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APPROXIMATION APPROACH TO MULTIPLE SINGULARITIES OF FLOW THROUGH A POROUS PIPE WITH DECELERATING WALL

R. A. Rouf [1], M. S. Alam[2] and M. A. H. Khan[3]

[1]School of Engineering and Computer Science, Independent University, Bangladesh, Email: rifat_rouf@yahoo.com
[2]Department of Mathematics, Jagannath University, Dhaka-1100, Bangladesh, Email: sarwardu75@gmail.com
[3]Department of Mathematics, Bangladesh University of Engineering and Technology, Dhaka-1000, Bangladesh
Email: mahkhan@math.buet.ac.bd

Abstract:

The multiple singularity behaviour of flow through a porous pipe with decelerating wall is numerically studied in the present paper. The steady axisymmetric flow of a viscous incompressible fluid driven along a pipe by the combined effect of the wall deceleration and suction is investigated. Our approach uses the power series in order to observe the instability of the problems. The series is then summed by using various generalizations of the Pade´-Hermite approximants. Analysis based on approximate method suggests that the convergence of the series of stream-function, skin friction and centerline axial velocity in powers of Reynolds number is limited by a number of singularities. The location and nature of the singularities in the real plane are presented. The bifurcations of skin friction and centerline axial velocity are also depicted graphically.

Keywords: Multiple singularities, porous pipe, decelerating wall, bifurcation, approximation methods.

1. Introduction

In many problems, the model of physical phenomena produced nonlinear differential equations whether a closed form of the exact solution is complicated. However, a power series solution can be considered, but the presence of singularity prevents rapid convergence of the series. Approximate methods are the techniques for summing power series, where a function is said to be approximant for a given series if its Taylor series expansion produces the first few terms of the series. *Pade´-Hermite* class introduced semi numerical approximate methods, such as Pade' approximants, algebraic and differential approximants etc. Later Drazin and Tourigny (1996) made some improvement to the algebraic approximant named as Drazin-Tourigny approximants where they extended the theory for large N. Khan (2002) introduced High-order differential approximant in his PhD thesis which is an extension of differential approximant. A comparative performance of High-order differential approximant and Drazin-Tourigny approximant to determine the multiple singularities of a nonlinear model problem in fluid dynamics is analyzed in this paper.

The flow in a pipe driven by suction or injection was first considered approximately fifty years ago. Berman (1953) first considered the steady Navier-Stokes equations to a fourth order ordinary differential equation. Brady (1984), Zaturska and Banks (1988) then considered various aspects of the flow, both steady and unsteady. Brady and Acrivos (1981) analysed the flow in a pipe with accelerating wall.

The flow in porous pipe with decelerating wall has significant effect in physical point of view. Practically it is found that there is no real solution to the steady similarity equation for a range of Reynolds number, whereas this absence of solution and the bifurcation study leads to the motivation for this study.

However, the steady axisymmetric flow of a viscous incompressible fluid driven along a pipe by the combined effect of the wall deceleration and suction is considered by Makinde (1999). He solved the non-linear 2D Navier-Stokes equations modeling the flow using a perturbation method, applying a special type of *Pade´-Hermite* approximation method Makinde (1999) obtained the first 54 coefficients of the solution series and a bifurcation study was also performed.

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Alam and Khan (2010) investigated the critical behaviour of the steady two-dimensional nonlinear flow of viscous incompressible fluid through convergent-divergent channels with the effect of external magnetic field using various generalizations of *Pade´-Hermite* approximations.

In the present problem we have studied the temporal stability of the flow in a porous pipe with decelerating wall using various generalizations of the approximation methods. The non-dimensional equation is solved into a series solution of 54 terms in terms of similarity parameter with the help of perturbation theory and MAPLE. The series is then analyzed to show the convergence of critical values and the bifurcation graph for R with the help of approximation method Khan (2002), Drazin and Tourigny (1996). In our analysis, it is found that the results are more accurate and uniform in comparison with Makinde (1999).

