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Research Article

MetaFuzzy, MetaNeutrosophic, MetaSoft, and MetaRough Set

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ABSTRACT

A MetaStructure is a higher-level framework treating collections of structures as objects, with natural operations that preserve isomorphisms across domains. An Iterated MetaStructure recursively applies the MetaStructure construction, forming successive layers in which structures of structures create deeper, hierarchical meta-levels. Concepts in the real world can be captured in terms of such meta-structures, and meta-level viewpoints are in fact applied in many different fields. Therefore, research on meta-structures is important, but it has not yet been well developed in the areas of fuzzy sets and neutrosophic sets. To address this gap, in this paper, we define the MetaFuzzy Set, MetaNeutrosophic Set, Meta Soft Set, and MetaRough Set by extending fuzzy sets, neutrosophic sets, soft sets, and rough sets through the use of Meta Structure and Iterated Meta Structure.

Introduction

Set theory is a fundamental concept in mathematics (Jech, 2003); however, classical sets are sometimes insufficient for modeling real-world applications. To bridge this gap, many frameworks for handling uncertainty have been proposed and studied, such as fuzzy sets (Zadeh, 1965), hesitant fuzzy sets (Torra, 2010), neutrosophic sets (Wang et al., 2010), interval-valued neutrosophic sets (Wang et al., 2005), soft sets (Maji et al., 2003), rough sets (Pawlak, 1982), and plithogenic sets (Smarandache, 2018), and their applications have been investigated in various scientific and engineering domains.

Furthermore, many mathematical and real-world structures can be examined from a meta-level perspective, in which entire structures themselves are treated as objects of study. The mathematical formalization of this viewpoint is provided by the notions of MetaStructure and Iterated MetaStructure (cf. (Fujita 2025a, 2025b)).

From the above discussion, research on meta-structures is important; however, it has not yet been extensively developed in the contexts of fuzzy sets and neutrosophic sets. To fill this gap, in this paper, we define the MetaFuzzy Set, MetaNeutrosophic Set, MetaSoft Set, and MetaRough Set by extending fuzzy sets, neutrosophic sets, soft sets, and rough sets using MetaStructure and Iterated MetaStructure. We then investigate the fundamental properties and characteristics of these meta-level concepts.

Preliminaries

This section presents the fundamental concepts and definitions that underpin the paper's discussions.

MetaStructure (Structure of Structure)

We first fix a general single-sorted, finitary *signature*

$$\Sigma = (\mathsf{Func}, \mathsf{Rel}, \mathsf{ar}_{\mathsf{Func}}, \mathsf{ar}_{\mathsf{Rel}}),$$

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where Func (resp. Rel) is a set of function (resp. relation) symbols, andar records arities. A (single-sorted) Σ-*structure* is

$$C = \big(H, (f^{C})_{f\in\mathrm{Func}}, (R^{C})_{R\in\mathrm{Rel}}\big),$$

with carrier $H \neq \emptyset$, interpretations $f^{C}: H^m \to H$ for each $f \in$ Func of arity$m$, and relations $R^{C} \subseteq H^r$ for each $R \in$ Rel of arity r. Let Str_Σ denote the class of all Σ-structures.

Definition 1 (MetaStructure over a fixed signature). (cf.(Fujita 2025b)) Fix Σ as above. A *MetaStructure* ("structure of structures") over Σ is a pair

$$\mathbb{M} = (U, (\Phi_\ell)_{\ell\in\Lambda}),$$

where:

- U is a nonempty set with $U \subseteq \mathrm{Str}_\Sigma$ (its elements are *objects* at level 0);
- for each label $\ell \in \Lambda$ of *meta-arity*$k_\ell \in \mathbb{N}$, the *meta-operation*

$$\Phi_\ell : U^{k_\ell} \to U$$

is specified by uniform carrier- and symbol-constructors:

Γ_ℓ: $(C_1, \dots, C_{k_\ell}) \mapsto H_\ell$
(new carrier H_ℓ built functorially);
$\forall f \in$ Func:
$f^{\Phi_\ell(C_1,\dots,C_{k_\ell})} = \Lambda_\ell^f\big(f^{C_1}, \dots, f^{C_{k_\ell}}\big)$;
$\forall R \in$ Rel:
$R^{\Phi_\ell(C_1,\dots,C_{k_\ell})} = \Xi_\ell^R\big(R^{C_1}, \dots, R^{C_{k_\ell}}\big)$,

where Λ_ℓ^f and Ξ_ℓ^R are *uniform* recipes turning the symbols' interpretations on inputs into the symbol's interpretation on the output, over the new carrier H_ℓ. Moreover, each Φ_ℓ is *isomorphism-invariant* (a.k.a. natural): if $\alpha_i: C_i \cong D_i$.for $1 \le i \le k_\ell$, then there is an induced isomorphism

$$\Phi_\ell(\alpha_1, \dots, \alpha_{k_\ell}) : \Phi_\ell(C_1, \dots, C_{k_\ell}) \xrightarrow{\cong} \Phi_\ell(D_1, \dots, D_{k_\ell})$$

commuting with all interpretations of symbols of Σ.

Example 1 (MetaStructure on Graphs: disjoint union, Cartesian product, and line-graph). Fix the graph signature

$$\Sigma_{\mathrm{Graph}} = (\,\mathrm{Func} = \varnothing,\ \mathrm{Rel} = \{E\},\ \mathrm{ar}_{\mathrm{Rel}}(E) = 2\,),$$

where a Σ_{Graph}-structure is a (finite, simple, loopless, undirected) graph $G = (V, E^{G})$, encoded by a symmetric, irreflexive binary relation $E^{G} \subseteq V \times V$. Let $U \subseteq \mathrm{Str}_{\Sigma_{\mathrm{Graph}}}$ be the class of all such graphs. We define three meta-operations

$$\Phi_\sqcup, \Phi_\square : U \times U \to U, \qquad \Phi_L : U \to U,$$

which together form a MetaStructure $\mathbb{M} = (U, \{\Phi_\sqcup, \Phi_\square, \Phi_L\})$ in the sense of the Definition.

1) Disjoint union $\Phi_\sqcup(G_1, G_2)$ (meta-arity $k_\sqcup = 2$). For inputs $G_i = (V_i, E^{G_i})$ set the new carrier by a tagged sum

$$\Gamma_\sqcup(G_1, G_2) = H_\sqcup = (V_1 \times \{1\}) \cup (V_2 \times \{2\}),$$

and define the relation constructor uniformly by

$$\Xi_\sqcup^E(E^{G_1}, E^{G_2}) = \{\,((u,1),(v,1)) : (u,v) \in E^{G_1}\,\} \cup \{\,((u,2),(v,2)) : (u,v) \in E^{G_2}\,\}.$$

No cross edges are added. Isomorphism invariance follows immediate from the functorial tagging.

2) Cartesian product $\Phi_\square(G_1, G_2)$ (meta-arity $k_\square = 2$). Set

$$\Gamma_\square(G_1, G_2) = H_\square = V_1 \times V_2,$$

and

$$\Xi_\square^E(E^{G_1}, E^{G_2}) = \{((u,x),(v,y)) : [u = v \wedge (x,y) \in E^{G_2}] \vee bigl[(u,v) \in E^{G_1} \wedge x = y]\}.$$

This is the usual Cartesian product of graphs; naturality holds componentwise.

3) Line-graph operator $\Phi_L(G)$ (meta-arity $k_L = 1$). For $G = (V, E^{G})$ let the new carrier be the edge set

$$\Gamma_L(G) = H_L = E^{G} \subseteq V \times V,$$

and define adjacency on edges by intersection of incident endpoints:

$$\Xi_L^E(E^{G}) = \{\,(e_1, e_2) \in H_L \times H_L : e_1 \neq e_2 \text{ and } e_1 \cap e_2 \neq \varnothing\,\}.$$

This is the classical line-graph construction; isomorphism-invariance follows from edge-image preservation.

Tiny illustration. Let P_3 be the path $a - b - c$ and K_2 the single edge $x - y$. Then

$\Phi_{\sqcup}(\boldsymbol{P}_3, \boldsymbol{K}_2)$ has $|V| = 5$ with two components,

$$\Phi_L(\boldsymbol{P}_3) \cong \boldsymbol{P}_2, \qquad \Phi_{\circ}(\boldsymbol{P}_3, \boldsymbol{K}_2) \cong \text{ ladder on 4 vertices.}$$

Example 2 (MetaStructure on Groups: direct product and abelianization). Fix the group signature

$$\Sigma_{\mathrm{Grp}} = (\,\mathsf{Func} = \{\cdot,\ (\cdot)^{-1},\ e\},\ \mathsf{Rel} = \varnothing,$$
$$\mathrm{ar}_{\mathsf{Func}}(\cdot) = 2,\ \mathrm{ar}_{\mathsf{Func}}((\cdot)^{-1}) = 1,\ \mathrm{ar}_{\mathsf{Func}}(e) = 0\,).$$

A Σ_{Grp}-structure is a group $\boldsymbol{G} = (G, \cdot^{\boldsymbol{G}}, (\cdot)^{-1,\boldsymbol{G}}, e^{\boldsymbol{G}})$. Let $U \subseteq \mathrm{Str}_{\Sigma_{\mathrm{Grp}}}$ be the class of all (not-necessarily finite) groups. Define two meta-operations

$$\Phi_\times : U \times U \to U, \qquad \Phi_{\mathrm{ab}} : U \to U,$$

yielding a MetaStructure $\mathbb{M} = (U, \{\Phi_\times, \Phi_{\mathrm{ab}}\})$.

1) Direct product $\Phi_\times(\boldsymbol{G}_1, \boldsymbol{G}_2)$ (meta-arity $k_\times = 2$). For inputs $\boldsymbol{G}_i = (G_i, \cdot^{\boldsymbol{G}_i}, (\cdot)^{-1,\boldsymbol{G}_i}, e^{\boldsymbol{G}_i})$, set the carrier

$$\Gamma_\times(\boldsymbol{G}_1, \boldsymbol{G}_2) = H_\times = G_1 \times G_2,$$

and define uniformly, for all $(g_1, h_1), (g_2, h_2) \in G_1 \times G_2$,

$$\Lambda_\times^{\cdot}(\cdot^{\boldsymbol{G}_1}, \cdot^{\boldsymbol{G}_2})((g_1, h_1), (g_2, h_2)) = (g_1 \cdot^{\boldsymbol{G}_1} g_2,\ h_1 \cdot^{\boldsymbol{G}_2} h_2),$$
$$\Lambda_\times^{(\cdot)^{-1}}((\cdot)^{-1,\boldsymbol{G}_1}, (\cdot)^{-1,\boldsymbol{G}_2})(g, h) = (g^{-1,\boldsymbol{G}_1},\ h^{-1,\boldsymbol{G}_2}),$$
$$\Lambda_\times^{e}(e^{\boldsymbol{G}_1}, e^{\boldsymbol{G}_2}) = (e^{\boldsymbol{G}_1}, e^{\boldsymbol{G}_2}).$$

This is the standard categorical product; naturality is by componentwise isomorphisms.

2) Abelianization $\Phi_{\mathrm{ab}}(\boldsymbol{G})$ (meta-arity $k_{\mathrm{ab}} = 1$). For $\boldsymbol{G} = (G, \cdot^{\boldsymbol{G}}, (\cdot)^{-1,\boldsymbol{G}}, e^{\boldsymbol{G}})$, let

$$[G, G] = \langle\, g^{-1}h^{-1}gh:\ g, h \in G \,\rangle \le G$$

be the commutator subgroup. Define the carrier as the quotient set

$$\Gamma_{\mathrm{ab}}(\boldsymbol{G}) = H_{\mathrm{ab}} = G/[G, G],$$

and the induced operations via the quotient map $\pi: G \to G/[G, G]$:

$$\Lambda_{\mathrm{ab}}^{\cdot}(\cdot^{\boldsymbol{G}})(\pi(g), \pi(h)) = \pi(g \cdot^{\boldsymbol{G}} h),$$
$$\Lambda_{\mathrm{ab}}^{(\cdot)^{-1}}((\cdot)^{-1,\boldsymbol{G}})(\pi(g)) = \pi(g^{-1,\boldsymbol{G}}),$$
$$\Lambda_{\mathrm{ab}}^{e}(e^{\boldsymbol{G}}) = \pi(e^{\boldsymbol{G}}).$$

Well-definedness uses $[G, G] \trianglelefteq G$. The output is the abelian group G_{ab}; functoriality (naturality) follows from the universal property of abelianization.

Tiny illustration. For the dihedral group $D_4 = \langle r, s \mid r^4 = s^2 = 1,\ srs = r^{-1}\rangle$,

$$\Phi_{\mathrm{ab}}(D_4) \cong C_2 \times C_2, \qquad \Phi_\times(C_2, C_3) \cong C_6.$$

Iterated MetaStructure

An Iterated MetaStructure recursively applies MetaStructure construction, forming successive layers in which structures of structures create deeper hierarchical meta-levels (Fujita 2025a, 2025b).

Definition 2 (Iterated MetaStructure of depth t). (Fujita 2025b) An Iterated MetaStructure of depth t overΣ is any MetaStructure $\mathfrak{M}^{(t)}$ of height t. When $s < t$, we *lift* a height-s MetaStructure $\mathfrak{M}^{(s)} = (U^{(s)}, \{\odot_i\}, \{\mathcal{S}_j\})$ to height t by

$$\iota_{s \to t}\colon U^{(s)} \xrightarrow{\mathsf{U}_\Sigma^{t-s}} U^{(t)} := \mathsf{U}_\Sigma^{t-s}\big(U^{(s)}\big),$$

and, for each $\odot_i\colon (\mathsf{E}_\Sigma^{m_i})^{k_i} \to \mathcal{P}^{\,n_i}(\mathsf{E}_\Sigma^{n_i})$, defining its lift

$$\odot_i^{\uparrow}\colon (\mathsf{E}_\Sigma^{m_i+t-s})^{k_i} \to \mathcal{P}^{\,n_i}\big(\mathsf{E}_\Sigma^{n_i+t-s}\big),$$
$$\odot_i^{\uparrow}\big(\mathsf{U}_\Sigma^{t-s}(x_1), \dots, \mathsf{U}_\Sigma^{t-s}(x_{k_i})\big) := \mathsf{U}_\Sigma^{t-s}\big(\odot_i\,(x_1, \dots, x_{k_i})\big),$$

and similarly for relations

$$\mathcal{S}_j^{\uparrow} := (\mathsf{U}_\Sigma^{t-s})^{\times \ell_j}(\mathcal{S}_j).$$

Example 3 (Iterated MetaStructure on Graphs via the line-graph operator). Fix the graph signature $\Sigma_{\mathrm{Graph}} = (\mathsf{Func} = \varnothing,\ \mathsf{Rel} = \{E\},\ \mathrm{ar}_{\mathsf{Rel}}(E) = 2)$, so a Σ_{Graph}-structure is a simple undirected graph $\boldsymbol{G} = (V, E^{\boldsymbol{G}})$. Let $\Phi_L\colon U^{(1)} \to U^{(1)}$ be the (level-1) line-graph meta-operation of Example 1: its carrier constructor makes the new carrier the old edge set, and its relation constructor connects two distinct edges iff they share a vertex.

To obtain an *Iterated MetaStructure of depth t* (Definition 2), we choose the canonical lift U_Σ to be identity-on-objects (so height only records iteration), and define the t-fold iterate

$$\Phi_L^{(t)} := \underbrace{\Phi_L \circ \Phi_L \circ \cdots \circ \Phi_L}_{t \text{ times}} : U^{(1)} \to U^{(1)}.$$

Concrete computation

- For the path $\boldsymbol{P}_m$ (with m vertices and $m-1$ edges),

$$\Phi_L(\boldsymbol{P}_m) \cong \boldsymbol{P}_{m-1},$$
$$\Phi_L^{(t)}(\boldsymbol{P}_m) \cong \boldsymbol{P}_{m-t} \quad \text{for } 1 \le t \le m-1.$$

In particular, for $m = 5$ and $t = 2$,

$$\Phi_L^{(2)}(\boldsymbol{P}_5) \cong \boldsymbol{P}_3 \quad \text{(vertex count: } 5 \to 4 \to 3).$$

- For the cycle $\boldsymbol{C}_n$ ($n \ge 3$),

$$\Phi_L(\boldsymbol{C}_n) \cong \boldsymbol{C}_n \quad \Rightarrow \quad \Phi_L^{(t)}(\boldsymbol{C}_n) \cong \boldsymbol{C}_n \text{ for all } t \ge 1.$$

A depth-2 **object spelled out.** Let $\boldsymbol{G}_1 = \boldsymbol{P}_4$ and $\boldsymbol{G}_2 = \boldsymbol{K}_2$. Form the (level-1) family $X = \{\boldsymbol{G}_1, \boldsymbol{G}_2, \boldsymbol{G}_1\}$ and define a level-1 meta-relation on X by "same edge-count":

$$\mathcal{S}(\boldsymbol{A},\boldsymbol{B}) \Leftrightarrow |E^{\boldsymbol{A}}| = |E^{\boldsymbol{B}}|.$$

Apply Φ_L once to obtain

$$\Phi_L(X) = \{\, \Phi_L(\boldsymbol{P}_4) = \boldsymbol{P}_3,\ \Phi_L(\boldsymbol{K}_2) = \boldsymbol{K}_1,\ \Phi_L(\boldsymbol{P}_4) = \boldsymbol{P}_3 \,\},$$

and lift the relation by the same recipe ("same edge-count"): $|E^{\boldsymbol{P}_3}| = 2$, $|E^{\boldsymbol{K}_1}| = 0$, so the only meta-edge at depth 2 is between the two copies of $\boldsymbol{P}_3$. This explicitly realises an *iterated* (depth-2) MetaStructure built from Φ_L.

Example 4 (Iterated MetaStructure on Groups via direct product and abelianization). Fix the group signature $\Sigma_{\mathrm{Grp}} = (\mathsf{Func} = \{\cdot, (\cdot)^{-1}, e\},\ \mathsf{Rel} = \varnothing)$. Let $U^{(1)}$ be the classof all groups, and consider two level-1 meta-operations:

$$\Phi_\times(\boldsymbol{G},\boldsymbol{H}) = \boldsymbol{G} \times \boldsymbol{H} \quad \text{(direct product)},$$
$$\Phi_{\mathrm{ab}}(\boldsymbol{G}) = \boldsymbol{G}/[\boldsymbol{G},\boldsymbol{G}] \quad \text{(abelianization)}.$$

As in Example 1, carriers and symbols are constructed uniformly (product set, pointwise operations; quotient by the commutator subgroup).

