \geq b$ and $y \vee b \geq a$.

Also $x \vee a = x \vee a \vee b \geq z \vee a \vee b \geq c$.

This implies $x \in < a, c >_d$. Similarly, $y \in < b, c >_d$.

Hence $z = x \wedge y \in < a, c >_d \vee < b, c >_d$ and so

$< a \vee b, c >_d \subseteq < a, c >_d \vee < b, c >_d$.

Since the reverse inequality is obvious, so (v) holds.

(v) $\Rightarrow$ (i). Suppose (v) holds.

Let $x \in [a, b]$, $a < b$. Suppose x^{0d} denotes the relatively dual pseudocomplemented of *x* in $[a, b]$.

Then clearly $[x^{0d}) = [x)^{0d} = \{t \in [a, b] : t \vee x = b,$ the largest element of $[a,b]\}$.

It is easy to see that $[x)^{0d} = < x, b >_d \cap [a, b]$.

Now suppose $x, y \in [a, b]$ with $x \vee y = b$,

Then by (v), $[x^{0d} \wedge y^{0d}) = [x^{0d}) \vee [y^{0d})$

$= [x)^{0d} \vee [y)^{0d}$

$= (< x, b >_d \cap [a, b]) \vee (< y, b >_d \cap [a, b])$

$= (< x, b >_d \vee < y, b >_d) \cap [a, b]$ (by lemma 1.2(iii))

$= < x \vee y, b >_d \cap [a, b]$

$= < b, b >_d \cap [a, b]$

$= (n] \cap [a, b] = [a, b]$.

This implies $x^{0d} \wedge y^{0d} = a$.

Hence by Theorem 1.6, $[a, b]$ is dual Stone and so (n] is relatively dual Stone. □

Following Theorems are due to Akhter [7] which will be used to prove the main result of this paper.

Theorem 1.8. *Let* S *be a distributive medial nearlattice with an upper element* n *and let* I, J *be two* n- *ideals of* S. *Then for any* $x \in I \vee J$, $x \vee n = i_1 \vee j_1$ *and* $x \wedge n = i_2 \wedge j_2$ *for some* $i_1, i_2 \in I,\ j_1, j_2 \in J$ *with* $i_1, j_1 \geq n$ *and* $i_2, j_2 \leq n$. □

Theorem 1.9. *For an element* n *of a nearlattice* S, *the following conditions are equivalent :*

(i) n *is central in* S

(ii) n *is upper and the map* $\Phi : P_n(S) \to (n]^d \times [n)$ *defined by*

$\Phi(< a >_n) = (a \wedge n, a \vee n)$ *is an isomorphism, where* $(n]^d$ *represents the dual of the lattice* (n]. □

Now we prove our main results of this paper, which are generalizations of Cornish [2, Theorem 3.7], Mandelker [3, Theorem 5] and a result of Davey [8], also see Raihan et. al. [9]. These give characterizations of those $P_n(S)$ which are relatively Stone , when S is medial.

Theorem 1.10. *Let n be a central element of a distributive medial nearlattice and* $P_n(S)$ *be relatively pseudocomplemented. Suppose A, B are two* n *- ideals of S. Then for all,* $a, b, c \in S$ *the following conditions are equivalent :*

(i) $P_n(S)$ is relatively Stone ;

(ii) $<< a >_n, < b >_n > \vee << b >_n, < a >_n > = S$;

(iii) $<< c >_n, < a >_n \vee < b >_n > = << c >_n, < a >_n > \vee << c >_n, < b >_n >$, whenever $a \vee b$ exists ;

(iv) $<< c >_n, A \vee B > = << c >_n, A > \vee << c >_n, B >$;

(v) $<< m(a,n,b) >_n, < c >_n > =$

$$<< a >_n, < c >_n > \vee << b >_n >, < c >_n >$$

Proof. (i) $\Rightarrow$ (ii). Let $z \in S$.

Consider the interval $I = [< a >_n \cap < b >_n \cap < z >_n, < z >_n >]$ in $P_n(S)$. Then $< a >_n \cap < b >_n \cap < z >_n$ is the smallest element of the interval I.

By (i), I is Stone. Then by Theorem 1.5, there exist principal n-ideals $< p >_n, < q >_n \in I$ such that,

$$< a >_n \cap < z >_n \cap < p >_n = < a >_n \cap < b >_n \cap < z >_n$$
$$= < b >_n \cap < z >_n \cap < q >_n$$

and $< z >_n = < p >_n \vee < q >_n$.

Now,

$$< a >_n \cap < p >_n = < a >_n \cap < p >_n \cap < z >_n$$
$$= < a >_n \cap < b >_n \cap < z >_n \subseteq < b >_n$$

implies $< p >_n \subseteq << a >_n, < b >_n >$.

