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# JOULE HEATING EFFECT ON MAGNETOHYDRODYNAMIC NATURAL CONVECTION FLOW ALONG A VERTICAL WAVY SURFACE

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## Abstract:
*In this paper, the effect of Joule heating on magnetohydrodynamic natural convection flow of viscous incompressible fluid along a uniformly heated vertical wavy surface has been investigated. The governing boundary layer equations with associated boundary conditions for this phenomenon are converted to non-dimensional form using a suitable transformation. The equations are mapped into the domain of a vertical flat plate and then solved numerically employing the implicit finite difference method, known as the Keller-box scheme. Effects of pertinent parameters, such as the Joule heating parameter (J), Prandtl number (Pr), magnetic parameter (M) and the amplitude of the wavy surface α on the surface shear stress in terms of the skin friction coefficient ($C_{fx}$), the rate of heat transfer in terms of local Nusselt number ($Nu_x$), the streamlines and the isotherms are discussed. A comparison with previously published work is performed and the results show excellent agreement.*

***Keywords:*** Magnetohydrodynamics, Joule heating, natural convection, uniform surface temperature, Keller-box method, wavy surface

## NOMENCLATURE

| | |
|---|---|
| $C_{fx}$ | local skin friction coefficient |
| $C_p$ | specific heat at constant pressure ($Jkg^{-1}K^{-1}$) |
| $f$ | dimensionless stream function |
| $g$ | acceleration due to gravity ($ms^{-2}$) |
| $Gr$ | Grashof number |
| $J$ | Joule heating parameter |
| $k$ | thermal conductivity of fluid ($Wm^{-1}K^{-1}$) |
| $L$ | wavelength associated with the wavy surface (m) |
| $M$ | magnetic parameter |
| $Nu_x$ | local Nusselt number |
| $P$ | pressure of the fluid ($Nm^{-2}$) |
| Pr | Prandtl number |
| $T$ | temperature of the fluid in the boundary layer (K) |
| $T_w$ | temperature at the surface (K) |
| $T_\infty$ | temperature of the ambient fluid (K) |
| $u$, $v$ | dimensionless velocity components along the ($x$, $y$) axes ($ms^{-1}$) |
| $x$, $y$ | axis in the direction along and normal to the tangent of the surface |

### *Greek symbols*

| | |
|---|---|
| $\alpha$ | amplitude of the wavy surface |
| $\beta$ | volumetric coefficient of thermal expansion($K^{-1}$) |
| $\beta_0$ | applied magnetic field strength |
| $\eta$ | dimensionless similarity variable |
| $\theta$ | dimensionless temperature function |
| $\psi$ | stream function ($m^2s^{-1}$) |
| $\mu$ | dynamic viscosity of the fluid ($Kg\ m^{-1}s^{-1}$) |
| $\nu$ | kinematic viscosity of the fluid ($m^2s^{-1}$) |
| $\rho$ | density of the fluid ($Kg\ m^{-3}$) |
| $\sigma_0$ | electrical conductivity of the fluid ($\Omega^{-1}m^{-1}$) |
| $\tau_w$ | shearing stress |
| $\sigma(x)$ | surface profile function defined in equation (1) |

### *Subscripts*

| | |
|---|---|
| $w$ | wall conditions |
| $\infty$ | ambient conditions |

### *Superscripts*

| | |
|---|---|
| $'$ | differentiation with respect to $\eta$ |

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## 1. Introduction

Laminar natural convection boundary layer flow and heat transfer problem from a vertical wavy surface get a great deal of attention in various branches of engineering. Along with the natural convection flow the phenomenon of the boundary layer flow of an electrically conducting fluid in the presence of Joule heating and magnetic field are also very common because of their applications in nuclear engineering in connection with the cooling of reactors. If the surface is roughened the flow is disturbed by the surface and this alters the rate of heat transfer. These types of roughened surface are taken into account in several heat transfer collectors, flat plate condensers in refrigerators and heat exchanger. One common example of a heat exchanger is the radiator used in car, in which the heat generated from engine transferred to air flowing through the radiator. The interface between concurrent or countercurrent two-phase flow is another example remotely related to this problem. Such an interface is always wavy and momentum transfer across it is by no means similar to that across a smooth, flat surface and neither is the heat transfer. Also a wavy interface can have an important effect on the condensation process.

The effects of nonuniformities of surface waviness on the natural convection boundary layer flow of a Newtonian fluid have studied by Yao (1983) and Moulic and Yao (1989). Yao used an extended Prantdl's transposition theorem and a finite-difference scheme. He proposed a simple transformation to study the natural convection heat transfer for an isothermal vertical sinusoidal surface. These simple coordinate transformations method to change the wavy surface into a flat plate. Hossain (1992) analyzed the viscous and Joule heating effects on MHD free convection flow with variable plate temperature. Rees and Pop (1994) investigated the natural convection boundary layer induced by vertical and horizontal wavy surface exhibiting small amplitude waves embedded in a porous medium. The magnetohydrodynamic boundary layer flow and heat transfer from a continuous moving wavy surface have been investigated by Hossain and Pop (1996). Alam et al. (1997) studied the problem of free convection from a wavy vertical surface in presence of a transverse magnetic field. On the other hand, the combined effects of thermal and mass diffusion on the natural convection flow of a viscous incompressible fluid along a vertical wavy surface have been investigated by Hossain and Rees (1999). In this paper the effect of waviness of the surface on the heat and mass flux is investigated in combination with the species concentration for a fluid having Prandtl number equal to 0.7. Cheng (2000) investigated the natural convection heat and mass transfer near a vertical wavy surface with constant wall temperature and concentration in a porous medium. Hossain et al. (2002) considered the problem of natural convection of fluid with temperature dependent viscosity along a heated vertical wavy surface. Amin (2003) analyzed combined effect of viscous dissipation and Joule heating on MHD forced convection over a non isothermal horizontal cylinder embedded in a fluid saturated porous medium. Molla et al. (2004) investigated natural convection flow along a vertical wavy surface with uniform surface temperature in presence of heat generation/absorption. They found the effect of varying the heat generation/absorption on the heat transfer rate in terms of local Nusselt number as well as on the streamlines and isotherm patterns for very small Prandtl number Pr ranging from 0.001 to 1.0. They concluded that the velocity and temperature distributions for the case of heat generation higher than that of the heat absorption case. Yao (2006) considered natural convection flow along a vertical complex wavy surface. Alim et al. (2007) investigated Joule heating effect on the coupling of conduction with MHD free convection flow from a vertical flat plate. Combined effects of viscous dissipation and Joule heating on the coupling of conduction and free convection along a vertical flat plate have also been studied by Alim et al. (2008). Nasrin and Alim (2009) studied combined effects of viscous dissipation and temperature dependent thermal conductivity on MHD free convection flow with conduction and Joule heating along a vertical flat plate. Very recently, Parveen and Alim (2011) analyzed the effect of temperature dependent thermal conductivity on magnetohydrodynamic natural convection flow along a vertical wavy surface. At the same time Parveen and Alim (2011) also studied the effect of temperature dependent variable viscosity on magnetohydrodynamic natural convection flow along a vertical wavy surface. The above literature survey shows that the Joule heating effect on magnetic field is an interesting macroscopic physical phenomenon in fluid dynamics. None of the above investigations considered the effect of Joule heating on MHD natural convection flow along wavy surface. Joule heating in electronics and in physics refers to the increase in temperature of a conductor as a result of resistance to an electrical current flowing through it.