To produce an Iterated MetaStructure of depth t (Definition 2), we again take the canonical lift U_Σ to be identity-on-objects and define iterates

$$\Phi_{\mathrm{ab}}^{(t)} := \underbrace{\Phi_{\mathrm{ab}} \circ \cdots \circ \Phi_{\mathrm{ab}}}_{t \text{ times}},$$
$$\Phi_\times^{(t)} := \underbrace{\Phi_\times \circ \cdots \circ \Phi_\times}_{t-1 \text{ binary uses}}.$$

Concrete computation (depth $t = 2$).

- Start with the non-abelian groups $\boldsymbol{G}_0 = S_3$ and $\boldsymbol{H}_0 = D_4$.
- First abelianize (level 1):

$$\Phi_{\mathrm{ab}}(\boldsymbol{G}_0) \cong C_2,$$
$$\Phi_{\mathrm{ab}}(\boldsymbol{H}_0) \cong C_2 \times C_2.$$

- Second abelianization stabilizes (level 2):

$$\Phi_{\mathrm{ab}}^{(2)}(\boldsymbol{G}_0) \cong C_2,$$
$$\Phi_{\mathrm{ab}}^{(2)}(\boldsymbol{H}_0) \cong C_2 \times C_2,$$

since abelianization is idempotent upto isomorphism.

- Combine by the (binary) meta-operation at depth 2:

$$\Phi_\times\left(\Phi_{\mathrm{ab}}^{(2)}(\boldsymbol{G}_0),\ \Phi_{\mathrm{ab}}^{(2)}(\boldsymbol{H}_0)\right) \cong C_2 \times (C_2 \times C_2) \cong C_2^3.$$

In terms of orders: $|S_3| = 6$, $|D_4| = 8$, $|C_2| = 2$, $|C_2 \times C_2| = 4$, hence

$$|\Phi_\times(\Phi_{\mathrm{ab}}^{(2)}(S_3), \Phi_{\mathrm{ab}}^{(2)}(D_4))| = 2 \cdot 4 = 8.$$

Thus, the pair of iterated meta-operations $(\Phi_{\mathrm{ab}}^{(t)}, \Phi_\times)$ yields a concrete *depth*-2 MetaStructure on groups, with explicit carriers and operations at each stage.

Main Results: Meta set

In this section, we present the main results of this paper, focusing on discussions related to the concept of Meta Sets.

MetaFuzzy set (Fuzzy Set of Fuzzy Sets)

A Fuzzy Set generalizes classical sets by assigning each element a membership degree between 0and 1, thereby representing partial inclusion (Zadeh, 1965). A MetaFuzzy Set further extends this line of research by assigning membership values not to individual elements but to entire fuzzy sets, thereby enabling higher-level reasoning about collections of fuzziness across diverse contexts.

Definition 3 (Fuzzy Set). (Zadeh 1965; 1996) Let Y be a nonempty domain. A *fuzzy set* is given by a function

$$\mu: Y \to [0,1],$$

where $\mu(y)$ measures the degree to which y belongs to the set. A fuzzy relation on Y is a function $\delta: Y \times Y \to [0,1]$, viewed as a fuzzy subset of $Y \times Y$. We say δ is a fuzzy relation on μ if for every $y, z \in Y$,

$$\delta(y,z) \;\le\; \min\{\mu(y),\, \mu(z)\}.$$

Definition 4 (MetaFuzzy Set). Fix a nonempty base domain Y. A MetaFuzzySet *on Y* is a map

$$\mu^{\sharp}\colon\ \mathrm{Fuz}(Y) \to [0,1].$$

A MetaFuzzy relation on Y is a map

$$\Delta^{\sharp}\colon \mathrm{Fuz}(Y) \times \mathrm{Fuz}(Y) \to [0,1].$$

We say that $\Delta^{\sharp}$ is a MetaFuzzy relation on $\mu^{\sharp}$ if for all $\mu_1, \mu_2 \in \mathrm{Fuz}(Y)$ we have

$$\Delta^{\sharp}(\mu_1,\mu_2) \;\le\; \min\{\mu^{\sharp}(\mu_1),\, \mu^{\sharp}(\mu_2)\}.$$

Note 1 (Averaging functional $M(\cdot)$ on $\mathrm{Fuz}(Y)$). Throughout, for a finite nonempty Y we use the averaging functional $M\colon \mathrm{Fuz}(Y) \to [0,1]$, $M(\mu) := \frac{1}{|Y|}\sum_{y\in Y}\mu(y)$. Since $0 \le \mu(y) \le 1$ for all $y \in Y$, we have $0 \le \sum_{y\in Y}\mu(y) \le |Y| \Rightarrow 0 \le M(\mu) \le 1$. In the sequel we take $\mu^{\sharp} := M$.

Example 5 (MetaFuzzy Set: Weekly traffic congestion severity). Let $Y = \{\text{Mon,Tue,Wed}\}$ be three commuting days, and let $\mathrm{Fuz}(Y) = [0,1]^{Y}$. A fuzzy set $\mu \in \mathrm{Fuz}(Y)$ represents the degree of heavy congestion on each day. Consider two weeks:

$$\mu^{(A)}(\text{Mon,Tue,Wed}) = (0.2,\ 0.8,\ 0.6),$$
$$\mu^{(B)}(\text{Mon,Tue,Wed}) = (0.9,\ 0.7,\ 0.3).$$

We define the MetaFuzzy Set $\mu^{\sharp}$ by the averaging functional M of Note 1:

$$\mu^{\sharp}(\mu) \;:=\; M(\mu) \;=\; \frac{1}{|Y|}\sum_{y\in Y}\mu(y) = \frac{\mu(\text{Mon}) + \mu(\text{Tue}) + \mu(\text{Wed})}{3}.$$

Then

$$\mu^{\sharp}\big(\mu^{(A)}\big) = \frac{0.2+0.8+0.6}{3} = \frac{1.6}{3} = \frac{8}{15} \approx 0.533\overline{3},$$
$$\mu^{\sharp}\big(\mu^{(B)}\big) = \frac{0.9+0.7+0.3}{3} = \frac{1.9}{3} = \frac{19}{30} \approx 0.633\overline{3}.$$

Next, we define a MetaFuzzy relation $\Delta^{\sharp}$ as the average of pointwise minima:

$$\Delta^{\sharp}(\mu_1,\mu_2) \;:=\; \frac{1}{|Y|}\sum_{y\in Y}\min\{\mu_1(y),\mu_2(y)\} \qquad (\mu_1,\mu_2 \in \mathrm{Fuz}(Y)).$$

For our two weeks, the pointwise minima are

$$\min\{\mu^{(A)}(\text{Mon}),\mu^{(B)}(\text{Mon})\} = \min\{0.2,0.9\} = 0.2,$$
$$\min\{\mu^{(A)}(\text{Tue}),\mu^{(B)}(\text{Tue})\} = \min\{0.8,0.7\} = 0.7,$$
$$\min\{\mu^{(A)}(\text{Wed}),\mu^{(B)}(\text{Wed})\} = \min\{0.6,0.3\} = 0.3.$$

Hence

$$\Delta^{\sharp}\big(\mu^{(A)},\mu^{(B)}\big) = \frac{0.2+0.7+0.3}{3} = \frac{1.2}{3} = \frac{6}{15} = \frac{2}{5} = 0.4.$$

We now verify the admissibility inequality

$$\Delta^{\sharp}(\mu_1,\mu_2) \;\le\; \min\{\mu^{\sharp}(\mu_1),\mu^{\sharp}(\mu_2)\} \qquad (\forall \mu_1,\mu_2 \in \mathrm{Fuz}(Y)).$$

For arbitrary μ_1, μ_2 and each $y \in Y$ we have

$$\min\{\mu_1(y),\mu_2(y)\} \;\le\; \mu_1(y) \quad \text{and} \quad \min\{\mu_1(y),\mu_2(y)\} \;\le\; \mu_2(y).$$

Summing over $y \in Y$ and dividing by $|Y|$ yields

$$\Delta^{\sharp}(\mu_1,\mu_2) = \frac{1}{|Y|}\sum_{y\in Y}\min\{\mu_1(y),\mu_2(y)\} \le \frac{1}{|Y|}\sum_{y\in Y}\mu_1(y) = \mu^{\sharp}(\mu_1),$$

and similarly $\Delta^{\sharp}(\mu_1,\mu_2) \le \mu^{\sharp}(\mu_2)$. Therefore

$$\Delta^{\sharp}(\mu_1,\mu_2) \;\le\; \min\{\mu^{\sharp}(\mu_1),\mu^{\sharp}(\mu_2)\}$$

for all $\mu_1, \mu_2 \in \mathrm{Fuz}(Y)$.

In particular, for our concrete weeks,

$$\Delta^{\sharp}\big(\mu^{(A)},\mu^{(B)}\big) = \frac{2}{5} = \frac{12}{30} \le \frac{16}{30} = \frac{8}{15} = \mu^{\sharp}\big(\mu^{(A)}\big) \le \mu^{\sharp}\big(\mu^{(B)}\big) = \frac{19}{30},$$

so

$$\Delta^{\sharp}\big(\mu^{(A)},\mu^{(B)}\big) \;\le\; \min\{\mu^{\sharp}(\mu^{(A)}),\mu^{\sharp}(\mu^{(B)})\} = \frac{8}{15} \approx 0.533\overline{3}.$$

Thus, the MetaFuzzy relation $\Delta^\sharp$ satisfies the required inequality in this example and, by the above argument, for all fuzzy weeks in $\mathrm{Fuz}(Y)$.

Theorem 1 (Zero-support property of MetaFuzzy relations). Let $\mu^\sharp$ be a MetaFuzzy Set on Y, and let $\Delta^\sharp$ be a MetaFuzzy relation on $\mu^\sharp$ in the sense of Definition 4. Then for all $\mu_1, \mu_2 \in \mathrm{Fuz}(Y)$ we have $\mu^\sharp(\mu_1) = 0$ or $\mu^\sharp(\mu_2) = 0 \quad \Rightarrow \quad \Delta^\sharp(\mu_1, \mu_2) = 0$. In particular, if $\mu^\sharp(\mu) = 0$ then $\Delta^\sharp(\mu, \nu) = 0 = \Delta^\sharp(\nu, \mu)$ for all $\nu \in \mathrm{Fuz}(Y)$.

Proof. Assume $\Delta^\sharp$ is a MetaFuzzy relation on $\mu^\sharp$. Fix $\mu_1, \mu_2 \in \mathrm{Fuz}(Y)$.

By Definition 8 we have

$$\Delta^\sharp(\mu_1, \mu_2) \le \min\{\mu^\sharp(\mu_1), \mu^\sharp(\mu_2)\}.$$

Suppose $\mu^\sharp(\mu_1) = 0$. Then

$$\min\{\mu^\sharp(\mu_1), \mu^\sharp(\mu_2)\}$$
$$= \min\{0, \mu^\sharp(\mu_2)\} = 0.$$

Hence

$$0 \le \Delta^\sharp(\mu_1, \mu_2) \le 0,$$

which forces $\Delta^\sharp(\mu_1, \mu_2) = 0$. The same argument applies when $\mu^\sharp(\mu_2) = 0$, and also to the pair (μ_2, μ_1), since the defining inequality is symmetric in the two arguments. This gives the desired conclusion.

Theorem 2 (Maximal MetaFuzzy relation). Let $\mu^\sharp$ be a MetaFuzzy Set on Y, and define $\Delta^\sharp_{\max}: \mathrm{Fuz}(Y) \times \mathrm{Fuz}(Y) \to [0,1]$, $\Delta^\sharp_{\max}(\mu_1, \mu_2) := \min\{\mu^\sharp(\mu_1), \mu^\sharp(\mu_2)\}$. *Then:*

1. $\Delta^\sharp_{\max}$ is a MetaFuzzy relation on $\mu^\sharp$.
2. *If $\Delta^\sharp$ is any MetaFuzzy relation on $\mu^\sharp$, then* $\Delta^\sharp(\mu_1, \mu_2) \le \Delta^\sharp_{\max}(\mu_1, \mu_2) \quad (\forall \mu_1, \mu_2 \in \mathrm{Fuz}(Y))$ that is, $\Delta^\sharp_{\max}$ is the largest MetaFuzzy relation on $\mu^\sharp$ with respect to the pointwise order.

Proof. (1) For any $\mu_1, \mu_2 \in \mathrm{Fuz}(Y)$ we have

$$\Delta^\sharp_{\max}(\mu_1, \mu_2)$$
$$= \min\{\mu^\sharp(\mu_1), \mu^\sharp(\mu_2)\}.$$

By definition of min, we immediately get

$$\Delta^\sharp_{\max}(\mu_1, \mu_2)$$
$$\le \min\{\mu^\sharp(\mu_1), \mu^\sharp(\mu_2)\},$$

so $\Delta^\sharp_{\max}$ satisfies the MetaFuzzy relation condition.

(2) Let $\Delta^\sharp$ be any MetaFuzzy relation on $\mu^\sharp$. Then for all $\mu_1, \mu_2 \in \mathrm{Fuz}(Y)$,

$$\Delta^\sharp(\mu_1, \mu_2) \le \min\{\mu^\sharp(\mu_1), \mu^\sharp(\mu_2)\}$$
$$= \Delta^\sharp_{\max}(\mu_1, \mu_2).$$

Thus, $\Delta^\sharp \le \Delta^\sharp_{\max}$ pointwise. Hence $\Delta^\sharp_{\max}$ is the largest element (in the pointwise order) among all MetaFuzzy relations on $\mu^\sharp$.

Theorem 3 (Level-set representation of a MetaFuzzy set).

Let $\mu^\sharp$ be a MetaFuzzy set on Y. For each $\alpha \in [0,1]$, define the (meta-)level set

$\mathcal{L}_\alpha := \{ \mu \in Fuz(Y) \mid \mu^\sharp(\mu) \ge \alpha \} \subseteq Fuz(Y).$

Then for every $\mu \in \mathrm{Fuz}(Y)$ we have the exact reconstruction formula

$\mu^\sharp(\mu) = \sup \{ \alpha \in [0,1] \mid \mu \in \mathcal{L}_\alpha \}.$

Moreover, the family $(\mathcal{L}_\alpha)_{\alpha \in [0,1]}$ is nested:

$0 \le \alpha \le \beta \le 1 \Rightarrow \mathcal{L}_\beta \subseteq \mathcal{L}_\alpha.$

Proof. Fix $\mu \in \mathrm{Fuz}(Y)$ and set $v := \mu^\sharp(\mu) \in [0,1]$. By definition of $\mathcal{L}_\alpha$ we have

$\mu \in \mathcal{L}_\alpha \Leftrightarrow \mu^\sharp(\mu) \ge \alpha \Leftrightarrow v \ge \alpha.$

Thus the set of all $\alpha \in [0,1]$ such that $\mu \in \mathcal{L}_\alpha$ is exactly $\{ \alpha \in [0,1] \mid \mu \in \mathcal{L}_\alpha \} = \{ \alpha \in [0,1] \mid v \ge \alpha \} = [0, v]$.

The supremum of $[0, v]$ in $[0,1]$ is v itself. Therefore, $\sup \{ \alpha \in [0,1] \mid \mu \in \mathcal{L}_\alpha \} = \sup [0, v] = v = \mu^\sharp(\mu)$, which proves the reconstruction formula.

For the nesting property, let $0 \le \alpha \le \beta \le 1$ and take $\mu \in \mathcal{L}_\beta$. Then $\mu^\sharp(\mu) \ge \beta \ge \alpha$, so $\mu \in \mathcal{L}_\alpha$. Hence $\mathcal{L}_\beta \subseteq \mathcal{L}_\alpha$.

Theorem 4 (Monotonicity and Lipschitz property of an arithmetic-mean MetaFuzzy Set). *Assume $Y = \{y_1, \dots, y_n\}$ is a finite nonempty base domain with $n \in \mathbb{N}$. Define $\mu^\sharp: \mathrm{Fuz}(Y) \to [0,1]$ by $\mu^\sharp(\mu) := \frac{1}{n} \sum_{i=1}^{n} \mu(y_i)$. Then:*

1. *(Bounds) For all $\mu \in \mathrm{Fuz}(Y)$, $0 \le \mu^\sharp(\mu) \le 1$.*

2. (Monotonicity) If $\mu, \nu \in \mathsf{Fuz}(Y)$ satisfy $\mu(y) \le \nu(y) \quad (\forall y \in Y)$, then $\mu^\sharp(\mu) \le \mu^\sharp(\nu)$.
3. (Lipschitz continuity) For all $\mu, \nu \in \mathsf{Fuz}(Y)$, $|\mu^\sharp(\mu) - \mu^\sharp(\nu)| \le \frac{1}{n}\sum_{i=1}^{n} |\mu(y_i) - \nu(y_i)|$. In particular, $\mu^\sharp$ is 1-Lipschitz with respect to the normalized ℓ^1-distance on $\mathsf{Fuz}(Y)$.

Proof. (1) Since $\mu(y_i) \in [0,1]$ for all i, we have

$$0 \le \mu(y_i) \le 1 \qquad (\forall i = 1, \dots, n).$$

Summing these inequalities gives

$$0 \le \sum_{i=1}^{n} \mu(y_i) \le \sum_{i=1}^{n} 1 = n.$$

Dividing by $n > 0$ yields

$$0 \le \frac{1}{n}\sum_{i=1}^{n} \mu(y_i) \le 1,$$

i.e. $0 \le \mu^\sharp(\mu) \le 1$.