Also, $< b >_n \cap < q >_n = < b >_n \cap < z >_n \cap < q >_n$

$$=<a>_n \cap <b>_n \cap <z>_n \subseteq <a>_n$$

implies $<q>_n \subseteq << b>_n, <a>_n>$.

Thus $<z>_n \subseteq << a>_n, <b>_n> \vee << b>_n, <a>_n>$ and

so $z \in << a>_n, <b>_n> \vee << b>_n, <a>_n>$.

Hence $<< a>_n, <b>_n> \vee << b>_n, <a>_n> = S$.

(ii) $\Rightarrow$ (iii). Suppose (ii) holds and $a \vee b$ exists.

For (iii), $R.H.S \subseteq L.H.S$ is obvious.

Now, let $z \in << c>_n, <a>_n \vee <b>_n>$.

Then $z \vee n \in << c>_n, <a>_n \vee <b>_n>$ and

so $m(z \vee n, n, c) \in <a>_n \vee <b>_n$.

That is $m(z \vee n, n, c) \in [a \wedge b \wedge n, a \vee b \vee n]$.

This implies $(z \vee n) \wedge (c \vee n) \leq a \vee b \vee n$.

Now by (ii), $z \vee n \in << a>_n, <b>_n> \vee << b>_n, <a>_n>$.

So $z \vee n \leq (p \vee n) \vee (q \vee n)$ for some $p \vee n \in << a>_n, <b>_n>$ and $q \vee n \in << b>_n, <a>_n>$.

Hence, $z \vee n = ((z \vee n) \wedge (p \vee n)) \vee ((z \vee n) \wedge (q \vee n)) = r \vee t$ (say).

Now, $m(p \vee n, n, a) = (p \vee n) \wedge (a \vee n) \leq (b \vee n)$.

So $b \wedge n \leq r \wedge (a \vee n) \leq b \vee n$.

Hence,
$$\begin{aligned} r \wedge (c \vee n) &= r \wedge (z \vee n) \wedge (c \vee n) \\ &\leq r \wedge (a \vee b \vee n) \\ &= (r \wedge (a \vee n)) \vee (r \wedge (b \vee n)) \\ &\leq (b \vee n). \end{aligned}$$

This implies $r \in << c>_n, <b>_n>$. Similarly, $t \in << c>_n, <a>_n>$.

Hence, $z \vee n \in << c>_n, <a>_n> \vee << c>_n, <b>_n>$.

Again $z \in << c>_n, <a>_n \vee <b>_n>$

implies $z \wedge n \in << c>_n, <a>_n \vee <b>_n>$.

Then a dual calculation of above shows that

$z \wedge n \in << c>_n, <a>_n> \vee << c>_n, <b>_n>$.

Thus by convexity, $z \in << c>_n, <a>_n> \vee << c>_n, <b>_n>$ and

so $L.H.S \subseteq R.H.S$.

Hence (iii) holds.

(iii) $\Rightarrow$ (iv). Suppose (iii) holds.

In (iv), $R.H.S \subseteq L.H.S$ is obvious.

Now let $x \in << c>_n, A \vee B>$. Then $x \vee n \in << c>_n, A \vee B>$.

Thus $m(x \vee n, n, c) \in A \vee B.$

Now $m(x \vee n, n, c) = (x \vee n) \wedge (n \vee c) \geq n$ implies
$m(x \vee n, n, c) \in (A \vee B) \cap [n)$.

Hence by Theorem 1.1(ii), $x \vee n \in \langle\langle c \rangle_n, (A \cap [n)) \vee (B \cap [n)) \rangle$

$$= \vee_{r \in (A \cap [n)) \vee (B \cap [n))} \langle\langle c \rangle_n, \langle r \rangle_n \rangle.$$

But by Theorem 1.8, $r \in (A \cap [n)) \vee (B \cap [n))$ implies $r = s \vee t$ for some
$s \in A$, $t \in B$ and $s, t \geq n.$

Then by (iii), $\langle\langle c \rangle_n, \langle r \rangle_n \rangle = \langle\langle c \rangle_n, \langle s \vee t \rangle_n \rangle$

$$= \langle\langle c \rangle_n, \langle s \rangle_n \vee \langle t \rangle_n \rangle$$

$$= \langle\langle c \rangle_n, \langle s \rangle_n \rangle \vee \langle\langle c \rangle_n, \langle t \rangle_n \rangle$$

$$\subseteq \langle\langle c \rangle_n, A \rangle \vee \langle\langle c \rangle_n, B \rangle.$$

Hence $x \vee n \in \langle\langle c \rangle_n, A \rangle \vee \langle\langle c \rangle_n, B \rangle$.

Also $x \in \langle\langle c \rangle_n, A \vee B \rangle$ implies $x \wedge n \in \langle\langle c \rangle_n, A \vee B \rangle$.

Since $m(x \wedge n, n, c) = (x \wedge n) \vee (n \wedge c) \leq n$, so $x \wedge n \in \langle\langle c \rangle_n, (A \vee B) \cap (n] \rangle$.