In this paper, attention has been given to study the Joule heating effect in presence of magnetic field of electrically conducting fluid with free convection boundary layer flow along a vertical wavy surface. It is assumed that the wavy surface is electrically insulated and is maintained at a uniform temperature $T_w$. Far above the wavy plate, the fluid is stationary and is kept at a temperature $T_\infty$, where $T_w > T_\infty$. Using the appropriate transformations, the boundary layer equations are reduced to non-similar partial differential forms. The transformed boundary layer equations are solved numerically using implicit finite difference method known as the Keller box technique (1978). Consideration is given to the situation where the buoyancy force assist the natural convection flow for various values of the Joule heating parameter $J$, Prandtl number Pr, magnetic parameter $M$ and the amplitude of the wavy surface $\alpha$.

## 2. Formulation of the Problem

The boundary layer analysis outlined below allows $\bar{\sigma}(\bar{x})$ being arbitrary, but our detailed numerical work assumed that the surface exhibits sinusoidal deformations. The wavy surface may be described by

$$\bar{y}_w = \bar{\sigma}(\bar{x}) = \alpha \sin\left(\frac{n\pi\bar{x}}{L}\right) \tag{1}$$

where $L$ is the wavelength associated with the wavy surface.

The geometry of the wavy surface and the two-dimensional Cartesian coordinate system are shown in Fig. 1.

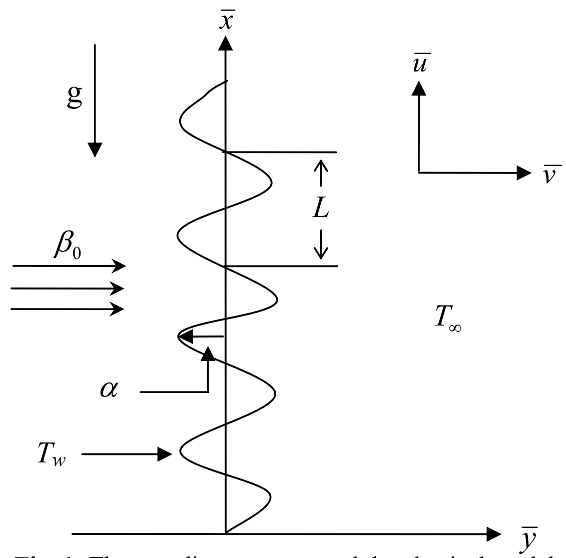

**Fig. 1:** The coordinate system and the physical model

Under the usual Boussinesq approximation, the equations governing the flow can be written as:

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0 \tag{2}$$

$$\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial \bar{x}} + \nu\nabla^2\bar{u} + g\beta(T - T_\infty) - \frac{\sigma_0\beta_0^2}{\rho}\bar{u} \tag{3}$$

$$\bar{u}\frac{\partial \bar{v}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{v}}{\partial \bar{y}} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial \bar{y}} + \nu\nabla^2\bar{v} \tag{4}$$

$$\bar{u}\frac{\partial T}{\partial \bar{x}} + \bar{v}\frac{\partial T}{\partial \bar{y}} = \frac{k}{\rho C_p}\nabla^2 T + \frac{\sigma_0\beta_0^2}{\rho C_p}\bar{u}^2 \tag{5}$$

where $(\bar{x}, \bar{y})$ are the dimensional coordinates along and normal to the tangent of the surface and $(\bar{u}, \bar{v})$ are the velocity components parallel to $(\bar{x}, \bar{y})$, $\nabla^2(= \partial^2/\partial x^2 + \partial^2/\partial y^2)$ is the Laplacian operator, $g$ is the acceleration due to gravity, $\bar{p}$ is the dimensional pressure of the fluid, $\rho$ is the density, $\beta_0$ is the strength of magnetic field, $\sigma_0$ is the electrical conduction, $k$ is the thermal conductivity, $\beta$ is the coefficient of thermal expansion, $\nu$ ( = $\mu/\rho$) is the kinematic viscosity, $\mu$ is the dynamic viscosity of the fluid in the boundary layer and $C_p$ is the specific heat due to constant pressure.

The boundary conditions relevant to the present problem are

$$\bar{u} = 0, \bar{v} = 0, T = T_w \quad at\ \bar{y} = \bar{y}_w = \bar{\sigma}(\bar{x}) \tag{6a}$$

$$\bar{u} = 0, \quad T = T_\infty, \quad \bar{p} = p_\infty \quad as \quad \bar{y} \to \infty \tag{6b}$$

where $T_w$ is the surface temperature, $T_\infty$ is the ambient temperature of the fluid and $p_\infty$ is the pressure of fluid outside the boundary layer.

Following Yao (1983), we now introduce the following nondimensional variables:

$$x = \frac{\bar{x}}{L}, \quad y = \frac{\bar{y} - \bar{\sigma}}{L} Gr^{\frac{1}{4}} \; p = \frac{L^2}{\rho\nu^2} Gr^{-1} \bar{p}$$

$$u = \frac{L}{\nu} Gr^{-1/2} \bar{u}, \quad v = \frac{L}{\nu} Gr^{-1/4} (\bar{v} - \sigma_x \bar{u}), \; \theta = \frac{T - T_\infty}{T_w - T_\infty} \tag{7}$$

$$\sigma_x = \frac{d\bar{\sigma}}{d\bar{x}} = \frac{d\sigma}{dx}, \; Gr = \frac{g\beta(T_w - T_\infty)}{\nu^2} L^3$$

where $\theta$ is the dimensionless temperature function and ($u$, $v$) are the dimensionless velocity components parallel to ($x$, $y$). Here ($x$, $y$) are not orthogonal, but a regular rectangular computational grid can be easily fitted in the transformed coordinates. It is also worthwhile to point out that ($u$, $v$) are the velocity components parallel to ($x$, $y$) which are not parallel to the wavy surface.

Introducing the above dimensionless dependent and independent variables into Equations (2)–(5), the following dimensionless form of the governing equations are obtained after ignoring terms of smaller orders of magnitude in $Gr$, the Grashof number defined in (7).