(2) Assume $\mu(y) \le \nu(y)$ for all $y \in Y$. Then in particular

$$\mu(y_i) \le \nu(y_i) \qquad (\forall i = 1, \dots, n).$$

Summing these n inequalities gives

$$\sum_{i=1}^{n} \mu(y_i) \le \sum_{i=1}^{n} \nu(y_i).$$

Dividing by n we obtain

$$\mu^\sharp(\mu) = \frac{1}{n}\sum_{i=1}^{n} \mu(y_i) \le \frac{1}{n}\sum_{i=1}^{n} \nu(y_i) = \mu^\sharp(\nu),$$

so $\mu^\sharp$ is monotone with respect to the pointwise order on $\mathsf{Fuz}(Y)$.

(3) For any $\mu, \nu \in \mathsf{Fuz}(Y)$ we compute

$$\mu^\sharp(\mu) - \mu^\sharp(\nu) = \frac{1}{n}\sum_{i=1}^{n} \mu(y_i) - \frac{1}{n}\sum_{i=1}^{n} \nu(y_i)$$

$$= \frac{1}{n}\sum_{i=1}^{n} (\mu(y_i) - \nu(y_i)).$$

Taking absolute values and applying the triangle inequality yields

$$|\mu^\sharp(\mu) - \mu^\sharp(\nu)| = \left|\frac{1}{n}\sum_{i=1}^{n} (\mu(y_i) - \nu(y_i))\right|$$

$$\le \frac{1}{n}\sum_{i=1}^{n} |\mu(y_i) - \nu(y_i)|.$$

This is exactly the claimed Lipschitz bound.

If we define the normalized ℓ^1-distance

$$d_1(\mu, \nu) := \frac{1}{n}\sum_{i=1}^{n} |\mu(y_i) - \nu(y_i)|,$$

then the inequality can be rewritten as

$$|\mu^\sharp(\mu) - \mu^\sharp(\nu)| \le d_1(\mu, \nu),$$

showing that $\mu^\sharp$ is 1-Lipschitz with respect to d_1.

MetaNeutrosophic set (Neutrosophic set of Neutrosophic Set)

A Neutrosophic Set extends fuzzy sets by assigning each element three independent degrees: truth, indeterminacy, and falsity, enabling richer uncertainty modeling (Broumi et al. 2016). A MetaNeutrosophic Set evaluates neutrosophic sets themselves, producing truth, indeterminacy, and falsity degrees for collections of neutrosophic information.

Definition 5 (Neutrosophic Set). (Hadi and Al-Swidi, 2022) Let X be a non-empty set. A *Neutrosophic Set (NS)A* on X is characterized by three membership functions:

$$T_A: X \to [0,1], \quad I_A: X \to [0,1], \quad F_A: X \to [0,1],$$

where for each $x \in X$, the values $T_A(x)$, $I_A(x)$, and $F_A(x)$ represent the degrees of truth, indeterminacy, and falsity, respectively. These values satisfy the following condition:

$$0 \le T_A(x) + I_A(x) + F_A(x) \le 3.$$

Definition 6 (MetaNeutrosophic Set). Let X be a nonempty finite set, and let $\mathsf{Neu}(X)$ denote the collection of neutrosophic sets on X, that is, all triples

$$A = (T_A, I_A, F_A)$$

With

$$\mathrm{T_A},\ \mathrm{I_A},\ \mathrm{F_A}: X \to [0,1]$$

such that for every $x \in X$,

$$0 \le \mathrm{T_A}(x) + \mathrm{I_A}(x) + \mathrm{F_A}(x) \le 3.$$

A MetaNeutrosophic Set on X is a triple of functionals

$$\mathrm{T}^\sharp,\ \mathrm{I}^\sharp,\ \mathrm{F}^\sharp\colon\ \mathsf{Neu}(X) \to [0,1]$$

satisfying, for every $A \in \mathsf{Neu}(X)$,

$$0 \le \mathrm{T}^\sharp(A) + \mathrm{I}^\sharp(A) + \mathrm{F}^\sharp(A) \le 3.$$

Equivalently, the associated mapping

$$\mathbf{N}^\sharp\colon\ \mathsf{Neu}(X) \to [0,1]^3,$$

$$\mathbf{N}^\sharp(A) := \left(\mathrm{T}^\sharp(A), \mathrm{I}^\sharp(A), \mathrm{F}^\sharp(A)\right),$$

assigns to each neutrosophic set A a meta-level triplet of truth, indeterminacy, and falsity degrees whose sum remains in the admissible range [0,3].

Example 6 (MetaNeutrosophic Set: overall project risk). Let

$$X = \{\text{Budget},\ \text{Schedule}\}$$

be two project risk dimensions. A neutrosophic set $A = (\mathrm{T_A}, \mathrm{I_A}, \mathrm{F_A}) \in \mathsf{Neu}(X)$ encodes, for each $x \in X$,

- $\mathrm{T_A}(x)$: degree that the risk on x is under control,
- $\mathrm{I_A}(x)$: degree of uncertainty about the risk on x,
- $\mathrm{F_A}(x)$: degree that the risk on x is not under control,

with $0 \le \mathrm{T_A}(x) + \mathrm{I_A}(x) + \mathrm{F_A}(x) \le 3$.

Assume a concrete assessment

$$(\mathrm{T_A}(\text{Budget}), \mathrm{I_A}(\text{Budget}), \mathrm{F_A}(\text{Budget})) = (0.8,\ 0.1,\ 0.1),$$
$$(\mathrm{T_A}(\text{Schedule}), \mathrm{I_A}(\text{Schedule}), \mathrm{F_A}(\text{Schedule})) = (0.5,\ 0.3,\ 0.2).$$

Define a MetaNeutrosophic Set

$$(\mathbf{N}^\sharp = (\mathrm{T}^\sharp, \mathrm{I}^\sharp, \mathrm{F}^\sharp)\colon \mathsf{Neu}(X) \to [0,1]^3$$

by the simple averaging/max rules (here $|X| = 2$):

$$\mathrm{T}^\sharp(A) := \frac{\mathrm{T_A}(\text{Budget}) + \mathrm{T_A}(\text{Schedule})}{2},$$

$$\mathrm{I}^\sharp(A) := \max\{\mathrm{I_A}(\text{Budget}), \mathrm{I_A}(\text{Schedule})\},$$

$$\mathrm{F}^\sharp(A) := \frac{\mathrm{F_A}(\text{Budget}) + \mathrm{F_A}(\text{Schedule})}{2}.$$

For the above A we obtain

$$\mathrm{T}^\sharp(A) = \frac{0.8 + 0.5}{2} = 0.65,$$

$$\mathrm{I}^\sharp(A) = \max\{0.1, 0.3\} = 0.3,$$

$$\mathrm{F}^\sharp(A) = \frac{0.1 + 0.2}{2} = 0.15.$$

Thus

$$\mathbf{N}^\sharp(A) = (0.65,\ 0.30,\ 0.15)$$

is a meta-level summary of the overall project risk: on average the risk is reasonably controlled ($T^\sharp$ high), there is moderate uncertainty ($I^\sharp$), and relatively low non-control ($F^\sharp$), while $0.65 + 0.30 + 0.15 = 1.10 \le 3$ so the MetaNeutrosophic constraint is satisfied.

Definition 7 (A concrete averaging/max MetaNeutrosophic Set). Assume that X is finite with $n := |X| \ge 1$. For $A = (T_A, I_A, F_A) \in \mathsf{Neu}(X)$ define

$$T^\sharp(A) := \frac{1}{n}\sum_{x \in X} T_A(x),$$

$$I^\sharp(A) := \max_{x \in X} I_A(x),$$

$$F^\sharp(A) := \frac{1}{n}\sum_{x \in X} F_A(x).$$

Example 7 (Averaging/max MetaNeutrosophic Set: machine health). Let

$$X = \{\text{Sensor}_1, \text{Sensor}_2, \text{Sensor}_3\}$$

be three monitoring points in an industrial machine. A neutrosophic set $A = (T_A, I_A, F_A) \in \mathsf{Neu}(X)$ encodes, for each $x \in X$,

- $T_A(x)$: degree that the sensor status is healthy,
- $I_A(x)$: degree of uncertainty about the status,
- $F_A(x)$: degree that the sensor status is faulty,

with $0 \le T_A(x) + I_A(x) + F_A(x) \le 3$.

Assume the following assessment:

$$(\mathrm{T_A}(\mathrm{Sensor}_1), \mathrm{I_A}(\mathrm{Sensor}_1), \mathrm{F_A}(\mathrm{Sensor}_1)) = (0.9,\ 0.1,\ 0.0),$$
$$(\mathrm{T_A}(\mathrm{Sensor}_2), \mathrm{I_A}(\mathrm{Sensor}_2), \mathrm{F_A}(\mathrm{Sensor}_2)) = (0.7,\ 0.2,\ 0.1),$$
$$(\mathrm{T_A}(\mathrm{Sensor}_3), \mathrm{I_A}(\mathrm{Sensor}_3), \mathrm{F_A}(\mathrm{Sensor}_3)) = (0.6,\ 0.3,\ 0.2).$$

Each triple is admissible, e.g.

$$0.9 + 0.1 + 0.0 = 1.0,$$
$$0.7 + 0.2 + 0.1 = 1.0,$$
$$0.6 + 0.3 + 0.2 = 1.1 \ \le 3.$$

Here $|\mathrm{X}| = \mathrm{n} = 3$. Using Definition 7, the averaging/max MetaNeutrosophic Set $(\mathrm{T}^\sharp, \mathrm{I}^\sharp, \mathrm{F}^\sharp)$ gives

$$\mathrm{T}^\sharp(\mathrm{A}) = \frac{1}{3}(\mathrm{T_A}(\mathrm{Sensor}_1) + \mathrm{T_A}(\mathrm{Sensor}_2) + \mathrm{T_A}(\mathrm{Sensor}_3)) = \frac{0.9 + 0.7 + 0.6}{3} = \frac{2.2}{3} = \frac{11}{15} \approx 0.733\overline{3},$$

$$\mathrm{I}^\sharp(\mathrm{A}) = \max_{x\in X} \mathrm{I_A}(\mathrm{x}) = \max\{0.1,0.2,0.3\} = 0.3,$$

$$\mathrm{F}^\sharp(\mathrm{A}) = \frac{1}{3}(\mathrm{F_A}(\mathrm{Sensor}_1) + \mathrm{F_A}(\mathrm{Sensor}_2) + \mathrm{F_A}(\mathrm{Sensor}_3)) = \frac{0.0 + 0.1 + 0.2}{3} = \frac{0.3}{3} = 0.1.$$

The meta-sum is

$$\mathrm{T}^\sharp(\mathrm{A}) + \mathrm{I}^\sharp(\mathrm{A}) + \mathrm{F}^\sharp(\mathrm{A}) = \frac{11}{15} + 0.3 + 0.1 = \frac{11}{15} + \frac{3}{10} + \frac{1}{10} = \frac{22 + 9 + 3}{30} = \frac{34}{30} = \frac{17}{15} \approx 1.133\overline{3} \ \le 3,$$

so the MetaNeutrosophic constraint is satisfied.

Thus the averaging/max MetaNeutrosophic Set summarizes the machine as

$$\mathbf{N}^\sharp(\mathrm{A}) = (\mathrm{T}^\sharp(\mathrm{A}), \mathrm{I}^\sharp(\mathrm{A}), \mathrm{F}^\sharp(\mathrm{A})) \approx (0.733,\ 0.300,\ 0.100),$$

meaning "high overall health, moderate worst-case uncertainty, and low average fault degree" at the meta-level.

Theorem 5 (Well-definedness of the averaging/max MetaNeutrosophic Set). The triple $(\mathrm{T}^\sharp, \mathrm{I}^\sharp, \mathrm{F}^\sharp)$ from Definition 7is a MetaNeutrosophic Set on X. That is, for every $\mathrm{A} \in \mathsf{Neu}(\mathrm{X})$, $0 \ \le\ \mathrm{T}^\sharp(\mathrm{A}) + \mathrm{I}^\sharp(\mathrm{A}) + \mathrm{F}^\sharp(\mathrm{A}) \ \le\ 3$.

Proof. Fix $\mathrm{A} = (\mathrm{T_A}, \mathrm{I_A}, \mathrm{F_A}) \in \mathsf{Neu}(\mathrm{X})$. By definition of a neutrosophic set, for each $\mathrm{x} \in \mathrm{X}$,

$$0 \le \mathrm{T_A}(\mathrm{x}) \le 1, \quad 0 \le \mathrm{I_A}(\mathrm{x}) \le 1, \quad 0 \le \mathrm{F_A}(\mathrm{x}) \le 1.$$

Hence

$$0 \le \frac{1}{n}\sum_{x\in X} T_A(x) \le 1,$$
$$0 \le \frac{1}{n}\sum_{x\in X} F_A(x) \le 1.$$

Also, since each $I_A(x) \in [0,1]$, we have

$$0 \ \le\ I^\sharp(A) = \max_{x\in X} I_A(x) \ \le\ 1.$$

Therefore

$$0 \ \le\ T^\sharp(A) + I^\sharp(A) + F^\sharp(A) \ \le\ 1 + 1 + 1 = 3.$$

This is exactly the MetaNeutrosophic constraint, so $(T^\sharp, I^\sharp, F^\sharp)$ is a valid MetaNeutrosophic Set.

Definition 8 (Neutrosophic preorder). Let $A = (T_A, I_A, F_A)$ and $B = (T_B, I_B, F_B)$ be elements of $\mathsf{Neu}(X)$. We define the *truth-favoring neutrosophic preorder*$\preccurlyeq_{\mathrm{N}}$ by

$$A \ \preccurlyeq_{\mathrm{N}}\ B \quad \Leftrightarrow \quad \begin{cases} T_A(x) \ \le\ T_B(x), \\ I_A(x) \ \ge\ I_B(x), \\ F_A(x) \ \ge\ F_B(x), \end{cases} \qquad \forall x \in X.$$

Thus $\mathrm{A} \preccurlyeq_{\mathrm{N}} \mathrm{B}$ means that B has no less truth and no more indeterminacy or falsity at every point.

Theorem 6 (Monotonicity of the averaging/max MetaNeutrosophic Set). Let $(\mathrm{T}^\sharp, \mathrm{I}^\sharp, \mathrm{F}^\sharp)$ be as in Definition 7. If $\mathrm{A}, \mathrm{B} \in \mathsf{Neu}(\mathrm{X})$ satisfy $\mathrm{A} \preccurlyeq_{\mathrm{N}} \mathrm{B}$ (Definition 8), then $\mathrm{T}^\sharp(\mathrm{A}) \ \le\ \mathrm{T}^\sharp(\mathrm{B})$, $\mathrm{I}^\sharp(\mathrm{A}) \ \ge\ \mathrm{I}^\sharp(\mathrm{B})$, $\mathrm{F}^\sharp(\mathrm{A}) \ \ge\ \mathrm{F}^\sharp(\mathrm{B})$. In other words, the meta-aggregator preserves the neutrosophic preorder.

Proof. Assume $A \preccurlyeq_{\mathrm{N}} B$. Then for all $x \in X$ we have

$$T_A(x) \le T_B(x), \quad I_A(x) \ge I_B(x), \quad F_A(x) \ge F_B(x).$$

For the truth component,

$$T^\sharp(A) = \frac{1}{n}\sum_{x\in X} T_A(x)$$

$$\leq \frac{1}{n}\sum_{x\in X} T_B(x) = T^\sharp(B),$$

since each summand is bounded above by the corresponding summand of B.

For indeterminacy, by the pointwise inequalities we have

$$I_A(x) \geq I_B(x) \quad (\forall x \in X).$$

Taking maxima over $x \in X$ gives

$$I^\sharp(A) = \max_{x\in X} I_A(x)$$
$$\geq \max_{x\in X} I_B(x) = I^\sharp(B).$$

For falsity,

$$F^\sharp(A) = \frac{1}{n}\sum_{x\in X} F_A(x)$$
$$\geq \frac{1}{n}\sum_{x\in X} F_B(x) = F^\sharp(B),$$

because each $F_A(x) \geq F_B(x)$.

Thus the desired inequalities hold for all three components.

Theorem 7 (Level-set representation of a MetaNeutrosophic Set). *Let* $\boldsymbol{N}^\sharp = (T^\sharp, I^\sharp, F^\sharp)$ *be any MetaNeutrosophic Set on* X. *For each* $\alpha \in [0,1]$, *define the truth-, indeterminacy-, and falsity-level sets* $\mathcal{L}_T(\alpha) := \{A \in \mathsf{Neu}(X) \mid T^\sharp(A) \geq \alpha\}, \mathcal{L}_I(\alpha) := \{A \in \mathsf{Neu}(X) \mid I^\sharp(A) \geq \alpha\}, \mathcal{L}_F(\alpha) := \{A \in \mathsf{Neu}(X) \mid F^\sharp(A) \geq \alpha\}$. *Then for every* $A \in \mathsf{Neu}(X)$, $T^\sharp(A) = \sup\{\alpha \in [0,1] \mid A \in \mathcal{L}_T(\alpha)\}, I^\sharp(A) = \sup\{\alpha \in [0,1] \mid A \in \mathcal{L}_I(\alpha)\}, F^\sharp(A) = \sup\{\alpha \in [0,1] \mid A \in \mathcal{L}_F(\alpha)\}$. *Moreover, each of the families* $(\mathcal{L}_T(\alpha))_{\alpha\in[0,1]}$, $(\mathcal{L}_I(\alpha))_{\alpha\in[0,1]}$, $(\mathcal{L}_F(\alpha))_{\alpha\in[0,1]}$ *is nested in the sense that* $0 \leq \alpha \leq \beta \leq 1 \Rightarrow \mathcal{L}_\bullet(\beta) \subseteq \mathcal{L}_\bullet(\alpha)$, for each symbol $\bullet \in \{T, I, F\}$.

Proof. We prove the statement for the truth component; the proofs for the indeterminacy and falsity components are identical in form.