2. Review of Pade′-Hermite approximants

In 1893, Pade′ and Hermite introduced Pade′-Hermite class. All the one variable approximants that were used or discussed throughout this paper belong to the Pade′-Hermite class. In its most general form, this class is concerned with the simultaneous approximation of several independent series.

Let $d \in \mathrm{N}$ and let the $d+1$ power series $U_0(x), U_1(x), \ldots, U_d(x)$ are given.

Assume that the $(d+1)$ tuple of polynomials $P_N^{[0]}, P_N^{[1]}, \ldots, P_N^{[d]}$

where $\deg P_N^{[0]} + \deg P_N^{[1]} + \ldots + \deg P_N^{[d]} + d = N,$ (2.1)

is a Pade′-Hermite form of these series if

$$\sum_{i=0}^{d} P_N^{[i]}(x) U_i(x) = O\left(x^N\right) \text{ as } x \to 0. \tag{2.2}$$

Here $U_0(x), U_1(x), \ldots, U_d(x)$ may be independent series or different form of a unique series. We need to find the polynomials $P_N^{[i]}$ that satisfy Equations (2.1) and (2.2). These polynomials are completely determined by their coefficients. So, the total number of unknowns in Equation (2.2) is

$$\sum_{i=0}^{d} \deg P_N^{[i]} + d + 1 = N + 1 \tag{2.3}$$

Expanding the left hand side of Equation (2.2) in powers of *x* and equating the first *N* equations of the system equal to zero, we get a system of linear homogeneous equations. To calculate the coefficients of the Pade′-Hermite polynomials it require some sort of normalization, such as

$$P_N^{[i]}(0) = 1 \text{ for some } 0 \le i \le d \tag{2.4}$$

It is important to emphasize that the only input required for the calculation of the Pade′-Hermite polynomials are the first N coefficients of the series $U_0, \ldots, U_d$. Equation (2.3) simply ensures that the coefficient matrix associated with the system is square. One way to construct the Pade′-Hermite polynomials is to solve the system of linear equations by any standard method such as Gaussian elimination or Gauss-Jordan elimination.

Drazin-Tourigny approximants (1996) is a particular kind of algebraic approximants and Khan (2002) introduced High-order differential approximant as a special type of differential approximants. High-order partial differential approximants discussed in Rahman (2004) is a multivariable differential approximants.

An algebraic programming language Maple available on www.maplesoft.com was used to compute the series coefficients of non-dimensional governing equation of the problem.

3. Mathematical formulation of the problem:

Consider the steady axisymmetric flow of a viscous incompressible fluid driven through a porous pipe with decelerating wall. Let E be a parameter such that the axial velocity of the wall is E_z. It is assumed that $\frac{aE}{V} = O(1)$ and $V \neq 0$ ($V > 0$ represents suction velocity and $V < 0$ represents the injection velocity).

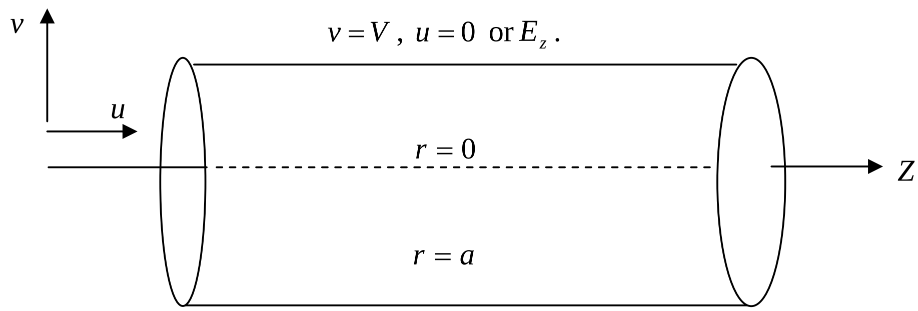

Fig.1: Schematic diagram of the problem.