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{8}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + Gr^{1/4}\sigma_x \frac{\partial p}{\partial y} + \left(1 + \sigma_x^2\right)\frac{\partial^2 u}{\partial y^2} - Mu + \theta \tag{9}$$

$$\sigma_x\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -Gr^{1/4}\frac{\partial p}{\partial y} + \sigma_x\left(1 + \sigma_x^2\right)\frac{\partial^2 u}{\partial y^2} - \sigma_{xx}u^2 \tag{10}$$

$$u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y} = \frac{1}{\Pr}\left(1 + \sigma_x^2\right)\frac{\partial^2 \theta}{\partial y^2} + Ju^2 \tag{11}$$

In the above equations Pr, $J$ and $M$ are respectively known as the Prandtl number, the Joule heating parameter and magnetic parameter, which are defined as

$$\Pr = \frac{\nu}{\alpha}, \; J = \frac{\sigma_0 \beta_0^2 \nu Gr^{1/2}}{\rho C_p \left(T_w - T_\infty\right)} \text{ and } M = \frac{\sigma_0 \beta_0^2 L^2}{\mu Gr^{1/2}} \tag{12}$$

It can easily be seen that the convection induced by the wavy surface is described by Equations (8)–(11). We further notice that, Equation (10) indicates that the pressure gradient along the $y$-direction is $O(Gr^{-1/4})$, which implies that lowest order pressure gradient along $x$ -direction can be determined from the inviscid flow solution. For the present problem this pressure gradient ($\partial p/\partial x = 0$) is zero. Equation (10) further shows that $Gr^{-1/4}\partial p/\partial y$ is $O(1)$ and is determined by the left-hand side of this equation. Thus, the elimination of $\partial p/\partial y$ from Equations (9) and (10) leads to

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\left(1+\sigma_x^2\right)\frac{\partial^2 u}{\partial y^2}-\frac{\sigma_x\sigma_{xx}}{1+\sigma_x^2}u^2-\frac{M}{1+\sigma_x^2}u+\frac{1}{1+\sigma_x^2}\theta \tag{13}$$

The corresponding boundary conditions for the present problem then turn into

$$\left.\begin{aligned} &u=v=0, \quad \theta=1 \qquad at \quad y=0 \\ &u=\theta=0, \quad p=0 \qquad as \quad y\to\infty \end{aligned}\right\} \tag{14}$$

Now we introduce the following transformations to reduce the governing equations to a convenient form:

$$\psi=x^{3/4}f(x,\eta), \quad \eta=yx^{-1/4}, \quad \theta=\theta(x,\eta) \tag{15}$$

where $\eta$ is the pseudo similarity variable and $\psi$ is the stream function that satisfies the Equation (8) and is defined by

$$u=\frac{\partial\psi}{\partial y}, \quad v=-\frac{\partial\psi}{\partial x}. \tag{16}$$

Introducing the transformations given in Equation (15) into Equations (13) and (11) the momentum and energy equations transformed into the new co-ordinate system. Thus the resulting equations are obtained

$$\begin{aligned} &\left(1+\sigma_x^2\right)f'''+\frac{3}{4}ff''-\left(\frac{1}{2}+\frac{x\sigma_x\sigma_{xx}}{1+\sigma_x^2}\right)f'^2+\frac{1}{1+\sigma_x^2}\theta-\frac{Mx^{1/2}}{1+\sigma_x^2}f' \\ &=x\left(f'\frac{\partial f'}{\partial x}-f''\frac{\partial f}{\partial x}\right) \end{aligned} \tag{17}$$

$$\frac{1}{\Pr}\left(1+\sigma_x^2\right)\theta''+\frac{3}{4}f\theta'+J\,x^{3/2}f'^2=x\left(f'\frac{\partial\theta}{\partial x}-\theta'\frac{\partial f}{\partial x}\right) \tag{18}$$

The boundary conditions (14) now take the following form:

$$\left.\begin{aligned} &f(x,o)=f'(x,o)=0, \qquad \theta(x,o)=1 \\ &f'(x,\infty)=0, \qquad \theta(x,\infty)=0 \end{aligned}\right\} \tag{19}$$

In the above equations prime denote the differentiation with respect to $\eta$.

However, it is important to calculate the values of the shearing stress $\tau_w$ and the rate of heat transfer in terms of the skin friction coefficients $C_{fx}$ and Nusselt number $Nu_x$ respectively, which can be written as

$$C_{fx}=\frac{2\tau_w}{\rho U^2} \quad \text{and} \quad Nu_x=\frac{q_w x}{k(T_w-T_\infty)} \tag{20}$$

$$\text{where} \quad \tau_w=(\mu\,\overline{n}.\nabla\overline{u})_{y=0} \text{ and } q_w=-k(\overline{n}.\nabla T)_{y=0} \tag{21}$$

Using the transformations (15) into Equation (20), the local skin friction coefficient, $C_{fx}$ and the rate of heat transfer in terms of the local Nusselt number, $Nu_x$ takes the following form:

$$C_{fx}(Gr/x)^{1/4}/2=\sqrt{1+\sigma_x^2}\,f''(x,o) \tag{22}$$

$$Nu_x(Gr/x)^{-1/4}=-\sqrt{1+\sigma_x^2}\,\theta'(x,o) \tag{23}$$

## 3. Method of Solution

This paper concerns the natural convection flow of viscous incompressible fluid along a uniformly heated vertical wavy surface in presence of Joule heating and magnetic field has been investigated using the very efficient implicit finite difference method known as the Keller box scheme developed by Keller (1978), which is well documented by Cebeci and Bradshaw (1984).

To apply the aforementioned method, Equations (17) and (18) their boundary condition (19) are first converted into the following system of first order equations. For this purpose we introduce new dependent variables $u(\xi,\eta)$, $v(\xi,\eta)$, $p(\xi,\eta)$ and $g(\xi,\eta)$ so that the transformed momentum and energy equations can be written as

$$f' = u \tag{24}$$

$$u' = v \tag{25}$$

$$g' = p \tag{26}$$

$$P_1 v' + P_2 f\, v - P_3 u^2 + P_4 g - P_5 u = \xi\left(u\frac{\partial u}{\partial \xi} - v\frac{\partial f}{\partial \xi}\right) \tag{27}$$

$$\frac{1}{\Pr} P_1 p' + P_2 f\, p + P_6 u^2 = \xi\left(u\,\frac{\partial g}{\partial \xi} - p\,\frac{\partial f}{\partial \xi}\right) \tag{28}$$

where $x = \xi$, $\theta = g$ and

$$P_1 = \left(1+\sigma_x^2\right),\quad P_2 = \frac{3}{4},\quad P_3 = \frac{1}{2} + \frac{x\sigma_x\sigma_{xx}}{1+\sigma_x^2},\quad P_4 = \frac{1}{1+\sigma_x^2},\quad P_5 = \frac{Mx^{1/2}}{1+\sigma_x^2} \text{ and } P_6 = J\,x^{3/2}$$

and the boundary conditions (19) are

$$\begin{aligned} &f(\xi,0) = 0,\; u(\xi,0) = 0, g(\xi,0) = 1 \\ &u(\xi,\infty) = 0,\; g\;(\xi,\infty) = 0 \end{aligned} \tag{29}$$