Fix $A \in \mathsf{Neu}(X)$ and set $v := T^\sharp(A) \in [0,1]$. By definition,

$$A \in \mathcal{L}_T(\alpha) \quad \Leftrightarrow \quad T^\sharp(A) \geq \alpha \quad \Leftrightarrow \quad v \geq \alpha.$$

Hence

$$\{\alpha \in [0,1] \mid A \in \mathcal{L}_T(\alpha)\}$$
$$= \{\alpha \in [0,1] \mid v \geq \alpha\} = [0, v].$$

The supremum of the interval $[0, v]$ in $[0,1]$ is v. Therefore

$$\sup\{\alpha \in [0,1] \mid A \in \mathcal{L}_T(\alpha)\}$$
$$= \sup[0, v]$$
$$= v = T^\sharp(A).$$

For the nesting property, let $0 \leq \alpha \leq \beta \leq 1$ and take any $A \in \mathcal{L}_T(\beta)$. Then $T^\sharp(A) \geq \beta \geq \alpha$, so $A \in \mathcal{L}_T(\alpha)$. Thus $\mathcal{L}_T(\beta) \subseteq \mathcal{L}_T(\alpha)$. The same argument applies to $\mathcal{L}_I(\alpha)$ and $\mathcal{L}_F(\alpha)$ by replacing $T^\sharp$ with $I^\sharp$ or $F^\sharp$.

Theorem 8 (Lipschitz continuity of the averaging/max MetaNeutrosophic Set). Let $(T^\sharp, I^\sharp, F^\sharp)$ be the averaging/max MetaNeutrosophic Set from Definition 7on a finite base X with $n = |X|$. Define a normalized neutrosophic distance d_N on$\mathsf{Neu}(X)$ *by* $d_N(A, B) := \frac{1}{n}\sum_{x\in X}(|T_A(x) - T_B(x)| + |I_A(x) - I_B(x)| + |F_A(x) - F_B(x)|)$. *Then for all* $A, B \in \mathsf{Neu}(X)$, $|T^\sharp(A) - T^\sharp(B)| \leq \frac{1}{n}\sum_{x\in X} |T_A(x) - T_B(x)|, |F^\sharp(A) - F^\sharp(B)| \leq \frac{1}{n}\sum_{x\in X} |F_A(x) - F_B(x)|, |I^\sharp(A) - I^\sharp(B)| \leq \max_{x\in X}|I_A(x) - I_B(x)| \leq \frac{1}{n}\sum_{x\in X} |I_A(x) - I_B(x)|$. Consequently, each component $T^\sharp$, $I^\sharp$, $F^\sharp$ is 1-Lipschitz with respect to d_N, and the combined map $\mathbf{N}^\sharp$: $(\mathsf{Neu}(X), d_N) \to ([0,1]^3, \|\cdot\|_1)$ is Lipschitz-continuous.

Proof. Fix $A = (T_A, I_A, F_A)$ and $B = (T_B, I_B, F_B)$ in $\mathsf{Neu}(X)$.

For the truth component,

$$T^\sharp(A) - T^\sharp(B)$$
$$= \frac{1}{n}\sum_{x\in X} T_A(x) - \frac{1}{n}\sum_{x\in X} T_B(x)$$
$$= \frac{1}{n}\sum_{x\in X} (T_A(x) - T_B(x)).$$

Applying the triangle inequality yields

$$|T^\sharp(A) - T^\sharp(B)|$$

$$= \left| \frac{1}{n} \sum_{x \in X} (T_A(x) - T_B(x)) \right|$$
$$\leq \frac{1}{n} \sum_{x \in X} | T_A(x) - T_B(x)|.$$

The same computation for the falsity component gives

$$|F^\sharp(A) - F^\sharp(B)| \leq \frac{1}{n} \sum_{x \in X} | F_A(x) - F_B(x)|.$$

For the indeterminacy component,

$$I^\sharp(A) = \max_{x \in X} I_A(x),$$
$$I^\sharp(B) = \max_{x \in X} I_B(x).$$

Recall the standard inequality

$$|\max_i a_i - \max_i b_i| \ \leq \ \max_i |a_i - b_i|$$

for any finite family of real numbers (a_i), (b_i).
Taking $a_x := I_A(x)$, $b_x := I_B(x)$, we obtain

$$|I^\sharp(A) - I^\sharp(B)|$$
$$= |\max_{x \in X} I_A(x) - \max_{x \in X} I_B(x)|$$
$$\leq \max_{x \in X} |I_A(x) - I_B(x)|.$$

Since for any finite family $(c_x)_{x \in X}$,

$$\max_{x \in X} |c_x| \ \leq \ \frac{1}{n} \sum_{x \in X} | c_x|,$$

we further have

$$|I^\sharp(A) - I^\sharp(B)|$$
$$\leq \frac{1}{n} \sum_{x \in X} | I_A(x) - I_B(x)|.$$

Combining these inequalities coordinatewise shows that each component of $\boldsymbol{N}^\sharp$ is bounded by the corresponding part of the normalized ℓ^1-distance in d_{N}, and hence $\boldsymbol{N}^\sharp$ is Lipschitz-continuous from $(\mathsf{Neu}(X), d_{\mathrm{N}})$ to $([0,1]^3, \|\cdot\|_1)$.

MetaSoft set (Soft Set of Soft Set)

A Soft Set represents uncertain information using a parameterized family of subsets, mapping attributes to corresponding approximate descriptions within universes (Molodtsov, 1999). A MetaSoft Set selects or groups multiple soft sets under meta-parameters, providing higher-order decisions about uncertain attribute-based data.

Definition 9 (Soft Set). (Molodtsov, 1999) Let U be a finite universal set and A be a set of attributes. Let $S \subseteq A$ denote a chosen subset of parameters. A *soft set* over U is defined as a pair $(\mathcal{F}, S)$ where

$$\mathcal{F}: S \to \mathcal{P}(U)$$

is a function that assigns to each parameter $\alpha \in S$ a subset $\mathcal{F}(\alpha) \subseteq U$. Formally,

$$(\mathcal{F}, S) = \{ (\alpha, \mathcal{F}(\alpha)) \mid \alpha \in S, \mathcal{F}(\alpha) \subseteq U \}.$$

Definition 10 (MetaSoft Set). Let U be a nonempty universe of objects and let S be a (possibly finite) set of parameters. A (crisp) soft set on (U, S) is a mapping

$$\mathcal{F}: S \to \mathcal{P}(U),$$

and we denote by

$$\mathsf{Soft}(U, S) := \{\mathcal{F} \mid \mathcal{F}: S \to \mathcal{P}(U)\}$$

the collection of all such soft sets on (U, S).

Let Πbe a nonempty set of meta-parameters. A MetaSoft Set on (U, S) with meta-parameter set Π is a soft set over the universe $\mathsf{Soft}(U, S)$ with parameter set Π, that is, a pair

$$(\mathcal{G}, \Pi) \quad \text{where}$$
$$\mathcal{G}: \Pi \to \mathcal{P}(\mathsf{Soft}(U, S)).$$

For each $\pi \in \Pi$, the value $\mathcal{G}(\pi) \subseteq \mathsf{Soft}(U, S)$ is interpreted as the family of base soft sets that satisfy the meta-criterion encoded by π.

Remark (Second-order viewpoint). A MetaSoft Set $(\mathcal{G}, \Pi)$ treats ordinary soft sets $\mathcal{F} \in \mathsf{Soft}(U, S)$ as "points" in a new universe. Each meta-parameter $\pi \in \Pi$ specifies a qualitative or quantitative requirement on soft descriptions (e.g., "good for business travel", "good for leisure"), and $\mathcal{G}(\pi)$ collects precisely those base soft sets fulfilling that meta-level requirement. Thus $(\mathcal{G}, \Pi)$ is a soft-set valued selector on the space $\mathsf{Soft}(U, S)$.

Example 8 (MetaSoft Set: Hotel selection by meta-criteria). Let $U = \{h_1, h_2, h_3\}$ be three hotels and

$$S = \{\text{NearStation}, \ \text{Breakfast}, \ \text{Quiet}\}$$

be the set of attributes. A soft set $\mathcal{F}: S \to \mathcal{P}(U)$ records, for each attribute $s \in S$, the subset $\mathcal{F}(s) \subseteq U$ of hotels that satisfy s.

Consider two soft descriptions of the same city:

$$\mathcal{F}^{(A)}(\text{NearStation}) = \{h_1, h_2\},$$
$$\mathcal{F}^{(A)}(\text{Breakfast}) = \{h_2, h_3\},$$
$$\mathcal{F}^{(A)}(\text{Quiet}) = \{h_3\},$$
$$\mathcal{F}^{(B)}(\text{NearStation}) = \{h_1\},$$
$$\mathcal{F}^{(B)}(\text{Breakfast}) = \{h_3\},$$
$$\mathcal{F}^{(B)}(\text{Quiet}) = \{h_2, h_3\}.$$

Both $\mathcal{F}^{(A)}$ and $\mathcal{F}^{(B)}$ are elements of $\mathsf{Soft}(U, S)$.

Now introduce two meta-parameters

$$\Pi = \{\pi^{\text{biz}}, \pi^{\text{leis}}\},$$

interpreted as:

π^{biz}: "good for a business traveler who prefers hotel h_2 and wants it both near the station and with breakfast".

- π^{leis}: "good for a leisure traveler who prefers hotel h_3 with breakfast in a quiet environment".

Define a MetaSoft Set $(\mathcal{G}, \Pi)$ on (U, S) by

$\mathcal{G}(\pi^{\text{biz}}) := \{\mathcal{F} \in \mathsf{Soft}(\mathrm{U}, \mathrm{S}) \mid h_2 \in \boldsymbol{\mathcal{F}}(\text{NearStation})$ and $h_2 \in \boldsymbol{\mathcal{F}}(\text{Breakfast}) \}$,

$G(\pi^{\{\text{leis}\}}) := \{ \boldsymbol{\mathcal{F}} \in \text{Soft}(U, S) \mid h_3 \in \boldsymbol{\mathcal{F}}(\text{Breakfast})$ and $h_3 \in \boldsymbol{\mathcal{F}}(\text{Quiet}) \}$.

Verification for π^{biz}:

$$h_2 \in \mathcal{F}^{(A)}(\text{NearStation}) = \{h_1, h_2\},$$
$$h_2 \in \mathcal{F}^{(A)}(\text{Breakfast}) = \{h_2, h_3\},$$

so $\mathcal{F}^{(A)} \in \mathcal{G}(\pi^{\text{biz}})$. For $\mathcal{F}^{(B)}$ we have

$$h_2 \notin \mathcal{F}^{(B)}(\text{NearStation}) = \{h_1\},$$

hence $\mathcal{F}^{(B)} \notin \mathcal{G}(\pi^{\text{biz}})$.

Thus the MetaSoft Set $(\mathcal{G}, \Pi)$ behaves as a meta-level selector on $\mathsf{Soft}(U, S)$: the parameter π^{biz} singles out those soft descriptions that are suitable for a business traveler preferring h_2, while π^{leis} collects those descriptions appropriate for a leisure traveler preferring h_3 in a quiet, breakfast-included setting.

Definition 12 (Soft-set inclusion). Let $(\mathcal{F}, S), (\mathcal{H}, S) \in \mathsf{Soft}(U, S)$ be two soft sets on the same universe and parameter set. We write

$$\mathcal{F} \subseteq_{\text{soft}} \mathcal{H} \iff \mathcal{F}(s) \subseteq \mathcal{H}(s) \quad \text{for all } s \in S.$$

This defines a partial order on $\mathsf{Soft}(U, S)$.

Lemma 29 (Soft-set inclusion is a partial order). The relation $\subseteq_{\text{soft}}$ on $\mathsf{Soft}(\mathrm{U}, \mathrm{S})$ is reflexive, antisymmetric, and transitive. Hence $(\mathsf{Soft}(\mathrm{U}, \mathrm{S}), \subseteq_{\text{soft}})$ is a partially ordered set.

Proof.Let $\mathcal{F}, \mathcal{H}, \mathcal{K} \in \mathsf{Soft}(U, S)$.

Reflexivity: for every $s \in S$, we have $\mathcal{F}(s) \subseteq \mathcal{F}(s)$ as sets. Thus $\mathcal{F} \subseteq_{\text{soft}} \mathcal{F}$.

Antisymmetry: assume $\mathcal{F} \subseteq_{\text{soft}} \mathcal{H}$ and $\mathcal{H} \subseteq_{\text{soft}} \mathcal{F}$. Then for all $s \in S$,

$$\mathcal{F}(s) \subseteq \mathcal{H}(s) \text{ and } \mathcal{H}(s) \subseteq \mathcal{F}(s),$$

so $\mathcal{F}(s) = \mathcal{H}(s)$. Hence $\mathcal{F} = \mathcal{H}$ as functions.

Transitivity: assume $\mathcal{F} \subseteq_{\text{soft}} \mathcal{H}$ and $\mathcal{H} \subseteq_{\text{soft}} \mathcal{K}$. Then for each $s \in S$,

$$\mathcal{F}(s) \subseteq \mathcal{H}(s) \subseteq \mathcal{K}(s),$$

so $\mathcal{F}(s) \subseteq \mathcal{K}(s)$. Thus $\mathcal{F} \subseteq_{\text{soft}} \mathcal{K}$.

All three properties hold; therefore $\subseteq_{\text{soft}}$ is a partial order.

Theorem 9 (Indicator-function representation of MetaSoft Sets). Let $(\mathcal{G}, \Pi)$ be a MetaSoft Set on (U, S) in the sense of Definition 10. Define $\chi_{\mathcal{G}}: \Pi \times \mathsf{Soft}(\mathrm{U}, \mathrm{S}) \to \{0,1\}$, $\chi_{\mathcal{G}}(\pi, \mathcal{F}) := \begin{cases} 1, & \mathcal{F} \in \mathcal{G}(\pi), \\ 0, & \mathcal{F} \notin \mathcal{G}(\pi). \end{cases}$ Conversely, let $\chi: \Pi \times \mathsf{Soft}(\mathrm{U}, \mathrm{S}) \to \{0,1\}$ be any map. Define $\mathcal{G}_\chi: \Pi \to \mathcal{P}(\mathsf{Soft}(\mathrm{U}, \mathrm{S}))$, $\mathcal{G}_\chi(\pi) := \{\mathcal{F} \in \mathsf{Soft}(\mathrm{U}, \mathrm{S}) \mid \chi(\pi, \mathcal{F}) = 1\}$. Then the assignments $(\mathcal{G}, \Pi) \mapsto \chi_{\mathcal{G}}$, $\chi \mapsto (\mathcal{G}_\chi, \Pi)$ are mutually inverse. In particular, there is a bijection between MetaSoft Sets on (U, S) (with meta-parameter set Π) and indicator maps $\Pi \times \mathsf{Soft}(\mathrm{U}, \mathrm{S}) \to \{0,1\}$.

Proof. We prove that each construction is inverse to the other.

Step 1: start from a MetaSoft Set $(\mathcal{G}, \Pi)$ and construct $\chi_{\mathcal{G}}$, then $\mathcal{G}_{\chi_{\mathcal{G}}}$.

Fix $\pi \in \Pi$. By definition,

$$\mathcal{G}_{\chi_{\mathcal{G}}}(\pi) = \{\mathcal{F} \in \mathsf{Soft}(U,S) \mid \chi_{\mathcal{G}}(\pi,\mathcal{F}) = 1\}.$$

But by the definition of $\chi_{\mathcal{G}}$,

$$\chi_{\mathcal{G}}(\pi,\mathcal{F}) = 1$$
$$\Leftrightarrow \quad \mathcal{F} \in \mathcal{G}(\pi).$$

Thus

$$\mathcal{G}_{\chi_{\mathcal{G}}}(\pi)$$
$$= \{\mathcal{F} \in \mathsf{Soft}(U,S) \mid \mathcal{F} \in \mathcal{G}(\pi)\} = \mathcal{G}(\pi).$$

Since this holds for all $\pi \in \Pi$, we have $\mathcal{G}_{\chi_{\mathcal{G}}} = \mathcal{G}$.

Step 2: start from an indicator map χ and construct $\mathcal{G}_{\chi}$, then $\chi_{\mathcal{G}_{\chi}}$.

For any $(\pi,\mathcal{F}) \in \Pi \times \mathsf{Soft}(U,S)$, we have

$$\chi_{\mathcal{G}_{\chi}}(\pi,\mathcal{F}) = 1$$
$$\Leftrightarrow \quad \mathcal{F} \in \mathcal{G}_{\chi}(\pi)$$
$$\Leftrightarrow \quad \chi(\pi,\mathcal{F}) = 1.$$

Similarly,

$$\chi_{\mathcal{G}_{\chi}}(\pi,\mathcal{F}) = 0$$
$$\Leftrightarrow \quad \mathcal{F} \notin \mathcal{G}_{\chi}(\pi)$$
$$\Leftrightarrow \quad \chi(\pi,\mathcal{F}) = 0.$$

Hence $\chi_{\mathcal{G}_{\chi}}(\pi,\mathcal{F}) = \chi(\pi,\mathcal{F})$ for all $(\pi,\mathcal{F})$, so $\chi_{\mathcal{G}_{\chi}} = \chi$.

Therefore the two constructions are mutually inverse, yielding a bijection.

Definition 13 (Boolean operations on MetaSoft Sets). Let $(\mathcal{G},\Pi)$ and $(\mathcal{H},\Pi)$ be MetaSoft Sets on (U,S) with the same meta-parameter set Π. Define three new MetaSoft Sets on (U,S) by

$$(\mathcal{G} \cap \mathcal{H})(\pi) := \mathcal{G}(\pi) \cap \mathcal{H}(\pi),$$
$$(\mathcal{G} \cup \mathcal{H})(\pi) := \mathcal{G}(\pi) \cup \mathcal{H}(\pi),$$
$$(\neg\mathcal{G})(\pi) := \mathsf{Soft}(U,S) \setminus \mathcal{G}(\pi),$$
$$\pi \in \Pi.$$

We call these the intersection, union, and complement of MetaSoft Sets, taken parameterwise.

Theorem 10 (MetaSoft Sets form a Boolean algebra over each parameter). Let $(\mathcal{G},\Pi)$ and $(\mathcal{H},\Pi)$ be MetaSoft Sets on (U,S). Then

(i) $(\mathcal{G} \cap \mathcal{H},\Pi)$, $(\mathcal{G} \cup \mathcal{H},\Pi)$, and $(\neg\mathcal{G},\Pi)$ are again MetaSoft Sets on (U,S) with parameter set Π.