Then, by Theorem 1.1(ii), $x \wedge n \in \langle\langle c \rangle_n, (A \cap (n]) \vee (B \cap (n]) \rangle$

$$= \vee_{l \in (A \cap (n]) \vee (B \cap (n])} \langle\langle c \rangle_n, \langle l \rangle_n \rangle.$$

Again, using Theorem 1.8, we see that $l = p \wedge q$ where $p \in A$, $q \in B$ and $p, q \leq n$.

Then by (iii), $\langle\langle c \rangle_n, \langle l \rangle_n \rangle = \langle\langle c \rangle_n, \langle p \wedge q \rangle_n \rangle$

$$= \langle\langle c \rangle_n, \langle p \rangle_n \vee \langle q \rangle_n \rangle$$

$$= \langle\langle c \rangle_n, \langle p \rangle_n \rangle \vee \langle\langle c \rangle_n, \langle q \rangle_n \rangle$$

$$\subseteq \langle\langle c \rangle_n, A \rangle \vee \langle\langle c \rangle_n, B \rangle.$$

Hence $x \wedge n \in \langle\langle c \rangle_n, A \rangle \vee \langle\langle c \rangle_n, B \rangle$.

Therefore, by convexity, $x \in \langle\langle c \rangle_n, A \rangle \vee \langle\langle c \rangle_n, B \rangle$ and so $L.H.S \subseteq R.H.S.$

Thus (iv) holds.

(iv) $\Rightarrow$ (iii) is trivial.

(ii) $\Rightarrow$ (v). Suppose (ii) holds. In (v), $R.H.S \subseteq L.H.S$ is obvious.

Now let $z \in \langle\langle m(a, n, b) \rangle_n, \langle c \rangle_n \rangle$,

which implies $z \vee n \in \langle\langle m(a, n, b) \rangle_n, \langle c \rangle_n \rangle$.

By (ii), $z \vee n \in \langle\langle a \rangle_n, \langle b \rangle_n \rangle \vee \langle\langle b \rangle_n, \langle a \rangle_n \rangle$.

Then by Theorem 1.8, $z \vee n = x \vee y$ for some $x \in \langle\langle a \rangle_n, \langle b \rangle_n \rangle$

and $y \in \langle\langle b \rangle_n, \langle a \rangle_n \rangle$ and $x, y \geq n$.

Thus, $\langle x \rangle_n \cap \langle a \rangle_n \subseteq \langle b \rangle_n$ and

$$so \ \langle x \rangle_n \cap \langle a \rangle_n = \langle x \rangle_n \cap \langle a \rangle_n \cap \langle b \rangle_n$$

$$\subseteq \langle z \vee n \rangle_n \cap \langle a \rangle_n \cap \langle b \rangle_n$$

$$= < z \vee n >_n \cap < m(a,n,b) >_n$$
$$\subseteq < c >_n .$$

This implies $x \in << a >_n, < c >_n >$. Similarly $y \in << b >_n, < c >_n >$ and so $z \vee n \in << a >_n, < c >_n > \vee << b >_n, < c >_n >$.

Similarly, a dual calculation of above shows that
$z \wedge n \in << a >_n, < c >_n > \vee << b >_n, < c >_n >$.

Thus by convexity,
$z \in << a >_n, < c >_n > \vee << b >_n, < c >_n >$ and so $L.H.S \subseteq R.H.S.$

Hence (v) holds.

(v) $\Rightarrow$ (i). Suppose (v) holds.

Let $a, b, c \geq n$.

By (v), $<< m(a,n,b) >_n, < c >_n > =$
$$<< a >_n, < c >_n > \vee << b >_n, < c >_n > .$$

But by Lemma 1.3(i), this is equivalent to $< a \wedge b, c > = < a, c > \vee < b, c >$.

Then by Rahman [10, Theorem 3.3], this shows that [n) is a relatively Stone.

Similarly, for $a, b, c \leq n$, using the Lemma 1.3(ii) and Theorem 1.7, we find that (n] is relatively dual Stone.

Therefore, by Theorem 1.9, $P_n(S)$ is relatively Stone.

Finally we need to prove that (iii) $\Rightarrow$ (i).

Suppose (iii) holds. Let $a, b, c \in S \cap [n)$.

By (iii), $<< c >_n, < a >_n \vee < b >_n > = << c >_n, < a >_n > \vee << c >_n, < b >_n >$.

But by Lemma 1.4(i), this is equivalent to $< c, a \vee b > = < c, a > \vee < c, b >$.

Then by Rahman [10, Theorem 3.3], this shows that [n) is relatively Stone.

Similarly for $a, b, c \leq n$, using the Lemma 1.4(ii) and Theorem 1.7, we find that $(n]$ is relatively dual Stone.

Therefore, by Theorem 1.9, $P_n(S)$ is relatively Stone. □

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