Now consider the net rectangle on the $(\xi,\eta)$ plane shown in the Fig. 2 and denote the net points by

$$\begin{aligned} &\xi^0 = 0,\; \xi^n = \xi^{n-1} + k_n,\; n = 1,2,\cdots\cdots\cdots\cdots, N \\ &\eta_0 = 0,\; \eta_j = \eta_{j-1} + h_j,\quad j = 1,2,\cdots\cdots\cdots\cdots, J \end{aligned} \tag{30}$$

Here $n$ and $j$ are just sequence of numbers on the $(\xi,\eta)$ plane, $k_n$ and $h_j$ are the variable mesh widths. Approximate the quantities $f$, $u$, $v$ and $p$ at the points $(\xi^n,\eta_j)$ of the net by $f_j^n$, $u_j^n$, $v_j^n$, $p_j^n$ which call net function. It is also employed that the notation $P_j^n$ for the quantities midway between net points shown in Fig. 2 and for any net function as

$$\xi^{n-1/2} = \frac{1}{2}(\xi^n + \xi^{n-1}) \tag{31}$$

$$\eta_{j-1/2} = \frac{1}{2}(\eta_j + \eta_{j-1}) \tag{32}$$

$$g_j^{\,n-1/2} = \frac{1}{2}(g_j^n + g_j^{n-1}) \tag{33}$$

$$g_{j-1/2}^n = \frac{1}{2}(g_j^n + g_{j-1}^n) \tag{34}$$

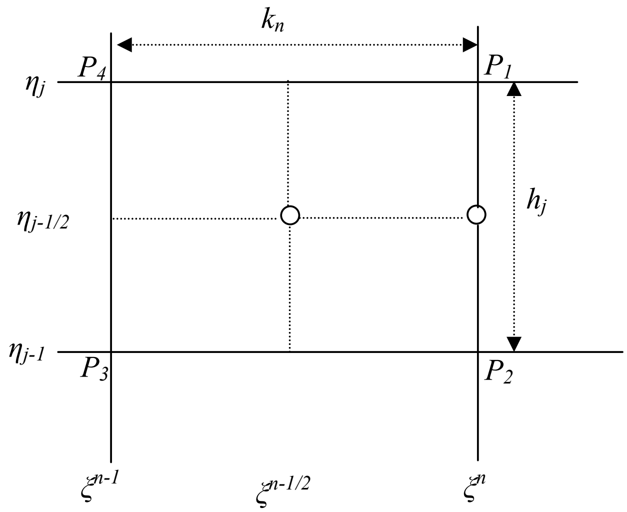


**Fig. 2:** Net rectangle of difference approximations for the Box scheme.

The finite difference approximations according to box method to the three first order ordinary differential equations (24) – (26) are written for the mid point $(\xi^n, \eta_{j-1/2})$ of the segment $P_1P_2$ shown in the Fig. 2.

$$\frac{f_j^n - f_{j-1}^n}{h_j} = u_{j-1/2}^n = \frac{u_{j-1}^n + u_j^n}{2} \tag{35}$$

$$\frac{u_j^n - u_{j-1}^n}{h_j} = v_{j-1/2}^n = \frac{v_{j-1}^n + v_j^n}{2} \tag{36}$$

$$\frac{g_j^n - g_{j-1}^n}{h_j} = p_{j-1/2}^n = \frac{p_{j-1}^n + p_j^n}{2} \tag{37}$$

The finite difference approximations to the two first order differential equations (27) and (28) are written for the mid point $(\xi^{n-1/2}, \eta_{j-1/2})$ of the rectangle $P_1P_2P_3P_4$. This procedure yields

$$\frac{1}{2}(P_1)_{j-1/2}^n\left(\frac{v_j^n - v_{j-1}^n}{h_j}\right) + \frac{1}{2}(P_1)_{j-1/2}^{n-1}\left(\frac{v_j^{n-1} - v_{j-1}^{n-1}}{h_j}\right) + \left(P_2 fv\right)_{j-1/2}^{n-1/2} - (P_3u^2)_{j-1/2}^{n-1}$$
$$+ (P_4 g)_{j-1/2}^{n-1} - (P_5 u)_{j-1/2}^{n-1} = \xi_{j-1/2}^{n-1/2}\left(u_{j-1/2}^{n-1/2}\frac{u_{j-1/2}^n - u_{j-1/2}^{n-1}}{k_n} - v_{j-1/2}^{n-1/2}\frac{f_{j-1/2}^n - f_{j-1/2}^{n-1}}{k_n}\right) \tag{38}$$

$$\frac{1}{2\Pr}\left\{(P_1)_{j-1/2}^n\right\}\left(\frac{p_j^n - p_{j-1}^n}{h_j}\right) + \frac{1}{2\Pr}\left\{(P_1)_{j-1/2}^{n-1}\right\}\left(\frac{p_j^{n-1} - p_{j-1}^{n-1}}{h_j}\right) + \left(P_2 fp\right)_{j-1/2}^{n-1/2}$$
$$+ \left(P_6 u^2\right)_{j-1/2}^{n-1/2} = \xi_{j-1/2}^{n-1/2}\left(u_{j-1/2}^{n-1/2}\frac{g_{j-1/2}^n - g_{j-1/2}^{n-1}}{k_n} - p_{j-1/2}^{n-1/2}\frac{f_{j-1/2}^n - f_{j-1/2}^{n-1}}{k_n}\right) \tag{39}$$

The above equations are to be linearized by using Newton's Quasi-linearization method. Then linear algebraic equations can be written in block matrix which form a coefficient matrix. The whole procedure, namely reduction to first order followed by central difference approximations, Newton's Quasi-linearization method and the block Thomas algorithm, is well known as the Keller-box method.

## 4. Results and Discussion

Here we have discussed the numerical results obtained from parabolic differential equations (17)-(18) using the method mentioned above. It can be seen that the solutions are affected by four parameters, namely the Joule heating parameter $J$, Prandtl number Pr, magnetic parameter $M$ and the amplitude of the wavy surface $\alpha$. Numerical values of local shearing stress and the rate of heat transfer are calculated from Equations (22) and (23) in terms of the skin friction coefficient $C_{fx}$ and Nusselt number $Nu_x$ respectively for a wide range of the axial distance $x$ starting from the leading edge. These are shown graphically in Figs. 3-7 for different values of the aforementioned parameters $J$, $M$, Pr and $\alpha$.