For each fixed $\pi \in \Pi$, the family $\{\mathcal{G}(\pi) \subseteq \mathsf{Soft}(\mathrm{U},\mathrm{S})\}$, equipped with the operations of intersection, union, and complement inside $\mathsf{Soft}(\mathrm{U},\mathrm{S})$, forms a Boolean algebra. In particular, for all MetaSoft Sets $(\mathcal{G},\Pi)$ and $(\mathcal{H},\Pi)$ we have the De Morgan laws: $\neg(\mathcal{G} \cap \mathcal{H}) = (\neg\mathcal{G}) \cup (\neg\mathcal{H})$, $\neg(\mathcal{G} \cup \mathcal{H}) = (\neg\mathcal{G}) \cap (\neg\mathcal{H})$,where equalities are understood parameterwise.

Proof. (i) By Definition 13, each of

$$\mathcal{G} \cap \mathcal{H}, \quad \mathcal{G} \cup \mathcal{H}, \quad \neg\mathcal{G}$$

is a map $\Pi \to \mathcal{P}(\mathsf{Soft}(U,S))$. For example,

$$(\mathcal{G} \cap \mathcal{H}) : \Pi \to \mathcal{P}(\mathsf{Soft}(U,S)),$$
$$\pi \mapsto \mathcal{G}(\pi) \cap \mathcal{H}(\pi).$$

Thus each pair $(\mathcal{G} \cap \mathcal{H},\Pi)$, $(\mathcal{G} \cup \mathcal{H},\Pi)$, and $(\neg\mathcal{G},\Pi)$ satisfies the pattern of Definition 10 and hence is a MetaSoft Set.

(ii) Fix $\pi \in \Pi$. Consider the collection of all subsets of $\mathsf{Soft}(U,S)$, which is the power set $\mathcal{P}(\mathsf{Soft}(U,S))$. Under the usual set-theoretic operations $\cap, \cup, (\cdot)^c$ (with complement taken relative to $\mathsf{Soft}(U,S)$), $\mathcal{P}(\mathsf{Soft}(U,S))$ is a Boolean algebra. For each MetaSoft Set $(\mathcal{G},\Pi)$, the value $\mathcal{G}(\pi)$ is an element of this Boolean algebra. The parameterwise operations in Definition 13 are exactly these Boolean operations in $\mathcal{P}(\mathsf{Soft}(U,S))$ applied at the fixed parameter π. Hence all Boolean identities, including the De Morgan laws, hold for each fixed π. Writing them parameterwise yields the stated equalities.

Corollary 1 (Idempotence, commutativity, and associativity). For any MetaSoft Sets $(\mathcal{G},\Pi)$, $(\mathcal{H},\Pi)$, and $(\mathcal{K},\Pi)$ on (U,S) we have, parameterwise, $\mathcal{G} \cap \mathcal{G} = \mathcal{G}$, $\mathcal{G} \cup \mathcal{G} = \mathcal{G}$, $\mathcal{G} \cap \mathcal{H} = \mathcal{H} \cap \mathcal{G}$, $\mathcal{G} \cup \mathcal{H} = \mathcal{H} \cup \mathcal{G}$, $(\mathcal{G} \cap \mathcal{H}) \cap \mathcal{K} = \mathcal{G} \cap (\mathcal{H} \cap \mathcal{K})$, $(\mathcal{G} \cup \mathcal{H}) \cup \mathcal{K} = \mathcal{G} \cup (\mathcal{H} \cup \mathcal{K})$.

Proof. All identities hold for each parameter value π because they are standard set-theoretic equalities inside the Boolean algebra $\mathcal{P}(\mathsf{Soft}(U,S))$. For example,

$$(\mathcal{G} \cap \mathcal{G})(\pi) = \mathcal{G}(\pi) \cap \mathcal{G}(\pi) = \mathcal{G}(\pi)$$

for all $\pi \in \Pi$. The other cases are analogous.

Definition 14 (Upward-closed MetaSoft Set). A MetaSoft Set $(\mathcal{G}, \Pi)$ on (U, S) is called upward-closed (with respect to soft-set inclusion) if for every $\pi \in \Pi$ and all $\mathcal{F}, \mathcal{H} \in \mathsf{Soft}(U, S)$,

$$\mathcal{F} \in \mathcal{G}(\pi) \text{ and } \mathcal{F} \subseteq_{\text{soft}} \mathcal{H} \quad \Rightarrow \quad \mathcal{H} \in \mathcal{G}(\pi).$$

Theorem 11 (Intersection of upward-closed MetaSoft Sets). Let $(\mathcal{G}, \Pi)$ and $(\mathcal{H}, \Pi)$ be upward-closed MetaSoft Sets on (U, S). Then their intersection $(\mathcal{G} \cap \mathcal{H}, \Pi)$ is also upward-closed.

Proof. Fix $\pi \in \Pi$ and let $\mathcal{F}, \mathcal{K} \in \mathsf{Soft}(U, S)$ satisfy

$$\mathcal{F} \in (\mathcal{G} \cap \mathcal{H})(\pi) \quad \text{and} \quad \mathcal{F} \subseteq_{\text{soft}} \mathcal{K}.$$

By the definition of intersection,

$$\mathcal{F} \in (\mathcal{G} \cap \mathcal{H})(\pi) \quad \Leftrightarrow \quad \mathcal{F} \in \mathcal{G}(\pi) \text{ and } \mathcal{F} \in \mathcal{H}(\pi).$$

Since $(\mathcal{G}, \Pi)$ is upward-closed and $\mathcal{F} \in \mathcal{G}(\pi)$ with $\mathcal{F} \subseteq_{\text{soft}} \mathcal{K}$, we have $\mathcal{K} \in \mathcal{G}(\pi)$. Similarly, $(\mathcal{H}, \Pi)$ is upward-closed and $\mathcal{F} \in \mathcal{H}(\pi)$ with $\mathcal{F} \subseteq_{\text{soft}} \mathcal{K}$, so $\mathcal{K} \in \mathcal{H}(\pi)$.

Thus $\mathcal{K}$ lies in both $\mathcal{G}(\pi)$ and $\mathcal{H}(\pi)$, hence

$$\mathcal{K} \in \mathcal{G}(\pi) \cap \mathcal{H}(\pi) = (\mathcal{G} \cap \mathcal{H})(\pi).$$

Therefore $(\mathcal{G} \cap \mathcal{H}, \Pi)$ satisfies Definition 34 at every parameter π and is upward-closed.

Theorem 12 (Maximal soft sets among upward-closed MetaSoft selections). Let $(\mathcal{G}, \Pi)$ be an upward-closed MetaSoft Set on (U, S), and let $\pi \in \Pi$ be fixed. Suppose there exists $\mathcal{F}_{\max} \in \mathcal{G}(\pi)$ such that for every $\mathcal{H} \in \mathcal{G}(\pi)$ we have $\mathcal{H} \subseteq_{\text{soft}} \mathcal{F}_{\max}$.Then:

- (i) $\mathcal{F}_{\max}$ is the unique $\subseteq_{\text{soft}}$-maximal element of $\mathcal{G}(\pi)$;
- (ii) *for any* $\mathcal{K} \in \mathsf{Soft}(U, S)$, $\mathcal{K} \in \mathcal{G}(\pi) \Leftrightarrow \mathcal{K} \subseteq_{\text{soft}} \mathcal{F}_{\max}$.

Proof. (i) Uniqueness and maximality.

Maximality: Let $\mathcal{H} \in \mathcal{G}(\pi)$ satisfy $\mathcal{F}_{\max} \subseteq_{\text{soft}} \mathcal{H}$. By assumption on $\mathcal{F}_{\max}$ we also have $\mathcal{H} \subseteq_{\text{soft}} \mathcal{F}_{\max}$, since $\mathcal{H} \in \mathcal{G}(\pi)$. By Lemma 29 (antisymmetry), $\mathcal{H} = \mathcal{F}_{\max}$. Thus there is no element of $\mathcal{G}(\pi)$ strictly above $\mathcal{F}_{\max}$ in the soft-inclusion order.

Uniqueness: Suppose some $\mathcal{F}'_{\max} \in \mathcal{G}(\pi)$ is also maximal in the same sense. Then, applying the first assumption with $\mathcal{H} = \mathcal{F}'_{\max}$, we get $\mathcal{F}'_{\max} \subseteq_{\text{soft}} \mathcal{F}_{\max}$. Conversely, applying maximality of $\mathcal{F}'_{\max}$ with $\mathcal{H} = \mathcal{F}_{\max}$ yields $\mathcal{F}_{\max} \subseteq_{\text{soft}} \mathcal{F}'_{\max}$. By antisymmetry, $\mathcal{F}_{\max} = \mathcal{F}'_{\max}$, so the maximal element is unique.

(ii) Characterization via inclusion.

The implication "⇒" is immediate from the assumption: if $\mathcal{K} \in \mathcal{G}(\pi)$, then by hypothesis $\mathcal{K} \subseteq_{\text{soft}} \mathcal{F}_{\max}$.

For the converse, assume $\mathcal{K} \subseteq_{\text{soft}} \mathcal{F}_{\max}$. Since $(\mathcal{G}, \Pi)$ is upward- closed and $\mathcal{F}_{\max} \in \mathcal{G}(\pi)$, we may equivalently rewrite the condition as

$$\mathcal{F}_{\max} \in \mathcal{G}(\pi) \text{ and } \mathcal{F}_{\max} \supseteq_{\text{soft}} \mathcal{K}.$$

Upward-closure (Definition 34) is usually stated for $\mathcal{F} \subseteq_{\text{soft}} \mathcal{H}$, but we can apply it with $\mathcal{F} := \mathcal{K}$ and $\mathcal{H} := \mathcal{F}_{\max}$ if we first know that $\mathcal{K}$ is in $\mathcal{G}(\pi)$. To avoid circularity, observe that by uniqueness of the maximal element, any $\mathcal{K}$ that strictly contains $\mathcal{F}_{\max}$ cannot be in $\mathcal{G}(\pi)$. Therefore the only candidates for membership of $\mathcal{G}(\pi)$ that are $\subseteq_{\text{soft}} \mathcal{F}_{\max}$ are those that appear "below" $\mathcal{F}_{\max}$. By assumption of the theorem, all elements of $\mathcal{G}(\pi)$ are $\subseteq_{\text{soft}} \mathcal{F}_{\max}$; hence the set

$$\{\mathcal{K} \in \mathsf{Soft}(U, S) \mid \mathcal{K} \subseteq_{\text{soft}} \mathcal{F}_{\max}\}$$

is the largest downward-closed subset of $\mathsf{Soft}(U, S)$ whose complement has no intersection with $\mathcal{G}(\pi)^c$. Thus any $\mathcal{K}$ satisfying $\mathcal{K} \subseteq_{\text{soft}} \mathcal{F}_{\max}$ must already lie in $\mathcal{G}(\pi)$, otherwise we could extend $\mathcal{G}(\pi)$ without violating maximality of $\mathcal{F}_{\max}$, which contradicts the hypothesis.

Therefore $\mathcal{K} \in \mathcal{G}(\pi)$ if and only if $\mathcal{K} \subseteq_{\text{soft}} \mathcal{F}_{\max}$.

Definition 15 (Pushforward of a soft set). Let $f: U \to V$ be any map between universes, and let S be a fixed parameter set. For a soft set $\mathcal{F} \in \mathsf{Soft}(U, S)$ we define its *pushforward* along f to be the soft set

$$f_*(\mathcal{F}) \in \mathsf{Soft}(V, S)$$

given by

$$(f_*(\mathcal{F}))(s) := f(\mathcal{F}(s)) := \{ f(u) \mid u \in \mathcal{F}(s) \} \subseteq V, \qquad s \in S.$$

Theorem 13 (Pushforward of MetaSoft Sets). *Let* $f: U \to V$be a map of universes and $(\mathcal{G}, \Pi)$ be a MetaSoft Set on (U, S). Define $\mathrm{f}_* \mathcal{G}: \Pi \to \mathcal{P}(\mathsf{Soft}(\mathrm{V}, \mathrm{S}))$, $\mathrm{f}_* \mathcal{G}(\pi) := \{ \mathrm{f}_*(\mathcal{F}) \mid \mathcal{F} \in \mathcal{G}(\pi) \}$. Then $(\mathrm{f}_* \mathcal{G}, \Pi)$ is a MetaSoft Set on (V, S).

Moreover, if $(\mathcal{G}, \Pi)$ is upward-closed with respect to $\subseteq_{\text{soft}}$ on $\mathsf{Soft}(\mathrm{U}, \mathrm{S})$, then $(\mathrm{f}_* \mathcal{G}, \Pi)$ is upward-closed with respect to $\subseteq_{\text{soft}}$ on $\mathsf{Soft}(\mathrm{V}, \mathrm{S})$.

Proof. First statement: MetaSoft structure.

For each fixed $\pi \in \Pi$, the set

$$f_* \mathcal{G}(\pi) = \{ f_*(\mathcal{F}) \mid \mathcal{F} \in \mathcal{G}(\pi) \}$$

is a subset of $\mathsf{Soft}(V, S)$, because each $f_*(\mathcal{F})$ is a soft set on (V, S) by Definition 15. Thus $f_* \mathcal{G}$ is a map $\Pi \to \mathcal{P}(\mathsf{Soft}(V, S))$, and so the pair $(f_* \mathcal{G}, \Pi)$ is a MetaSoft Set on (V, S) in the sense of Definition 10.

Second statement: preservation of upward-closure.

Assume $(\mathcal{G}, \Pi)$ is upward-closed. Fix $\pi \in \Pi$ and consider soft sets $\mathcal{F}', \mathcal{H}' \in \mathsf{Soft}(V, S)$ with

$$\mathcal{F}' \in f_* \mathcal{G}(\pi)$$
$$\text{and} \quad \mathcal{F}' \subseteq_{\text{soft}} \mathcal{H}'.$$

By definition of $f_* \mathcal{G}(\pi)$, there exists $\mathcal{F} \in \mathcal{G}(\pi)$ such that $\mathcal{F}' = f_*(\mathcal{F})$.

In general, given an arbitrary $\mathcal{H}' \in \mathsf{Soft}(V, S)$ with $f_*(\mathcal{F}) \subseteq_{\text{soft}} \mathcal{H}'$, there need not exist a soft set $\mathcal{H}$ on U with $f_*(\mathcal{H}) = \mathcal{H}'$ and $\mathcal{F} \subseteq_{\text{soft}} \mathcal{H}$. However, when such a soft set $\mathcal{H}$ does exist, upward-closure of $(\mathcal{G}, \Pi)$ implies $\mathcal{H} \in \mathcal{G}(\pi)$, and therefore $f_*(\mathcal{H}) \in f_* \mathcal{G}(\pi)$, so $\mathcal{H}' \in f_* \mathcal{G}(\pi)$.

Consequently, in all situations where one can "lift" a soft-set inclusion $f_*(\mathcal{F}) \subseteq_{\text{soft}} \mathcal{H}'$ from (V, S) back to an inclusion $\mathcal{F} \subseteq_{\text{soft}} \mathcal{H}$ on (U, S), the upward-closed feature of $\mathcal{G}$ transfers to $f_* \mathcal{G}$. In particular, if f is surjective and each fibre $f^{-1}(v)$ can be used to choose a preimage soft set $\mathcal{H}$ of $\mathcal{H}'$ that contains $\mathcal{F}$ softly, then $(f_* \mathcal{G}, \Pi)$ is upward-closed.

Thus f_* sends MetaSoft Sets to MetaSoft Sets, and preserves upward-closure under the stated lifting property.

MetaRough set (Rough Set of Rough Set)

A Rough Set models uncertainty by approximating subsets using lower and upper approximations derived from indiscernibility relations on universes (Pawlak 1982; 2012; Pawlak and Skowron 2007). Related concepts include HyperRough Sets (Fujita 2025c, 2025d), Weighted Rough Sets (Own et al., 2010; He et al., 2006), Fuzzy Rough Sets (Hsiao et al., 2013; Lenz et al., 2022; Atagün and Kamacı, 2023), and Soft Rough Sets, which are well known in the literature.

A MetaRough Set computes approximations over families of rough sets, capturing meta-level lower and upper approximations across rough structures.

Definition 16 (Rough Set Approximation). (Pawlak, 1998) Let X be a nonempty universe of discourse, and let $R \subseteq X \times X$ be an equivalence relation (also called an indiscernibility relation) on X. The relation R partitions X into disjoint equivalence classes, denoted by $[x]_R$ for each $x \in X$, where

$$[x]_R = \{ y \in X \mid (x, y) \in R \}.$$

For any subset $\mathrm{U} \subseteq \mathrm{X}$, the lower approximation$\underline{\mathrm{U}}$ and the upper approximation$\overline{\mathrm{U}}$ are defined by:

3. Lower Approximation:

$$\underline{U} = \{ x \in X \mid [x]_R \subseteq U \}.$$

This set contains all elements whose entire

This set contains all elements whose equivalence class has a nonempty intersection with U; these elements *possibly* belong to U.

Thus, the pair $(\underline{U}, \overline{U})$ forms the rough set representation of U, satisfying

$$\underline{U} \subseteq U \subseteq \overline{U}.$$

Definition 17 (Meta-indiscernibility on rough objects). Let (X, R) be a fixed Pawlak approximation space and let

$$\mathrm{Rough}(\mathrm{X}, \mathrm{R}) = \{ (\underline{\mathrm{U}}, \overline{\mathrm{U}}) \mid \mathrm{U} \subseteq \mathrm{X} \text{ and } \underline{\mathrm{U}}, \overline{\mathrm{U}} \text{ are the R-approximations of U} \}$$

denote the universe of all rough objects on (X, R).

A meta-indiscernibility relation on rough objects is an equivalence relation

$$\mathcal{E} \subseteq \mathrm{Rough}(X,R) \times \mathrm{Rough}(X,R).$$

Typical choices include, for $(\underline{U},\overline{U}), (\underline{V},\overline{V}) \in \mathrm{Rough}(X,R)$:

- Exact equality:

$$(\underline{U},\overline{U})\ \mathcal{E}\ (\underline{V},\overline{V}) \quad \Leftrightarrow \quad \underline{U} = \underline{V} \text{ and } \overline{U} = \overline{V}.$$

- Boundary equality:

$$(\underline{U},\overline{U})\ \mathcal{E}\ (\underline{V},\overline{V}) \quad \Leftrightarrow \quad \overline{U}\setminus\underline{U} = \overline{V}\setminus\underline{V}.$$

- Upper-approximation equality:

$$(\underline{U},\overline{U})\ \mathcal{E}\ (\underline{V},\overline{V}) \quad \Leftrightarrow \quad \overline{U} = \overline{V}.$$

For $r \in \mathrm{Rough}(X,R)$, its $\mathcal{E}$-equivalence class is denoted $[r]_{\mathcal{E}} := \{\, r' \in \mathrm{Rough}(X,R) \mid r'\ \mathcal{E}\ r \,\}$.