Assuming a similarity form for the solution of the Navier-Stokes equation, it is found by dimensional analysis that the velocity components (u, v) increasing in the directions of (z, r) respectively and vorticity ω of the flow may be expressed as Makinde (1999)

$$u = \frac{z}{r}\frac{dF}{dr}, v = -\frac{1}{r}F \text{ and } \omega = -zG.$$

and hence
$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(rG)\right] = R\left[\frac{G}{r}\frac{dF}{dr} - F\frac{d}{dr}\left(\frac{G}{r}\right)\right], G = \frac{d}{dr}\left(\frac{1}{r}\frac{dF}{dr}\right), \tag{3.1}$$

$$F = 0, \frac{d}{dr}\left(\frac{1}{r}\frac{dF}{dr}\right) = 0, \text{ on } r = 0,$$

$$F = -1, \frac{dF}{dr} = -1, \text{ on } r = 1.$$

When $R = 0$, the equation (3.1) can be solved easily and the solution is a parabolic Poiseulli flow. But for $R \neq 0$ an exact solution to this nonlinear system of equations is impossible.
Therefore, we seek a power series in terms of R

$$F(r) = F_0(r) + F_1(r)R + F_2(r)R^2 + F_3(r)R^3 + \cdots. \tag{3.2}$$

By substituting (3.2) into (3.1), we obtain recurrence relations for the unknown coefficients of this series (for small R)

$$\left[\frac{1}{r}(rG_n)'\right]' = R\sum_{i=0}^{n-1}\left[G_i\left(\frac{F'_{n-i-1}}{r}\right) - F_i\left(\frac{G_{n-i-1}}{r}\right)'\right], G_n = \left(\frac{1}{r}F_n'\right)', \tag{3.3}$$

$$F_n = 0, \left(\frac{1}{r}F_n'\right)' = 0 \text{ on } r = 0,$$

$$F_0 = -1, F_{n+1} = 0, F_0' = -1, F'_{n+1} = 0, \text{ on } r = 1, n = 0,1,2,...$$

Where the prime symbol denotes differentiation with respect to r .
In order to compute the series coefficients, let

$$F(r, R) = \sum_{i=0}^{\infty}\left(\sum_{j=1}^{2i+2} a_{0,2j} r^{2j}\right)R^i . \tag{3.4}$$

By substituting (3.4) to Equation (3.3), the recurrence relation for *F(r)* becomes

$$16 j(j+1)^2 (j+2) a_{2j+4} r^{2j-1}$$
$$= \sum_{j=1}^{k} 8\frac{(j+1)!}{(j-1)!}(k-j+1) a_{2j+2} a_{2k-2j-2} r^{2k-1} - \sum_{j=1}^{k} 8\frac{(k-j+3)!}{(k-j)!} a_{2j} a_{2k-2j+6} r^{2k+1}$$

We expand $F(r,R)$, $\beta = \left(\frac{F'}{r}\right)'$ at $r=1$ and $F''(0)$ (that is, stream function, skin friction and centerline axial velocity parameter) in powers of the Reynolds number, to obtain

$$F(r) = -\frac{1}{2}r^2(3-r^2) - \frac{1}{144}r^2(7-r^2)(1-r^2)^2 R + ... , \tag{3.5}$$

$$\beta = 4 - \frac{1}{3}R - \frac{2}{27}R^2 - \frac{17}{1008}R^3 - .... \tag{3.6}$$

and

$$F''(0) = -3 - \frac{7}{72}R - \frac{103}{4800}R^2 - \frac{760589}{152409600}R^3 - .... \quad \text{as } R \to 0 . \tag{3.7}$$

These expansions yield a single solution of the Equation (3.1).

Using a symbolic algebraic package Maple, we are managed to compute the first 54 coefficients of the solution. Then the algebraic and differential approximation methods are applied to these series.