The effect of magnetic parameter $M$ on the surface shear stress in terms of the local skin friction coefficient $C_{fx}$ and the rate of heat transfer in terms of the local Nusselt number $Nu_x$ are depicted graphically in Fig. 3 and Fig. 4 for $J = 0$ and $J > 0$, respectively while $\alpha = 0.2$ and Pr = 1.0. The skin friction coefficient $C_{fx}$ and local rate of heat transfer $Nu_x$ varies according to the slope of the wavy surface. This is due to the alignment of the buoyancy force $1/(1+\sigma_x^2)$, as shown in Equation (17), which drives the flow tangentially to the wavy surface. It is observed from Figs. 3 and 4 that with and without effects of Joule heating parameter the skin friction coefficient, the rate of heat transfer and their amplitude reduce at a great extent for increasing values of the magnetic parameter $M$. The magnetic field acts against the direction of fluid flow and reduce the skin friction and the rate of heat transfer. Again from Fig. 4 considering the Joule heating parameter $J$ the skin friction coefficient is higher and the rate of heat transfer becomes slower than that of not considering $J$, which is expected.

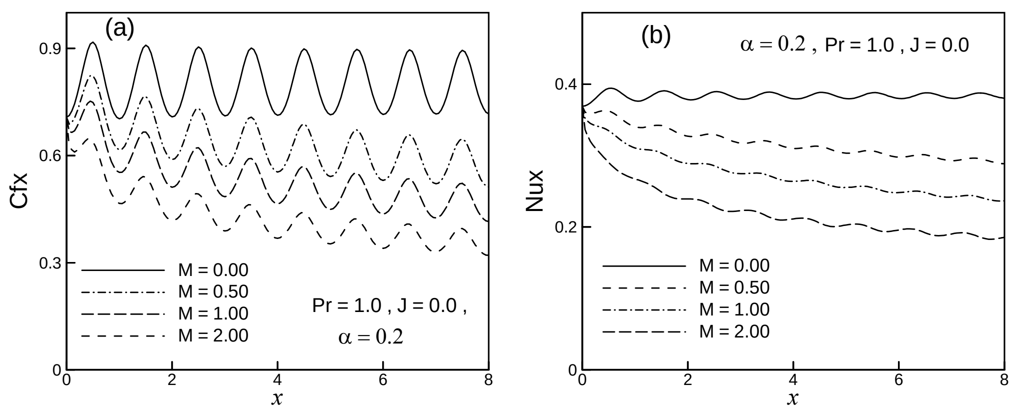


**Fig. 3:** Variation of (a) skin friction coefficient $C_{fx}$ and (b) rate of heat transfer $Nu_x$ for varying of magnetic parameter $M$ against $x$ while $J = 0.0$, Pr = 1.0 and $\alpha = 0.2$.

The effect of Joule heating parameter $J$ the local skin friction coefficient $C_{fx}$ and the rate of heat transfer in terms of the local Nusselt number $Nu_x$ from the wavy surface while $\alpha = 0.2$, $M = 0.01$ and Pr = 0.5 is illustrated in Fig. 5. From Fig. 5 it is noted that the skin friction coefficient increases slowly along the upstream direction of the surface and to decrease of the heat transfer rates. The maximum values of the skin friction coefficient $C_{fx}$ are 1.01264 and 1.01382 for $J = 0.00$ and 0.05 respectively which occurs at the same point $x = 0.5$. Furthermore, the maximum values of the rate of heat transfer $Nu_x$ are 0.30649 and 0.30379 for $J = 0.00$ and 0.05 respectively which occurs at the different position of $x$. It is observed that the skin friction coefficient increases by 0.12% and the rate of heat transfer decreases by 0.88% when $J$ increases from 0.00 to 0.05.

The variation of the local skin friction coefficient $C_{fx}$ and local rate of heat transfer $Nu_x$ for different values of Prandtl number Pr for $J = 0.001$, $M = 1.0$ and $\alpha = 0.2$ are depicted graphically in Fig. 6(a) and Fig. 6(b) respectively. The skin friction coefficient decreases and the rate of heat transfer increases for increasing value of the Prandtl number Pr. Increasing values of Prandtl number Pr, speed up the decay of the temperature field away from the heated surface with a consequent increase in the rate of heat transfer and reduces the skin friction coefficient. The maximum values of the local skin friction coefficient $C_{fx}$ are 0.78581, 0.75212, 0.64204 and 0.55692 for Pr = 0.70, 1.0, 3.0 and 7.0 respectively and each of which occurs at $x = 0.45$. It is noted that the skin friction coefficient decreases by 29.13% when Pr increases from 0.70 to 7.0. Furthermore, the maximum values of the rate of heat transfer $Nu_x$ are 0.32504, 0.36889, 0.53037 and 0.68546 for Pr = 0.70, 1.0, 3.0 and 7.0

respectively and each of which occurs at the surface. It is observed that the rate of heat transfer increases by 52.58% when Pr increases from 0.70 to 7.0.

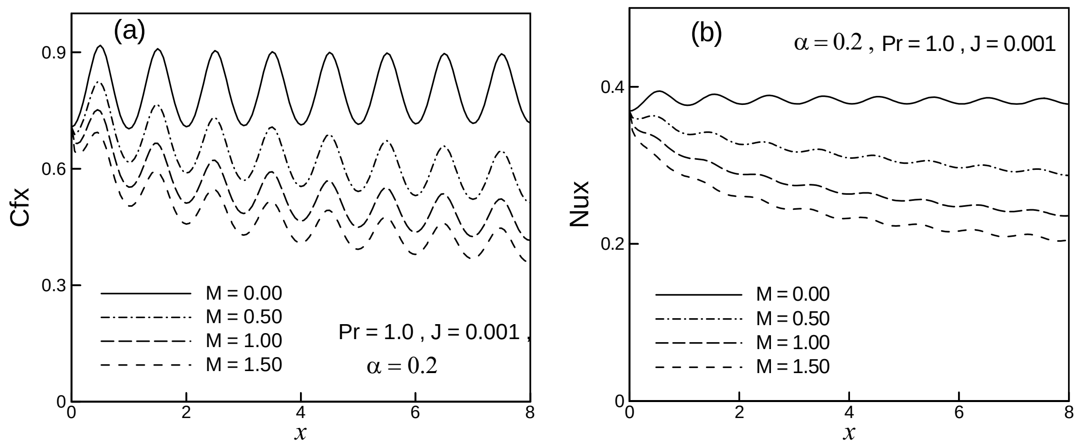


**Fig. 4:** Variation of (a) skin friction coefficient $C_{fx}$ and (b) rate of heat transfer $Nu_x$ for varying of magnetic parameter $M$ against $x$ while $J > 0$ ($J = 0.001$), Pr = 1.0 and $\alpha = 0.2$.