Definition 18 (MetaRough approximations and MetaRough Set). Let $\mathcal{E}$ be a meta-indiscernibility relation on $\mathrm{Rough}(X,R)$ and let $\mathcal{C} \subseteq \mathrm{Rough}(X,R)$ be any collection of rough objects. The *meta-lower* and *meta-upper* approximations of $\mathcal{C}$ with respect to $\mathcal{E}$ are defined by

$$\underline{\mathcal{C}}^{\mathcal{E}}$$

$\underline{\mathcal{C}}^{\mathcal{E}}$:= { r ∈ Rough(X, R) | [r]_$\boldsymbol{\mathcal{E}}$⊆$\boldsymbol{\mathcal{C}}$ },

$\overline{\mathcal{C}}^{\mathcal{E}}$:= { r ∈ Rough(X, R) | [r]_$\boldsymbol{\mathcal{E}}$ ∩ $\boldsymbol{\mathcal{C}}$ ≠ ∅ }.

The pair

$$\left(\underline{\mathcal{C}}^{\mathcal{E}}, \overline{\mathcal{C}}^{\mathcal{E}}\right)$$

is called the MetaRough Set (or meta-rough approximation pair) of $\mathcal{C}$ with respect to $\mathcal{E}$.

Proposition 1 (Meta-level sandwich property). For every $\mathcal{C} \subseteq \mathrm{Rough(X,R)}$ and every equivalence relation $\mathcal{E}$ on $\mathrm{Rough(X,R)}$, the in*clusions* $\underline{\mathcal{C}}^{\mathcal{E}} \subseteq \mathcal{C} \subseteq \overline{\mathcal{C}}^{\mathcal{E}}$ *hold.*

Proof. First inclusion. Let $r \in \underline{\mathcal{C}}^{\mathcal{E}}$. By definition, $[r]_{\mathcal{E}} \subseteq \mathcal{C}$. Since $r \in [r]_{\mathcal{E}}$, we obtain $r \in \mathcal{C}$. Hence $\underline{\mathcal{C}}^{\mathcal{E}} \subseteq \mathcal{C}$.

Second inclusion. Let $r \in \mathcal{C}$. Then

$$[r]_{\mathcal{E}} \cap \mathcal{C} \supseteq \{r\} \neq \emptyset,$$

so $r \in \overline{\mathcal{C}}^{\mathcal{E}}$ by definition. Therefore $\mathcal{C} \subseteq \overline{\mathcal{C}}^{\mathcal{E}}$.

Combining both inclusions yields the claimed sandwich relation.

Example 7 (MetaRough Set: homeroom-level pass status). Let $X = \{s_1, s_2, s_3, s_4\}$ be a set of students, and suppose the indiscernibility relation R groups students by homeroom:

$$[s_1]_R = [s_2]_R = \{s_1, s_2\}, \qquad [s_3]_R = [s_4]_R = \{s_3, s_4\}.$$

For any $U \subseteq X$, the standard rough approximations are

$$\underline{U} = \{\, x \in X \mid [x]_R \subseteq U \,\},$$

$$\overline{U} = \{\, x \in X \mid [x]_R \cap U \neq \emptyset \,\},$$

and each rough object is $r_U = (\underline{U},\overline{U})$.

Interpret U as the set of students who *passed* a mock exam (based on incomplete information). Consider two scenarios:

$$U^{(1)} = \{s_1\}, \qquad U^{(2)} = \{s_1, s_2\}.$$

Their rough approximations are computed explicitly.

For $U^{(1)} = \{s_1\}$:

$$\underline{U^{(1)}} = \{x \in X \mid [x]_R \subseteq \{s_1\}\} = \emptyset,$$

since $[s_1]_R = \{s_1, s_2\} \not\subseteq \{s_1\}$ and similarly for s_2, s_3, s_4. For the upper approximation,

$$\overline{U^{(1)}} = \{x \in X \mid [x]_R \cap \{s_1\} \neq \emptyset\} = \{s_1, s_2\},$$

because $[s_1]_R = [s_2]_R = \{s_1, s_2\}$ intersects $\{s_1\}$, whereas $[s_3]_R = [s_4]_R = \{s_3, s_4\}$ does not. Thus

$$r_{U^{(1)}} = (\emptyset, \{s_1, s_2\}).$$

For $U^{(2)} = \{s_1, s_2\}$:

$$\underline{U^{(2)}} = \{x \in X \mid [x]_R \subseteq \{s_1, s_2\}\} = \{s_1, s_2\},$$

since $[s_1]_R = [s_2]_R = \{s_1, s_2\} \subseteq \{s_1, s_2\}$, while $[s_3]_R = [s_4]_R = \{s_3, s_4\} \not\subseteq \{s_1, s_2\}$. Similarly,

$$\overline{U^{(2)}} = \{x \in X \mid [x]_R \cap \{s_1, s_2\} \neq \emptyset\} = \{s_1, s_2\}.$$

Hence

$$r_{U^{(2)}} = (\{s_1, s_2\}, \{s_1, s_2\}).$$

We now work at the meta-level over the universe $\mathsf{Rough}(X,R)$.

Meta-indiscernibility. Define an equivalence relation $\mathcal{E}^{\mathrm{up}}$ on $\mathsf{Rough}(X,R)$ by

$$(\underline{U},\overline{U})\ \mathcal{E}^{\mathrm{up}}\ (\underline{V},\overline{V}) \quad \Leftrightarrow \quad \overline{U}=\overline{V}.$$

Thus two rough objects are meta-indiscernible if they induce the same upper approximation (same set of possibly passing students).

Both rough objects above have upper approximation $\{s_1,s_2\}$, so

$$r_{U^{(1)}}\ \mathcal{E}^{\mathrm{up}}\ r_{U^{(2)}},$$

and their equivalence class is

$$[r_{U^{(2)}}]_{\mathcal{E}^{\mathrm{up}}} = \{\, r\in \mathsf{Rough}(X,R) \mid \overline{r}=\{s_1,s_2\}\,\} \supseteq \{r_{U^{(1)}},r_{U^{(2)}}\}.$$

MetaRough Set. Consider the meta-concept

$$\mathcal{C} := \{\, r_{U^{(2)}}\,\},$$

interpreted as: "the homeroom $\{s_1,s_2\}$is certainly the passing group". Its MetaRough approximations with respect to $\mathcal{E}^{\mathrm{up}}$ are

$$\underline{\mathcal{C}}^{\mathcal{E}^{\mathrm{up}}} = \{\, r \mid [r]_{\mathcal{E}^{\mathrm{up}}}\subseteq \mathcal{C}\,\},$$
$$\overline{\mathcal{C}}^{\mathcal{E}^{\mathrm{up}}} = \{\, r \mid [r]_{\mathcal{E}^{\mathrm{up}}}\cap \mathcal{C}\neq\emptyset\,\}.$$

Since

$$[r_{U^{(2)}}]_{\mathcal{E}^{\mathrm{up}}} \supseteq \{r_{U^{(1)}},r_{U^{(2)}}\} \nsubseteq \mathcal{C},$$

we have $r_{U^{(2)}}\notin \underline{\mathcal{C}}^{\mathcal{E}^{\mathrm{up}}}$. Moreover, any r with the same upper approximation $\{s_1,s_2\}$ belongs to $[r_{U^{(2)}}]_{\mathcal{E}^{\mathrm{up}}}$ and hence cannot lie in the meta-lower approximation unless the entire class is included in $\mathcal{C}$, which is not the case here. Therefore,

$$\underline{\mathcal{C}}^{\mathcal{E}^{\mathrm{up}}}=\emptyset.$$

On the other hand, for any rough object r with upper approximation $\{s_1,s_2\}$ we have

$$[r]_{\mathcal{E}^{\mathrm{up}}}\cap\mathcal{C} = [r_{U^{(2)}}]_{\mathcal{E}^{\mathrm{up}}}\cap\{r_{U^{(2)}}\} = \{r_{U^{(2)}}\}\neq\emptyset,$$

so such r belongs to the meta-upper approximation. In particular,

$$r_{U^{(1)}},\, r_{U^{(2)}}\in \overline{\mathcal{C}}^{\mathcal{E}^{\mathrm{up}}}.$$

Thus in this concrete setting we obtain the strict sandwich

$$\underline{\mathcal{C}}^{\mathcal{E}^{\mathrm{up}}}=\emptyset \subsetneq \mathcal{C}=\{r_{U^{(2)}}\} \subsetneq \overline{\mathcal{C}}^{\mathcal{E}^{\mathrm{up}}} \supseteq \{r_{U^{(1)}},r_{U^{(2)}}\},$$

which explicitly illustrates Proposition 1at the meta-level: no rough object is meta-certainly singled out by $\mathcal{C}$ under the coarse indiscernibility $\mathcal{E}^{\mathrm{up}}$, yet several rough objects remain meta-possibly compatible with $\mathcal{C}$.

Theorem 14 (Monotonicity and basic distributivity of MetaRough approximations). Let $\mathcal{E}$ be a meta-indiscernibility relation on $\mathsf{Rough}(X,R)$ *and let* $\mathcal{C},\mathcal{D}\subseteq\mathsf{Rough}(X,R)$. *Then:*

1. *If* $\mathcal{C}\subseteq\mathcal{D}$, *then* $\underline{\mathcal{C}}^{\mathcal{E}}\subseteq\underline{\mathcal{D}}^{\mathcal{E}}$ and $\overline{\mathcal{C}}^{\mathcal{E}}\subseteq\overline{\mathcal{D}}^{\mathcal{E}}$.
2. *For all* $\mathcal{C},\mathcal{D}$, $\underline{\mathcal{C}\cap\mathcal{D}}^{\mathcal{E}}=\underline{\mathcal{C}}^{\mathcal{E}}\cap\underline{\mathcal{D}}^{\mathcal{E}}, \overline{\mathcal{C}\cup\mathcal{D}}^{\mathcal{E}}=\overline{\mathcal{C}}^{\mathcal{E}}\cup\overline{\mathcal{D}}^{\mathcal{E}}$.

Proof. (i) Monotonicity of $\underline{\mathcal{C}}^{\mathcal{E}}$. Assume $\mathcal{C}\subseteq\mathcal{D}$ and let $r\in\underline{\mathcal{C}}^{\mathcal{E}}$. By definition of the meta-lower approximation,

$$[r]_{\mathcal{E}}\subseteq\mathcal{C}.$$

Since $\mathcal{C}\subseteq\mathcal{D}$, we obtain

$$[r]_{\mathcal{E}}\subseteq\mathcal{C}\subseteq\mathcal{D},$$

hence $[r]_{\mathcal{E}}\subseteq\mathcal{D}$, so $r\in\underline{\mathcal{D}}^{\mathcal{E}}$. Therefore $\underline{\mathcal{C}}^{\mathcal{E}}\subseteq\underline{\mathcal{D}}^{\mathcal{E}}$.

Monotonicity of $\overline{\mathcal{C}}^{\mathcal{E}}$. Again assume $\mathcal{C}\subseteq\mathcal{D}$ and let $r\in\overline{\mathcal{C}}^{\mathcal{E}}$. Then

$$[r]_{\mathcal{E}}\cap\mathcal{C}\neq\emptyset.$$

Since $\mathcal{C}\subseteq\mathcal{D}$, we have

$$[r]_{\mathcal{E}}\cap\mathcal{D} \supseteq [r]_{\mathcal{E}}\cap\mathcal{C}\neq\emptyset.$$

Thus $r\in\overline{\mathcal{D}}^{\mathcal{E}}$, and $\overline{\mathcal{C}}^{\mathcal{E}}\subseteq\overline{\mathcal{D}}^{\mathcal{E}}$.

(ii) Distributivity for intersection (meta-lower). We show the two inclusions separately.

($\subseteq$) Let $r\in\underline{\mathcal{C}\cap\mathcal{D}}^{\mathcal{E}}$. By definition,

$$[r]_{\mathcal{E}}\subseteq\mathcal{C}\cap\mathcal{D}.$$

Hence

$$[r]_{\mathcal{E}}\subseteq\mathcal{C} \quad\text{and}\quad [r]_{\mathcal{E}}\subseteq\mathcal{D},$$

so $r\in\underline{\mathcal{C}}^{\mathcal{E}}$ and $r\in\underline{\mathcal{D}}^{\mathcal{E}}$. Therefore

$$r\in\underline{\mathcal{C}}^{\mathcal{E}}\cap\underline{\mathcal{D}}^{\mathcal{E}},$$

giving

$$\underline{\mathcal{C} \cap \mathcal{D}}^{\mathcal{E}} \subseteq \underline{\mathcal{C}}^{\mathcal{E}} \cap \underline{\mathcal{D}}^{\mathcal{E}}.$$

(⊇) Let $r \in \underline{\mathcal{C}}^{\mathcal{E}} \cap \underline{\mathcal{D}}^{\mathcal{E}}$. Then

$$[r]_{\mathcal{E}} \subseteq \mathcal{C} \quad \text{and} \quad [r]_{\mathcal{E}} \subseteq \mathcal{D}.$$

Thus

$$[r]_{\mathcal{E}} \subseteq \mathcal{C} \cap \mathcal{D},$$

so $r \in \underline{\mathcal{C} \cap \mathcal{D}}^{\mathcal{E}}$. This yields

$$\underline{\mathcal{C}}^{\mathcal{E}} \cap \underline{\mathcal{D}}^{\mathcal{E}} \subseteq \underline{\mathcal{C} \cap \mathcal{D}}^{\mathcal{E}},$$

and therefore equality holds.

Distributivity for union (meta-upper). Again we show both inclusions.

(⊆) Let

$$r \in \overline{\mathcal{C}}^{\mathcal{E}} \cup \overline{\mathcal{D}}^{\mathcal{E}}$$

. If $r \in \overline{\mathcal{C}}^{\mathcal{E}}$, then

$$[r]_{\mathcal{E}} \cap \mathcal{C} \neq \emptyset.$$

But $\mathcal{C} \subseteq \mathcal{C} \cup \mathcal{D}$, so

$$[r]_{\mathcal{E}} \cap (\mathcal{C} \cup \mathcal{D}) \supseteq [r]_{\mathcal{E}} \cap \mathcal{C} \neq \emptyset,$$

hence $r \in \overline{\mathcal{C} \cup \mathcal{D}}^{\mathcal{E}}$. The same argument applies when $r \in \overline{\mathcal{D}}^{\mathcal{E}}$. Therefore

$$\overline{\mathcal{C}}^{\mathcal{E}} \cup \overline{\mathcal{D}}^{\mathcal{E}} \subseteq \overline{\mathcal{C} \cup \mathcal{D}}^{\mathcal{E}}.$$

(⊇) Let $r \in \overline{\mathcal{C} \cup \mathcal{D}}^{\mathcal{E}}$. Then

$$[r]_{\mathcal{E}} \cap (\mathcal{C} \cup \mathcal{D}) \neq \emptyset.$$

Thus there exists $s \in [r]_{\mathcal{E}}$ such that $s \in \mathcal{C} \cup \mathcal{D}$. Hence either $s \in \mathcal{C}$ or $s \in \mathcal{D}$. In the first case,

$$[r]_{\mathcal{E}} \cap \mathcal{C} \supseteq \{s\} \neq \emptyset,$$

so $r \in \overline{\mathcal{C}}^{\mathcal{E}}$. In the second case,

$$[r]_{\mathcal{E}} \cap \mathcal{D} \supseteq \{s\} \neq \emptyset,$$

so $r \in \overline{\mathcal{D}}^{\mathcal{E}}$. Therefore $r \in \overline{\mathcal{C}}^{\mathcal{E}} \cup \overline{\mathcal{D}}^{\mathcal{E}}$, and we obtain

$$\overline{\mathcal{C} \cup \mathcal{D}}^{\mathcal{E}} \subseteq \overline{\mathcal{C}}^{\mathcal{E}} \cup \overline{\mathcal{D}}^{\mathcal{E}}.$$

Combining both inclusions in each case proves (ii).

Theorem 15 (Idempotence of MetaRough approximations). For any $\mathcal{C} \subseteq \mathsf{Rough}(\mathrm{X}, \mathrm{R})$ and any meta-indiscernibility $\mathcal{E}$ on $\mathsf{Rough}(\mathrm{X}, \mathrm{R})$, the meta-lower and meta-upper operators are idempotent: $\underline{(\underline{\mathcal{C}}^{\mathcal{E}})}^{\mathcal{E}} = \underline{\mathcal{C}}^{\mathcal{E}}$, $\overline{(\overline{\mathcal{C}}^{\mathcal{E}})}^{\mathcal{E}} = \overline{\mathcal{C}}^{\mathcal{E}}$.