4. Results and Discussion:

The series (3.5), (3.6) and (3.7) of stream function, skin friction and centerline axial velocity in terms of flow Reynolds number respectively are considered for the investigation. Applying the High-order differential approximant to both the series of centerline axial velocity and skin friction, the two singular points R_1 and R_2 nearest to the origin are obtained. Table.1 displays that the singular point R_1 of centerline axial velocity converges to 15 decimal places at $d=8$ with $N=52$ and the other singular point R_2 of centerline axial velocity converges to 4 decimal places at $d=8$ with $N=52$. It is also observed that both the critical values R_1 and R_2 are compared with Makinde(1999) and the values of α confirm that R_1 and R_2 are branch points. Table.2 represents the singular point R_1 of skin friction converges to 13 decimal places at $d=8$ with $N=52$ and the other singular point R_2 also converges to 3 decimal places at $d=8$ with $N=52$. It is also noticed that both the values R_1 and R_2 are compared with Makinde(1999) and the values of α confirm that R_1 and R_2 are branch points.

Table. 1: Numerical values of R_1, R_2 and the critical exponent α of centerline axial velocity using High-order differential approximant. The results are compared with Makinde(1999).

d	N	R_1	α	R_2	α
2	7	3.1748622903786039	0.734469546787185866		
3	12	3.0768641905315211	0.511490960893766009		
4	18	3.0725166917148683	0.499719619856999405		
5	25	3.0724980065120133	0.499999902120348536	8.8164063517596622	0.4954596938484847968
6	33	3.0724980042445946	0.500000000098284071	8.8127995287711051	0.4991471015647478523
7	42	3.0724980042458199	0.499999999999926717	8.8131460358460523	0.4997653746760795187
8	52	3.0724980042458197	0.500000000000000001	8.8131097616625635	0.4999534119259680143
Makinde (1999)		3.0724980042458197		8.813114939	

Table. 2: Numerical values of R_1, R_2 and the critical exponent α of skin friction using High-order differential approximant. The results are compared with Makinde(1999).

d	N	R_1	α	R_2	α
2	7	3.169913601171531	0.1179160833837563		
3	12	3.078390933009397	0.4328071188188094		
4	18	3.072506610273949	0.4997190813662425	8.942239101069277	0.4060409189553583854
5	25	3.072498003441932	0.5000000089906489	8.441094619719760	2.7332819731928286414
6	33	3.072498004254962	0.4999999990682636	8.810061064410313	0.5076020616853571302
7	42	3.072498004245820	0.4999999999987227	8.813263038742537	0.4991927215986543196
8	52	3.072498004245819	0.4999999999999999	8.813083811310784	0.5003197469645002604
Makinde (1999)		3.0724980042458197		8.813114939	

Tables 1-2 show that the convergence of R_1 and R_2 increases very rapidly with the increase of d. It is remarkable that the secondary singularity also recovered from the information of a single series at the point of expansion.

Van Dyke (1974) argued that the convergence of the series may be limited by a singularity on the positive real axis. Using the graphical form of the D'Alembert's ratio test, Domb and Sykes (1957) together with Neville's extrapolation at $\frac{1}{n}=0$ (that is, $n \to \infty$) reveal that the radius of convergence is $R = 3.07249$ which supports our results.

The bifurcation diagrams for flow Reynolds number R are drawn in Figs. 2-3 applying Drazin-Tourigny method to the series (3.6) and (3.7).