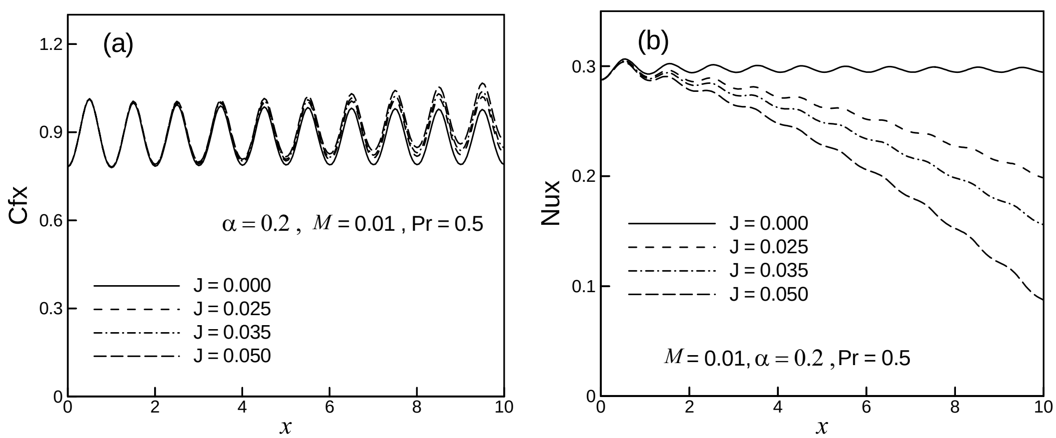


**Fig. 5:** Variation of (a) skin friction coefficient $C_{fx}$ and (b) rate of heat transfer $Nu_x$ for varying of Joule heating parameter $J$ against $x$ while $\alpha = 0.2$, $M = 0.01$ and Pr = 0.5.

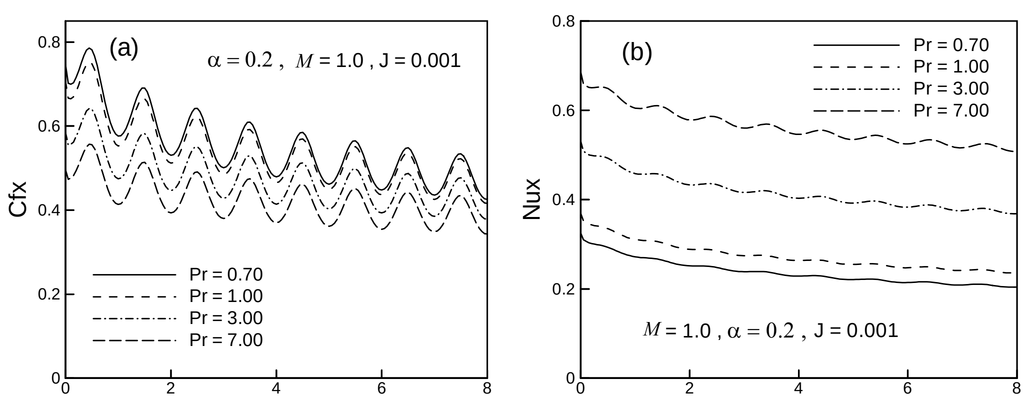


**Fig. 6:** Variation of (a) skin friction coefficient $C_{fx}$ and (b) rate of heat transfer $Nu_x$ for varying of Prandtl number Pr against $x$ while $M = 1.0$, $J = 0.001$ and $\alpha = 0.2$.

Figs. 7(a) and 7(b) show that increase in the value of the amplitude of wavy surface ($\alpha$ = 0.0, 0.1, 0.2, 0.3) leads to decrease the value of the skin friction coefficient and the rate of heat transfer in terms of the local Nusselt number while Prandtl number Pr = 1.0, magnetic parameter $M$ = 1.0 and Joule heating parameter $J$ = 0.001. Frictional force depends on the smoothness of the surface, temperature and nature of fluid. Surface becomes more roughened for increasing values of amplitude of the wavy surface. Velocity force decreases at the local points. It is seen that the skin friction coefficient and the heat transfer rate decrease by 16.84% and 18.86% respectively as $\alpha$ increases from 0.0 to 0.3.

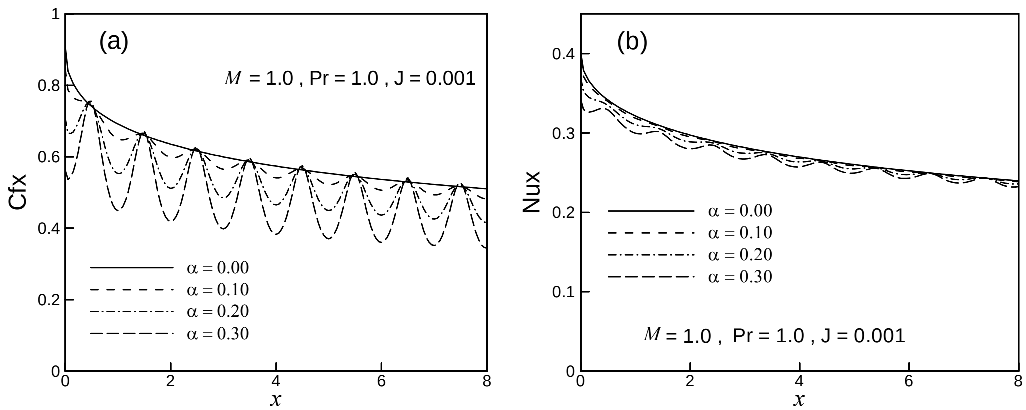


**Fig. 7:** Variation of (a) skin friction coefficient $C_{fx}$ and (b) rate of heat transfer $Nu_x$ for varying of amplitude of the wavy surface $\alpha$ against $x$ while $M$ = 1.0, $J$ = 0.001 and Pr = 1.0.