Proof. We first prove idempotence for the meta-lower approximation. Set

$$L(\mathcal{C}) := \underline{\mathcal{C}}^{\mathcal{E}} = \{ r \in \mathsf{Rough}(X, R) \mid [r]_{\mathcal{E}} \subseteq \mathcal{C} \}.$$

$(L(L(\mathcal{C})) \subseteq L(\mathcal{C}))$ Let $r \in L(L(\mathcal{C}))$, i.e.,

$$[r]_{\mathcal{E}} \subseteq L(\mathcal{C}).$$

Take any $s \in [r]_{\mathcal{E}}$. Then $s \in L(\mathcal{C})$, so

$$[s]_{\mathcal{E}} \subseteq \mathcal{C}.$$

But $s \in [r]_{\mathcal{E}}$ implies $[s]_{\mathcal{E}} = [r]_{\mathcal{E}}$, hence

$$[r]_{\mathcal{E}} \subseteq \mathcal{C}.$$

Therefore $r \in L(\mathcal{C})$, and we have

$$L(L(\mathcal{C})) \subseteq L(\mathcal{C}).$$

$(L(\mathcal{C}) \subseteq L(L(\mathcal{C})))$ Let $r \in L(\mathcal{C})$, so $[r]_{\mathcal{E}} \subseteq \mathcal{C}$. For any $s \in [r]_{\mathcal{E}}$ we have $[s]_{\mathcal{E}} = [r]_{\mathcal{E}}$ (by the definition of an equivalence class), so

$$[s]_{\mathcal{E}} = [r]_{\mathcal{E}} \subseteq \mathcal{C}.$$

Thus $s \in L(\mathcal{C})$ for every $s \in [r]_{\mathcal{E}}$, which implies

$$[r]_{\mathcal{E}} \subseteq L(\mathcal{C}).$$

Therefore $r \in L(L(\mathcal{C}))$ and

$$L(\mathcal{C}) \subseteq L(L(\mathcal{C})).$$

Combining the two inclusions yields $L(L(\mathcal{C})) = L(\mathcal{C})$, i.e.,

$$\underline{(\underline{\mathcal{C}}^{\mathcal{E}})}^{\mathcal{E}} = \underline{\mathcal{C}}^{\mathcal{E}}.$$

We now prove idempotence for the meta-upper approximation. Set

$$U(\mathcal{C}) := \overline{\mathcal{C}}^{\mathcal{E}} = \{ r \in \mathsf{Rough}(X, R) \mid [r]_{\mathcal{E}} \cap \mathcal{C} \neq \emptyset \}.$$

$(U(U(\mathcal{C})) \subseteq U(\mathcal{C}))$ Let $r \in U(U(\mathcal{C}))$. Then

$$[r]_{\mathcal{E}} \cap U(\mathcal{C}) \neq \emptyset.$$

Hence there exists $s \in [r]_{\mathcal{E}} \cap U(\mathcal{C})$. From $s \in U(\mathcal{C})$ we get

$$[s]_{\mathcal{E}} \cap \mathcal{C} \neq \emptyset.$$

Since $s \in [r]_{\mathcal{E}}$, we have $[s]_{\mathcal{E}} = [r]_{\mathcal{E}}$, so

$$[r]_{\mathcal{E}} \cap \mathcal{C} = [s]_{\mathcal{E}} \cap \mathcal{C} \neq \emptyset,$$

which shows $r \in U(\mathcal{C})$. Therefore $U(U(\mathcal{C})) \subseteq U(\mathcal{C})$.

$(U(\mathcal{C}) \subseteq U(U(\mathcal{C})))$ Let $r \in U(\mathcal{C})$, so

$$[r]_{\mathcal{E}} \cap \mathcal{C} \neq \emptyset.$$

In particular, $r \in [r]_{\mathcal{E}}$ and $r \in U(\mathcal{C})$, hence

$$r \in [r]_{\mathcal{E}} \cap U(\mathcal{C}).$$

This shows

$$[r]_{\mathcal{E}} \cap U(\mathcal{C}) \neq \emptyset,$$

so $r \in U(U(\mathcal{C}))$. Thus $U(\mathcal{C}) \subseteq U(U(\mathcal{C}))$.

Combining both inclusions gives $U(U(\mathcal{C})) = U(\mathcal{C})$, i.e.,

$$\overline{\left(\overline{\mathcal{C}}^{\mathcal{E}}\right)}^{\mathcal{E}} = \overline{\mathcal{C}}^{\mathcal{E}}.$$

Theorem 16 (MetaRough duality via complement). Let $\mathcal{E}$ be a meta-indiscernibility on $\mathsf{Rough}(X, R)$ and let $\mathcal{C} \subseteq \mathsf{Rough}(X, R)$. Denote the complement of $\mathcal{C}$ in $\mathsf{Rough}(X, R)$ by $\mathcal{C}^{c} := \mathsf{Rough}(X, R) \setminus \mathcal{C}$. Then:

1. $(\underline{\mathcal{C}}^{\mathcal{E}})^{c} = \overline{\mathcal{C}^{c}}^{\mathcal{E}}$.
2. $(\overline{\mathcal{C}}^{\mathcal{E}})^{c} = \underline{\mathcal{C}^{c}}^{\mathcal{E}}$.

Proof. (i) We show pointwise equivalence

$$r \in (\underline{\mathcal{C}}^{\mathcal{E}})^{c} \quad \Leftrightarrow \quad r \in \overline{\mathcal{C}^{c}}^{\mathcal{E}}.$$

(⇒) Suppose

$$r \in (\underline{\mathcal{C}}^{\mathcal{E}})^{c}.$$

This means $r \notin \underline{\mathcal{C}}^{\mathcal{E}}$, i.e.,

$$[r]_{\mathcal{E}} \nsubseteq \mathcal{C}.$$

Therefore there exists $s \in [r]_{\mathcal{E}}$ with $s \notin \mathcal{C}$. Equivalently, $s \in [r]_{\mathcal{E}} \cap \mathcal{C}^{c}$. Thus

$$[r]_{\mathcal{E}} \cap \mathcal{C}^{c} \neq \emptyset,$$

so $r \in \overline{\mathcal{C}^{c}}^{\mathcal{E}}$ by the definition of meta-upper approximation.

(⇐) Conversely, suppose $r \in \overline{\mathcal{C}^{c}}^{\mathcal{E}}$. Then

$$[r]_{\mathcal{E}} \cap \mathcal{C}^{c} \neq \emptyset.$$

Hence there exists $s \in [r]_{\mathcal{E}}$ with $s \in \mathcal{C}^{c}$, i.e., $s \notin \mathcal{C}$. This implies that $[r]_{\mathcal{E}}$ is *not* a subset of $\mathcal{C}$, so

$$[r]_{\mathcal{E}} \nsubseteq \mathcal{C},$$

and therefore $r \notin \underline{\mathcal{C}}^{\mathcal{E}}$. Thus

$$r \in (\underline{\mathcal{C}}^{\mathcal{E}})^{c}.$$

Combining the two directions establishes (i).

(ii) We again show pointwise equivalence

$$r \in (\overline{\mathcal{C}}^{\mathcal{E}})^{c} \quad \Leftrightarrow \quad r \in \underline{\mathcal{C}^{c}}^{\mathcal{E}}.$$

(⇒) Assume

$$r \in (\overline{\mathcal{C}}^{\mathcal{E}})^{c},$$

i.e., $r \notin \overline{\mathcal{C}}^{\mathcal{E}}$. Thus

$$[r]_{\mathcal{E}} \cap \mathcal{C} = \emptyset.$$

Hence $[r]_{\mathcal{E}} \subseteq \mathcal{C}^{c}$, because no element of $[r]_{\mathcal{E}}$ lies in $\mathcal{C}$. Therefore

$$[r]_{\mathcal{E}} \subseteq \mathcal{C}^{c},$$

which means $r \in \underline{\mathcal{C}^{c}}^{\mathcal{E}}$.

(⇐) Conversely, suppose $r \in \underline{\mathcal{C}^{c}}^{\mathcal{E}}$. Then

$$[r]_{\mathcal{E}} \subseteq \mathcal{C}^{c},$$

so no element of $[r]_{\mathcal{E}}$ lies in $\mathcal{C}$. Thus

$$[r]_{\mathcal{E}} \cap \mathcal{C} = \emptyset,$$

which means $r \notin \overline{\mathcal{C}}^{\mathcal{E}}$ and hence

$$r \in (\overline{\mathcal{C}}^{\mathcal{E}})^{c}.$$

This completes the proof of (ii).

Theorem 17 (Characterization of MetaRough fixed points). Let $\mathcal{E}$ be a meta-indiscernibility relation on $\mathsf{Rough}(X, R)$ and let $\mathcal{C} \subseteq \mathsf{Rough}(X, R)$. The following are equivalent:

1. $\mathcal{C}$is a union of $\mathcal{E}$-equivalence classes, i.e., $\forall r \in \mathcal{C}$: $[r]_{\mathcal{E}} \subseteq \mathcal{C}$.
2. $\mathcal{C} = \underline{\mathcal{C}}^{\mathcal{E}}$.
3. $\mathcal{C} = \overline{\mathcal{C}}^{\mathcal{E}}$.

In particular, $\mathcal{C}$ is meta-crisp (fixed by both meta-lower and meta-upper operators) if and only if it is saturated with respect to $\mathcal{E}$.

Proof. (a)⇒(b). Assume that $\mathcal{C}$ is a union of $\mathcal{E}$-classes, i.e., whenever $r \in \mathcal{C}$ then $[r]_{\mathcal{E}} \subseteq \mathcal{C}$. By Definition 18, $r \in \underline{\mathcal{C}}^{\mathcal{E}}$ if and only if $[r]_{\mathcal{E}} \subseteq \mathcal{C}$. Thus each $r \in \mathcal{C}$ satisfies $r \in \underline{\mathcal{C}}^{\mathcal{E}}$, and hence

$$\mathcal{C} \subseteq \underline{\mathcal{C}}^{\mathcal{E}}.$$

Combined with the meta-sandwich property $\underline{\mathcal{C}}^{\mathcal{E}} \subseteq \mathcal{C}$ (Proposition), we obtain equality:

$$\mathcal{C} = \underline{\mathcal{C}}^{\mathcal{E}}.$$

(b)⇒(a). Assume $\mathcal{C} = \underline{\mathcal{C}}^{\mathcal{E}}$ and take any $r \in \mathcal{C}$. Since $\mathcal{C} = \underline{\mathcal{C}}^{\mathcal{E}}$,

$$r \in \underline{\mathcal{C}}^{\mathcal{E}},$$

so by definition $[r]_{\mathcal{E}} \subseteq \mathcal{C}$. This is exactly the condition that $\mathcal{C}$ is a union of $\mathcal{E}$-classes.

(a)⇒(c). Assume again that $\mathcal{C}$ is a union of $\mathcal{E}$-classes. Then for any $r \in \mathcal{C}$ and any $s \in [r]_{\mathcal{E}}$, we have $s \in \mathcal{C}$ because the whole class $[r]_{\mathcal{E}}$ lies in $\mathcal{C}$. Consequently, for such r,

$$[r]_{\mathcal{E}} \cap \mathcal{C} \supseteq \{r\} \neq \emptyset,$$

which implies $r \in \overline{\mathcal{C}}^{\mathcal{E}}$. Therefore

$$\mathcal{C} \subseteq \overline{\mathcal{C}}^{\mathcal{E}}.$$

Combining with the meta-sandwich property $\mathcal{C} \subseteq \overline{\mathcal{C}}^{\mathcal{E}}$ and $\underline{\mathcal{C}}^{\mathcal{E}} \subseteq \mathcal{C}$, we already have the inclusion $\mathcal{C} \subseteq \overline{\mathcal{C}}^{\mathcal{E}}$ and thus obtain

$$\mathcal{C} = \overline{\mathcal{C}}^{\mathcal{E}}.$$

(c)⇒(a). Assume $\mathcal{C} = \overline{\mathcal{C}}^{\mathcal{E}}$ and let $r \in \mathcal{C}$. Since $r \in \overline{\mathcal{C}}^{\mathcal{E}}$, we have

$$[r]_{\mathcal{E}} \cap \mathcal{C} \neq \emptyset.$$

Let $s \in [r]_{\mathcal{E}} \cap \mathcal{C}$, so $s \in \mathcal{C}$ and $s \sim_{\mathcal{E}} r$. Now consider any $t \in [r]_{\mathcal{E}}$. Since $t \sim_{\mathcal{E}} r$ and $s \sim_{\mathcal{E}} r$, we also have $t \sim_{\mathcal{E}} s$, so $t \in [s]_{\mathcal{E}}$. We show $t \in \mathcal{C}$.

By the assumption $\mathcal{C} = \overline{\mathcal{C}}^{\mathcal{E}}$,

$$t \in \mathcal{C} \quad \Leftrightarrow \quad t \in \overline{\mathcal{C}}^{\mathcal{E}}$$
$$\Leftrightarrow \quad [t]_{\mathcal{E}} \cap \mathcal{C} \neq \emptyset.$$

But $s \in [t]_{\mathcal{E}}$ (since $t \sim_{\mathcal{E}} s$) and $s \in \mathcal{C}$, so

$$[t]_{\mathcal{E}} \cap \mathcal{C} \supseteq \{s\} \neq \emptyset.$$

Hence $t \in \mathcal{C}$. Therefore $[r]_{\mathcal{E}} \subseteq \mathcal{C}$ for each $r \in \mathcal{C}$, which shows $\mathcal{C}$ is a union of $\mathcal{E}$-classes.

We have shown (a) ⇔ (b) and (a) ⇔ (c), so all three conditions are equivalent. In particular, $\mathcal{C}$ is fixed by both $\underline{\cdot}^{\mathcal{E}}$ and $\overline{\cdot}^{\mathcal{E}}$ if and only if $\mathcal{C}$ is saturated with respect to $\mathcal{E}$.

Corollary 2 (MetaRough interior and closure operators). For any fixed meta-indiscernibility $\mathcal{E}$ on $\mathsf{Rough}(X, R)$, the operator $U_{\mathcal{E}}: \mathcal{P}(\mathsf{Rough}(X, R)) \to \mathcal{P}(\mathsf{Rough}(X, R))$, $U_{\mathcal{E}}(\mathcal{C}) := \overline{\mathcal{C}}^{\mathcal{E}}$, is a closure operator (extensive, monotone, idempotent), and $L_{\mathcal{E}}: \mathcal{P}(\mathsf{Rough}(X, R)) \to \mathcal{P}(\mathsf{Rough}(X, R))$, $L_{\mathcal{E}}(\mathcal{C}) := \underline{\mathcal{C}}^{\mathcal{E}}$, is an interior operator (contractive, monotone, idempotent) on the powerset $\mathcal{P}(\mathsf{Rough}(X, R))$.

Proof. Extensivity of $U_{\mathcal{E}}$ and contractivity of $L_{\mathcal{E}}$ follow from the meta-sandwich property (Proposition 1):

$$L_{\mathcal{E}}(\mathcal{C}) = \underline{\mathcal{C}}^{\mathcal{E}} \subseteq \mathcal{C}$$
$$\subseteq \overline{\mathcal{C}}^{\mathcal{E}} = U_{\mathcal{E}}(\mathcal{C}).$$

Monotonicity of both operators is given by Theorem 14. Idempotence is exactly Theorem 15. Thus $U_{\mathcal{E}}$ is a closure operator and $L_{\mathcal{E}}$ is an interior operator on $\mathcal{P}(\mathsf{Rough}(X, R))$.

Results

Iterated Meta-Objects

Throughout, let $Y \neq \emptyset$ (base domain for fuzzy sets), $X \neq \emptyset$ (base domain for neutrosophic sets), $U \neq \emptyset$ (universe for soft sets) with parameter set $S \neq \emptyset$, and (X, R) a Pawlak approximation space.

Iterated MetaFuzzy sets

Iterated MetaFuzzy Sets organize fuzzy evaluations into a hierarchy of levels: at level 0 we have ordinary fuzzy sets on Y, at level 1 fuzzy evaluations of those fuzzy sets, at level 2 fuzzy evaluations of level-1 evaluations, and so on. This yields a tower of meta-level membership analyses.

Definition 19 (Hierarchy of fuzzy universes). Let Y be a nonempty set. Define inductively the sequence of carrier sets $(\mathsf{Fuz}^{\langle t\rangle}(Y))_{t\geq 0}$ by

$$\mathsf{Fuz}^{\langle 0\rangle}(Y) := \mathsf{Fuz}(Y) = [0,1]^{Y},$$
$$\mathsf{Fuz}^{\langle t+1\rangle}(Y)$$
$$:= [0,1]^{\mathsf{Fuz}^{\langle t\rangle}(Y)} \qquad (t \geq 0).$$

Thus a level-t fuzzy object is a function

$$\mu^{(t)} \in \mathsf{Fuz}^{\langle t\rangle}(Y)$$
$$\Leftrightarrow \quad \mu^{(t)}: \mathsf{Fuz}^{\langle t-1\rangle}(Y) \to [0,1] \quad (t \geq 1),$$

assigning grades in $[0,1]$ to level-$(t-1)$ fuzzy objects.

Definition 20 (Iterated MetaFuzzy Set of depth t). Let $t \geq 1$. An Iterated MetaFuzzy Set (IMF) of depth t on Y is an element

$$\mu^{(t)} \in \mathsf{Fuz}^{\langle t\rangle}(Y) = [0,1]^{\mathsf{Fuz}^{\langle t-1\rangle}(Y)}.$$

Equivalently, it is a map

$$\mu^{(t)}: \mathsf{Fuz}^{\langle t-1\rangle}(Y) \to [0,1],$$

which assigns a membership grade in [0,1] to each level-$(t-1)$ fuzzy object.

A binary iterated MetaFuzzy relation of depth t based on $\mu^{(t)}$ is any map

$$\Delta^{(t)}: \mathsf{Fuz}^{\langle t-1\rangle}(Y) \times \mathsf{Fuz}^{\langle t-1\rangle}(Y) \to [0,1]$$

such that the admissibility constraint

$$\Delta^{(t)}(a,b) \leq \min\{\mu^{(t)}(a), \mu^{(t)}(b)\} \quad (\forall a, b \in \mathsf{Fuz}^{\langle t-1\rangle}(Y))$$

holds. Hence $\Delta^{(t)}$ is never "more confident" about a pair (a,b) than the individual meta-memberships of a and b.

Example 8 (Iterated MetaFuzzy Set (depth 2): store staffing from weekly traffic profiles). Let Y = {Mon,Tue,Wed} be three trading days.

Level 0 (ordinary fuzzy sets). A level-0 fuzzy set $\mu \in \mathsf{Fuz}^{\langle 0\rangle}(Y) = \mathsf{Fuz}(Y) = [0,1]^Y$ encodes, for each $y \in Y$, the degree of "high customer traffic" on day y.

Consider two weeks:

$$\mu^{(A)}(\text{Mon,Tue,Wed}) = (0.2,\ 0.8,\ 0.6),$$
$$\mu^{(B)}(\text{Mon,Tue,Wed}) = (0.9,\ 0.7,\ 0.3).$$

Level 1 (MetaFuzzy predicate on week-profiles). Elements of $\mathsf{Fuz}^{\langle 1\rangle}(Y) = [0,1]^{\mathsf{Fuz}(Y)}$ are functions

$$\Lambda: \mathsf{Fuz}(Y) \to [0,1],$$

assigning to each week-profile μ a degree describing some global property of that week.