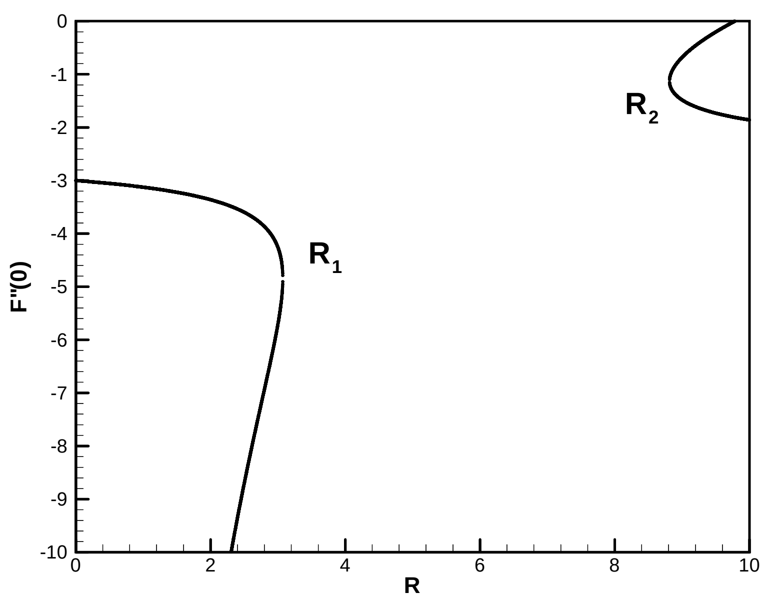

Fig 2(a): The bifurcation diagram of the first and second singular points R_1, R_2 in the $(F''(0), R)$ plane for the centerline axial velocity using the Drazin-Tourigny method with $d = 8$.

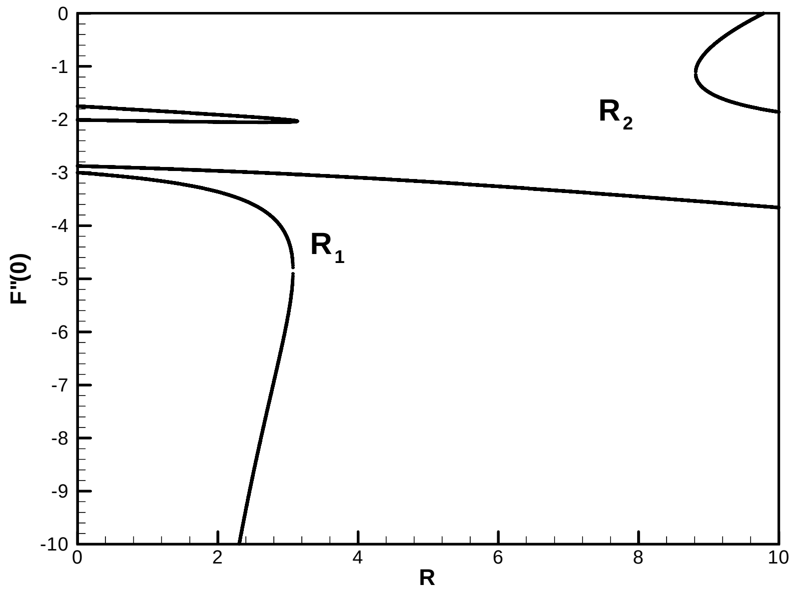

Fig 2(b): The bifurcation diagram of the first and second singular points R_1, R_2 in the $(F''(0), R)$ plane for the centerline axial velocity using the Drazin-Tourigny method with $d = 9$. Other curves are spurious.