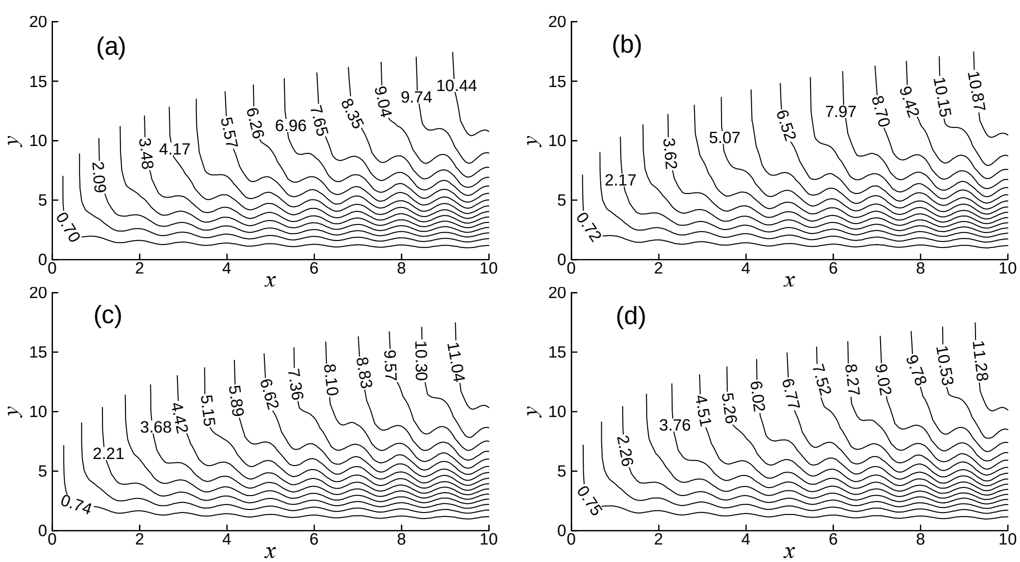


**Fig. 8:** Streamlines for (a) $J$ = 0.00 (b) $J$ = 0.025 (c) $J$ = 0.035 (d) $J$ = 0.05 while Pr = 0.5, $\alpha$ = 0.2 and $M$ = 0.01.

The influence of the Joule heating parameter $J$ on the development of streamlines and isotherms profile which are plotted for the amplitude of the wavy surface $\alpha$ = 0.2, Prandtl number Pr = 0.5 and $M$ = 0.01 are shown in Figs. 8 and 9 respectively. It is observed that as the value of $J$ increases, the maximum value of $\psi$ increase steadily. When $J$ = 0.0, $\psi_{max}$ = 10.44 and $J$ = 0.05, $\psi_{max}$ = 11.28. From Fig. 9, it is noted that the thermal boundary layer becomes thicker for increasing value of $J$.

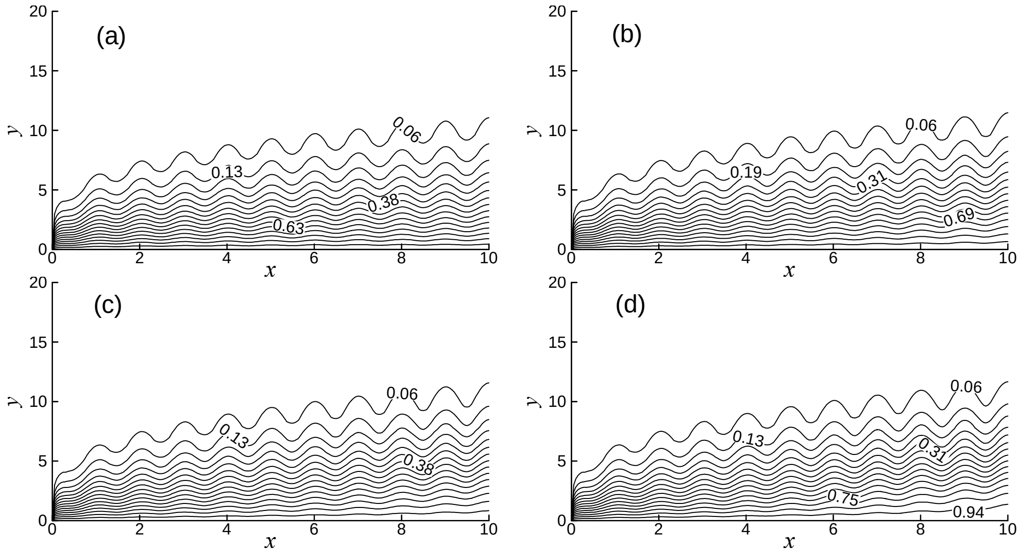


**Fig. 9:** Isotherms for (a) $J = 0.00$ (b) $J = 0.025$ (c) $J = 0.035$ (d) $J = 0.05$ while Pr = 0.5, $\alpha = 0.2$ and $M = 0.01$.

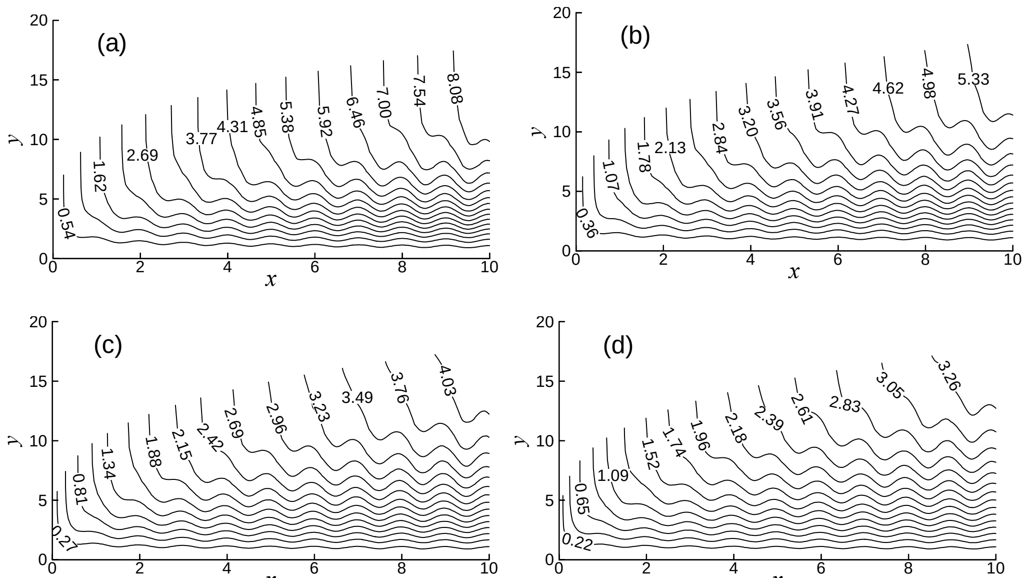


**Fig. 10:** Streamlines for (a) $M = 0.0$ (b) $M = 0.5$ (c) $M = 1.0$ (d) $M = 1.5$ while Pr = 1.0, $\alpha = 0.2$ and $J = 0.001$.

Figs. 10 and 11 illustrate the effect of magnetic parameter $M$ on the streamlines and isotherms profile for $\alpha$ = 0.2, $J$ = 0.001 and Pr = 1.0. For increasing values of the magnetic parameter $M$, the flow rate within the boundary layer decreases and the thermal boundary layer becomes thicker. Fig. 10 depicts that the maximum values of $\psi$ decreases steadily while the values of $M$ increases. The maximum values of $\psi$, that is, $\psi_{max}$ are 8.08, 5.33, 4.03 and 3.26 for $M$ = 0.0, 0.5, 1.0 and 1.5 respectively. The magnetic field acting along the horizontal direction retards the fluid velocity. For this there creates a Lorentz force by the interaction between the applied magnetic field and flow field. This force acts against the direction of fluid flow and reduces the velocity. The magnetic field decreases the temperature gradient at the surface and increases the temperature in the flow region due to the interaction. So the thermal boundary layer becomes higher.

The influence of the magnetic parameter $M$, on the local Nusselt number are illustrated in Figs. 12 and 13 respectively with Pr = 1.0, $\alpha$ = 0.0 and $J$ = 0.0. The results for without Joule heating ($J$ = 0.0) and a fluid having Pr = 1.0 are compared with those of Alam et al. (1997) and a very good agreement is found.

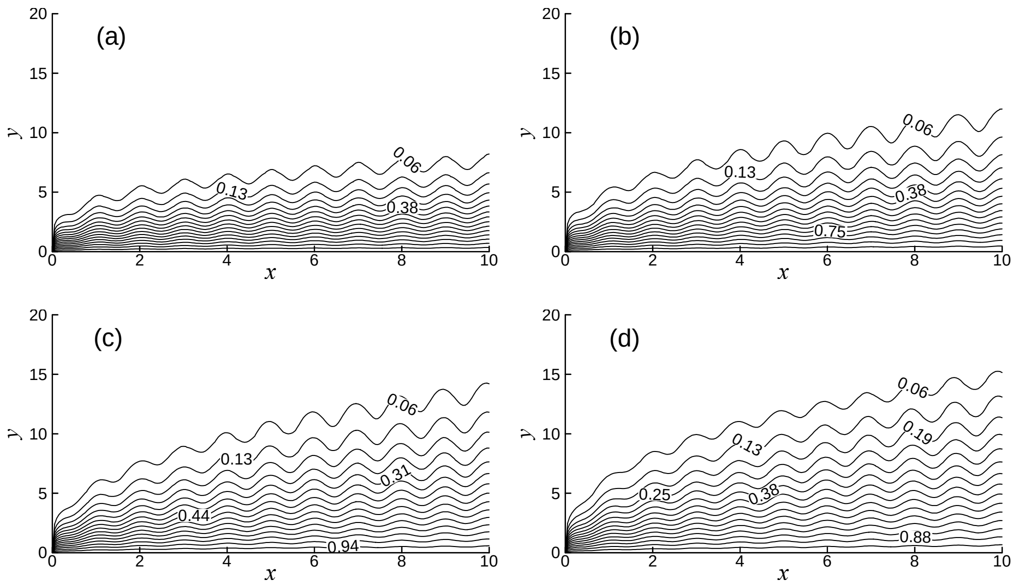


**Fig. 11**. Isotherms for (a) $M$ = 0.0 (b) $M$ = 0.5 (c) $M$ = 1.0 (d) $M$ = 1.5 while Pr = 1.0, $\alpha$ = 0.2 and $J$ = 0.001.

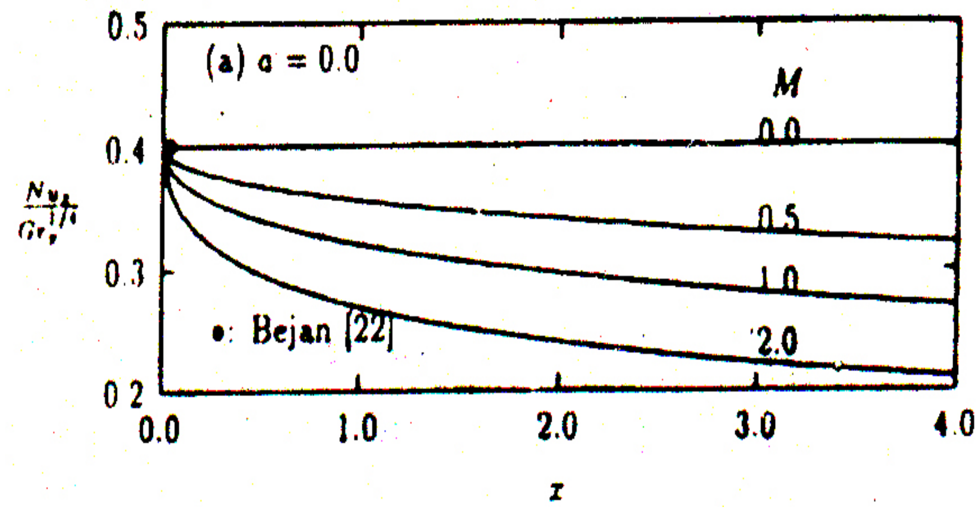


**Fig. 12.** Local Nusselt number for different values of magnetic parameter $M$ with Pr = 1.0, $\alpha$ = 0.0 and $J$ = 0.0 (Alam et al., 1997).

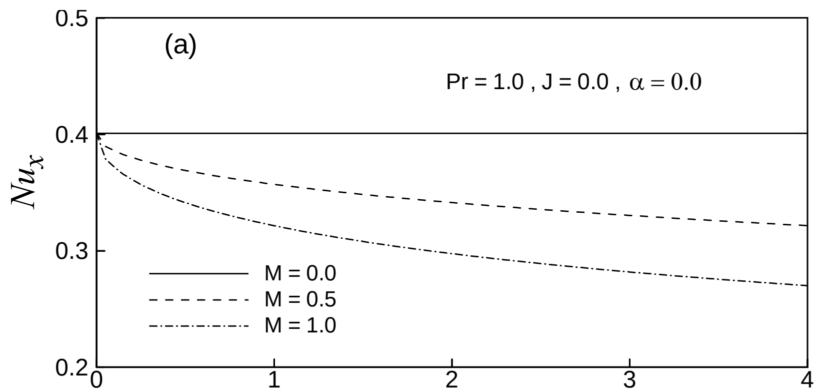


**Fig. 13.** Local Nusselt number for different values of magnetic parameter $M$ with Pr = 1.0, $\alpha$ = 0.0 and $J$ = 0.0 (present work).

## 5. Conclusion

The effect of Joule heating on natural convection flow of viscous incompressible fluid including the magnetic field along a uniformly heated vertical wavy surface has been studied. New variables to transform the complex geometry into a simple shape and were used a very efficient implicit finite difference method known as the Keller-box scheme was employed to solve the boundary layer equations. From the present investigation the following conclusions may be drawn:

- The skin friction coefficient decreases for increasing values of Prandtl number Pr, over the whole boundary layer but the significantly increase the rate of heat transfer.
- The effect of increasing Joule heating parameter $J$ results in decreasing the local rate of heat transfer $Nu_x$ and increasing the local skin friction coefficient $C_{fx}$.
- An increase in the values of M and $\alpha$ leads to decrease the local skin friction coefficient $C_{fx}$ and the local rate of heat transfer $Nu_x$.
- The velocity and the thermal boundary layer become thicker when Joule heating parameter J increases.
- The flow rate decreases and the thermal boundary layer grows thick when the effect of magnetic field is considered.

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