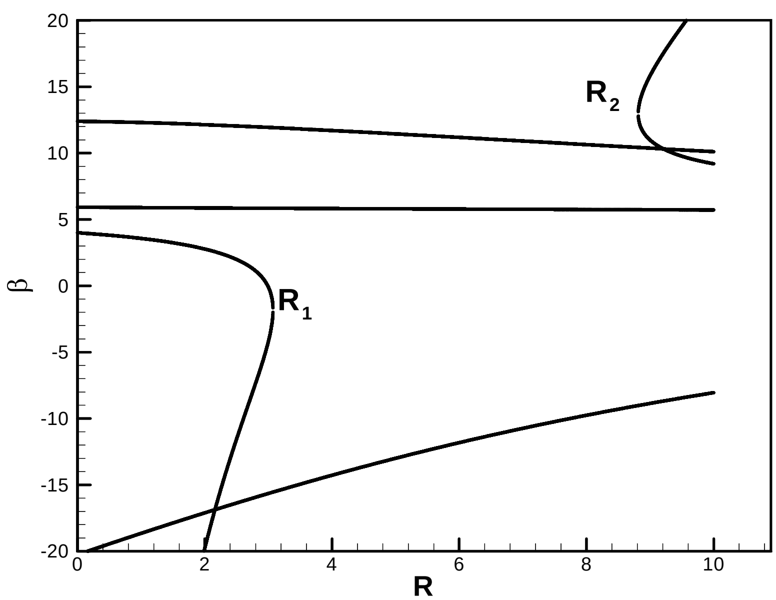

Fig. 3(a): The bifurcation diagram of the first and second singular points R_1, R_2 in the (β, R) plane for the skin friction using the Drazin-Tourigny method with $d = 7$. Other curves are spurious.

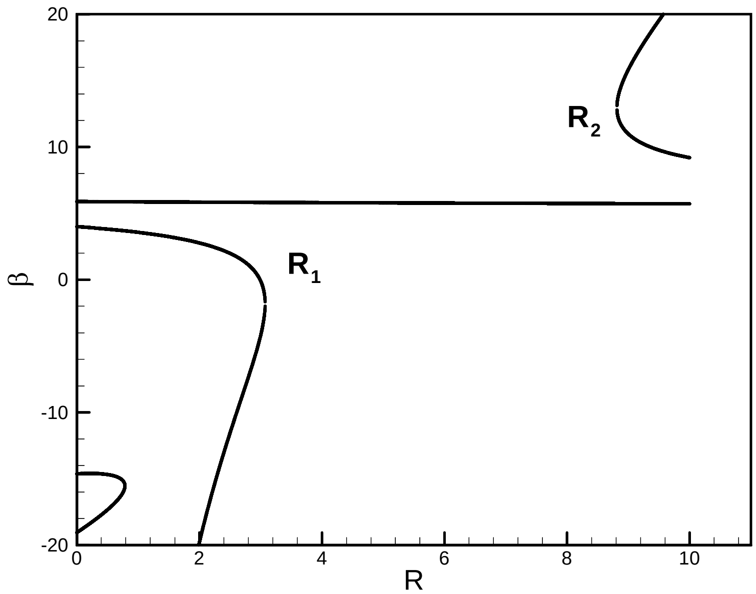

Fig. 3(b): The bifurcation diagram of the first and second singular points R_1, R_2 in the (β, R) plane for the skin friction using the Drazin-Tourigny method with $d = 8$. Other curves are spurious.

Fig. 2(a, b) illustrates that the two singular points R_1, R_2 of centerline axial velocity as the two solution branches at $d = 8$ and $d = 9$ respectively. Moreover, in Fig. 3(a, b) R_1, R_2 are also shown as a separation of the solution of skin friction for $d = 7$ and $d = 8$.

It is interesting to notice from the Figs 2-3 that the real solutions are absent for $R_1 < R < R_2$, and that $\beta \to 0$ as $R \to 2.828847,$ that is, reversal of the flow at the wall will occur.

5. Conclusion

The approximate solution behavior of the flow through a porous pipe with decelerating wall is investigated. By applying various types of Pade′-Hermite approximations, the accurate numerical approximations of the parameters of the flow are obtained, which involves physically the instability of the problem. The bifurcation diagrams of flow Reynolds number are significant results in comparison with Makinde(1999).

Moreover, we provide a basis for guidance about new approximants idea for summing power series should be chosen for many problems in fluid mechanics and similar subjects. The computing costs of finding the coefficients of a power series of this problem are higher than the costs of processing them by a summation method. So it behoves the user to exploit all the available information about the problem that gives rise to the series.

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