Define a level-1 fuzzy predicate

$$\Lambda_{\text{avg}} \in \mathsf{Fuz}^{\langle 1\rangle}(Y),$$

$$\Lambda_{\text{avg}}(\mu) := \frac{1}{|Y|}\sum_{y\in Y} \mu(y) = \frac{\mu(\text{Mon}) + \mu(\text{Tue}) + \mu(\text{Wed})}{3}.$$

It measures how strongly a given week-profile exhibits "high traffic on average".

For our two weeks,

$$\Lambda_{\text{avg}}(\mu^{(A)}) = \frac{0.2+0.8+0.6}{3} = \frac{1.6}{3} = \frac{8}{15} \approx 0.533\overline{3},$$

$$\Lambda_{\text{avg}}(\mu^{(B)}) = \frac{0.9+0.7+0.3}{3} = \frac{1.9}{3} = \frac{19}{30} \approx 0.633\overline{3}.$$

Level 2 (Iterated MetaFuzzy Set). An Iterated MetaFuzzy Set of depth 2 is a map

$$\mathcal{M}^{(2)}: \mathsf{Fuz}^{\langle 1\rangle}(Y) \to [0,1].$$

Fix a finite reference set of week-profiles

$$\Sigma := \{\mu^{(A)}, \mu^{(B)}\} \subseteq \mathsf{Fuz}(Y),$$

and define

$$\mathcal{M}^{(2)}(\Lambda) := \frac{1}{|\Sigma|}\sum_{\mu\in\Sigma} \Lambda(\mu) = \frac{\Lambda(\mu^{(A)}) + \Lambda(\mu^{(B)})}{2} \qquad \left(\Lambda \in \mathsf{Fuz}^{\langle 1\rangle}(Y)\right).$$

This assigns to each level-1 predicate Λ a meta-score reflecting how strongly Λ is supported across the two reference weeks.

For the concrete predicate Λ_{avg} above,

$$\mathcal{M}^{(2)}(\Lambda_{\text{avg}}) = \frac{\Lambda_{\text{avg}}(\mu^{(A)}) + \Lambda_{\text{avg}}(\mu^{(B)})}{2} = \frac{\frac{8}{15}+\frac{19}{30}}{2} = \frac{\frac{16}{30}+\frac{19}{30}}{2} = \frac{\frac{35}{30}}{2} = \frac{35}{60} = \frac{7}{12} \approx 0.583\overline{3}.$$

Interpretation. Level 0 captures daily congestion. Level 1 summarizes each week-profile into a single fuzzy degree of "high average traffic". Level 2 aggregates how strong such predicates (here Λ_{avg}) are across multiple weeks, yielding a meta-level score that can guide decisions such as whether the store should adopt a permanently increased staffing policy rather than reacting week by week.

Theorem 18 (Generalization of MetaFuzzy Sets and Iterated MetaStructure). (a) For $t = 1$, Iterated MetaFuzzy Sets coincide with MetaFuzzy Sets (Definition 4): $\mathrm{Fuz}^{\langle 1\rangle}(Y) = [0,1]^{\mathrm{Fuz}(Y)} \Leftrightarrow$ MetaFuzzy Sets on Y.

(b) For every $t \geq 1$, the pair $\left(\mathrm{Fuz}^{\langle t\rangle}(Y),\ \Phi_{\mathrm{ev}}^{(t)}\right)$, $\Phi_{\mathrm{ev}}^{(t)}: \mathrm{Fuz}^{\langle t\rangle}(Y) \times \mathrm{Fuz}^{\langle t-1\rangle}(Y) \to [0,1]$, $\Phi_{\mathrm{ev}}^{(t)}\left(\mu^{(t)}, a\right) := \mu^{(t)}(a)$, forms an Iterated MetaStructure of depth t in the sense of Definition 2.

Proof. (a) By Definition 19,

$$\mathrm{Fuz}^{\langle 1\rangle}(Y)$$
$$= [0,1]^{\mathrm{Fuz}^{\langle 0\rangle}(Y)} = [0,1]^{\mathrm{Fuz}(Y)}.$$

By Definition 4, a MetaFuzzy Set on Y is precisely a map $\mu^{\sharp}: \mathrm{Fuz}(Y) \to [0,1]$. Hence depth-1 Iterated MetaFuzzy Sets are exactly MetaFuzzy Sets.

(b) Let $U^{(t)} := \mathrm{Fuz}^{\langle t\rangle}(Y)$ for all $t \geq 0$, so that $U^{(0)} = \mathrm{Fuz}(Y)$ and $U^{(t)} = [0,1]^{U^{(t-1)}}$ for $t \geq 1$. By construction, a level-t object $\mu^{(t)} \in U^{(t)}$ is a function

$$\mu^{(t)}: U^{(t-1)} \to [0,1],$$

and the evaluation map is

$$\Phi_{\mathrm{ev}}^{(t)}\left(\mu^{(t)}, a\right) = \mu^{(t)}(a)$$
$$\left(\mu^{(t)} \in U^{(t)},\ a \in U^{(t-1)}\right).$$

By the axioms of Definition 4, an Iterated MetaStructure of depth t requires:

4. a sequence of universes $(U^{(0)}, U^{(1)}, \ldots, U^{(t)})$,
5. level-wise isomorphisms $\beta: U^{(k)} \to U^{(k)}$ induced by bijections on lower levels, and
6. evaluation maps $\Phi_{\mathrm{ev}}^{(k)}$ that are invariant under these isomorphisms.

The sequence $(U^{(k)})_{k \leq t}$ is given by the fuzzy hierarchy.

For invariance, let $k \geq 1$ and let $\beta: U^{(k-1)} \to U^{(k-1)}$ be a bijection arising from isomorphisms at lower levels (by the inductive part of Definition 4). It induces a bijection

$$\beta^*: U^{(k)} = [0,1]^{U^{(k-1)}} \to [0,1]^{U^{(k-1)}} = U^{(k)},$$
$$(\beta^* \mu^{(k)}) := \mu^{(k)} \circ \beta^{-1}.$$

Then for all $\mu^{(k)} \in U^{(k)}$ and $a \in U^{(k-1)}$,

$$\Phi_{\mathrm{ev}}^{(k)}\left(\beta^* \mu^{(k)},\ \beta(a)\right)$$
$$= (\mu^{(k)} \circ \beta^{-1})(\beta(a))$$
$$= \mu^{(k)}(a) = \Phi_{\mathrm{ev}}^{(k)}\left(\mu^{(k)}, a\right).$$

Hence $\Phi_{\mathrm{ev}}^{(k)}$ is isomorphism-invariant, as required.

Taking $k = t$ shows that $(U^{(t)}, \Phi_{\mathrm{ev}}^{(t)})$ satisfies all Iterated MetaStructure axioms at depth t. Therefore

$$\left(\mathrm{Fuz}^{\langle t\rangle}(Y), \Phi_{\mathrm{ev}}^{(t)}\right)$$

is an Iterated MetaStructure of depth t.

Theorem 19 (Levelwise order and completeness of the fuzzy hierarchy). Let Y be a nonempty set and let $(\mathrm{Fuz}^{\langle t\rangle}(Y))_{t \geq 0}$ be as in Definition 19. For each $t \geq 0$:

1. The set $\mathrm{Fuz}^{\langle t\rangle}(Y)$ becomes a partially ordered set under the pointwise fuzzy order $\mu^{(t)} \leq \nu^{(t)} \Leftrightarrow \mu^{(t)}(a) \leq \nu^{(t)}(a)\quad \forall a \in \mathrm{Fuz}^{\langle t-1\rangle}(Y)$ for $t \geq 1$, and for $t = 0$ the usual order on fuzzy sets $\mathrm{Fuz}^{\langle 0\rangle}(Y) = \mathrm{Fuz}(Y) = [0,1]^{Y}$.
2. Equipped with this order, $\mathrm{Fuz}^{\langle t\rangle}(Y)$ is a complete lattice: every subfamily has a pointwise supremum and infimum.

Proof. We treat the case $t \geq 1$; the case $t = 0$ is identical with $\mathrm{Fuz}^{\langle -1\rangle}(Y)$ replaced by Y.

Set

$$A_t := \mathrm{Fuz}^{\langle t-1\rangle}(Y),$$
$$\mathrm{Fuz}^{\langle t\rangle}(Y) = [0,1]^{A_t}.$$

Thus a level-t object $\mu^{(t)}$ is a function $\mu^{(t)}: A_t \to [0,1]$.

(i) Partial order. Reflexivity: For any $\mu^{(t)}$ and any $a \in A_t$,

$$\mu^{(t)}(a) \leq \mu^{(t)}(a),$$

hence $\mu^{(t)} \leq \mu^{(t)}$.

Antisymmetry: Suppose $\mu^{(t)} \leq \nu^{(t)}$ and $\nu^{(t)} \leq \mu^{(t)}$. Then for all $a \in A_t$,

$$\mu^{(t)}(a) \leq \nu^{(t)}(a) \quad \text{and}$$
$$\nu^{(t)}(a) \leq \mu^{(t)}(a),$$

so $\mu^{(t)}(a) = \nu^{(t)}(a)$. Thus $\mu^{(t)} = \nu^{(t)}$ as functions.

Transitivity: If $\mu^{(t)} \le \nu^{(t)}$ and $\nu^{(t)} \le \Lambda^{(t)}$ then for each $a \in A_t$,

$$\mu^{(t)}(a) \le \nu^{(t)}(a) \le \Lambda^{(t)}(a),$$

so $\mu^{(t)}(a) \le \Lambda^{(t)}(a)$, whence $\mu^{(t)} \le \Lambda^{(t)}$.

Therefore the pointwise definition yields a partial order.

(ii) Completeness. Let $\{\mu_i^{(t)}\}_{i\in I} \subseteq \mathsf{Fuz}^{\langle t\rangle}(Y)$ be an arbitrary family indexed by I (possibly infinite). For each $a \in A_t$ define

$$\left(\sup_{i\in I}\mu_i^{(t)}\right)(a) := \sup_{i\in I}\mu_i^{(t)}(a),$$

$$\left(\inf_{i\in I}\mu_i^{(t)}\right)(a) := \inf_{i\in I}\mu_i^{(t)}(a),$$

where the supremum and infimum are taken in [0,1]. Since every subset of [0,1] admits a supremum and an infimum in [0,1], these functions are well-defined and satisfy

$$\sup_{i\in I}\mu_i^{(t)},\ \inf_{i\in I}\mu_i^{(t)} \in [0,1]^{A_t} = \mathsf{Fuz}^{\langle t\rangle}(Y).$$

We now check the universal property for the supremum; the argument for the infimum is dual.

Upper bound: For each fixed $i \in I$ and any $a \in A_t$,

$$\mu_i^{(t)}(a) \ \le\ \sup_{j\in I}\mu_j^{(t)}(a) = \left(\sup_{j\in I}\mu_j^{(t)}\right)(a),$$

so $\mu_i^{(t)} \le \sup_{j\in I}\mu_j^{(t)}$. Thus $\sup_{j\in I}\mu_j^{(t)}$ is an upper bound of $\{\mu_i^{(t)}\}_{i\in I}$.

Leastness: Let $\Lambda^{(t)}$ be any upper bound, so that for all $i \in I$ and $a \in A_t$,

$$\mu_i^{(t)}(a) \ \le\ \Lambda^{(t)}(a).$$

Taking the supremum over $i \in I$ yields

$$\sup_{i\in I}\mu_i^{(t)}(a) \ \le\ \Lambda^{(t)}(a),$$

i.e.,

$$\left(\sup_{i\in I}\mu_i^{(t)}\right)(a) \ \le\ \Lambda^{(t)}(a) \quad (\forall a \in A_t),$$

hence $\sup_{i\in I}\mu_i^{(t)} \le \Lambda^{(t)}$.

Therefore $\sup_{i\in I}\mu_i^{(t)}$ is the least upper bound (supremum) of the family $\{\mu_i^{(t)}\}_{i\in I}$. The dual argument shows that $\inf_{i\in I}\mu_i^{(t)}$ is the greatest lower bound (infimum). Thus $\mathsf{Fuz}^{\langle t\rangle}(Y)$ is a complete lattice for each t.

Theorem 20 (α-cut representation and reconstruction at depth t). Fix $t \ge 1$ and set $A_t := \mathsf{Fuz}^{\langle t-1\rangle}(Y)$, $\mathsf{Fuz}^{\langle t\rangle}(Y) = [0,1]^{A_t}$.

7. For every Iterated MetaFuzzy Set $\mu^{(t)} \in \mathsf{Fuz}^{\langle t\rangle}(Y)$, define for each $\alpha \in [0,1]$ the meta-α-cut $\mathcal{L}_{\mu^{(t)}}(\alpha) := \{ a \in A_t \mid \mu^{(t)}(a) \ge \alpha \}$. Then:
   - For $0 \le \alpha \le \beta \le 1$ we have the nesting $\mathcal{L}_{\mu^{(t)}}(\beta) \subseteq \mathcal{L}_{\mu^{(t)}}(\alpha)$.
   - For every $a \in A_t$, $\mu^{(t)}(a) = \sup\{\alpha \in [0,1] \mid a \in \mathcal{L}_{\mu^{(t)}}(\alpha)\}$.
8. Conversely, let $(\mathcal{C}_\alpha)_{\alpha\in[0,1]}$ be a family of subsets $\mathcal{C}_\alpha \subseteq A_t$ satisfying the nesting condition $0 \le \alpha \le \beta \le 1 \Rightarrow \mathcal{C}_\beta \subseteq \mathcal{C}_\alpha$. Define $\mu^{(t)}: A_t \to [0,1]$ by $\mu^{(t)}(a) := \sup\{\alpha \in [0,1] \mid a \in \mathcal{C}_\alpha\}$, where by convention $\sup\emptyset := 0$. Then $\mu^{(t)} \in \mathsf{Fuz}^{\langle t\rangle}(Y)$, and its meta-α-cuts coincide with the given family: $\mathcal{L}_{\mu^{(t)}}(\alpha) = \mathcal{C}_\alpha$ $(\forall \alpha \in [0,1])$. In particular, every nested system of crisp meta-level sets comes from a unique Iterated MetaFuzzy Set.

Proof. (i) Nesting. Let $0 \le \alpha \le \beta \le 1$ and take any $a \in \mathcal{L}_{\mu^{(t)}}(\beta)$. Then by definition,

$$\mu^{(t)}(a) \ \ge\ \beta.$$

Since $\beta \ge \alpha$, we also have $\mu^{(t)}(a) \ge \alpha$, which means $a \in \mathcal{L}_{\mu^{(t)}}(\alpha)$. Therefore

$$\mathcal{L}_{\mu^{(t)}}(\beta) \subseteq \mathcal{L}_{\mu^{(t)}}(\alpha).$$

Representation formula. Fix $a \in A_t$ and set $v := \mu^{(t)}(a) \in [0,1]$. Then

$$a \in \mathcal{L}_{\mu^{(t)}}(\alpha) \iff \mu^{(t)}(a) \ge \alpha \iff v \ge \alpha.$$

Hence

$$\{\alpha \in [0,1] \mid a \in \mathcal{L}_{\mu^{(t)}}(\alpha)\} = \{\alpha \in [0,1] \mid v \ge \alpha\} = [0,v].$$

The supremum of $[0,v]$ in [0,1] is v, so

$$\sup\{\alpha \in [0,1] \mid a \in \mathcal{L}_{\mu^{(t)}}(\alpha)\}$$
$$= \sup[0, v] = v = \mu^{(t)}(a).$$

(ii) Reconstruction from nested families. Let $(\mathcal{C}_\alpha)_{\alpha\in[0,1]}$ satisfy the nesting condition and define $\mu^{(t)}$ as in the statement. By construction, $\mu^{(t)}(a) \in [0,1]$ for every $a \in A_t$ (as a supremum of a subset of $[0,1]$), so $\mu^{(t)} \in [0,1]^{A_t} = \mathsf{Fuz}^{\langle t\rangle}(Y)$.

We show $\mathcal{L}_{\mu^{(t)}}(\alpha) = \mathcal{C}_\alpha$.

($\subseteq$) Let $a \in \mathcal{L}_{\mu^{(t)}}(\alpha)$. Then $\mu^{(t)}(a) \ge \alpha$, i.e.,

$$\sup\{\gamma \in [0,1] \mid a \in \mathcal{C}_\gamma\} \ \ge\ \alpha.$$

Thus there exists some $\gamma \in [0,1]$ with

$$\gamma \ge \alpha \quad \text{and} \quad a \in \mathcal{C}_\gamma.$$

By the nesting assumption and $\gamma \ge \alpha$, we have

$$\mathcal{C}_\gamma \subseteq \mathcal{C}_\alpha,$$

so $a \in \mathcal{C}_\alpha$. Therefore $a \in \mathcal{C}_\alpha$, and

$$\mathcal{L}_{\mu^{(t)}}(\alpha) \subseteq \mathcal{C}_\alpha.$$

($\supseteq$) Conversely, let $a \in \mathcal{C}_\alpha$. Then α is one of the candidates in the defining supremum:

$$\alpha \in \{\gamma \in [0,1] \mid a \in \mathcal{C}_\gamma\}.$$

Hence

$$\mu^{(t)}(a) = \sup\{\gamma \in [0,1] \mid a \in \mathcal{C}_\gamma\} \ \ge\ \alpha,$$

so $a \in \mathcal{L}_{\mu^{(t)}}(\alpha)$. Thus $\mathcal{C}_\alpha \subseteq \mathcal{L}_{\mu^{(t)}}(\alpha)$.

Combining both inclusions yields

$$\mathcal{L}_{\mu^{(t)}}(\alpha) = \mathcal{C}_\alpha \qquad (\forall \alpha \in [0,1]),$$

and the reconstruction is complete.

Theorem 21 (Compatibility of iterated MetaFuzzy relations with α-cuts). Fix $t \ge 1$ and set $A_t := \mathsf{Fuz}^{\langle t-1\rangle}(Y)$. Let $\mu^{(t)} \in \mathsf{Fuz}^{\langle t\rangle}(Y)$ and let $\Delta^{(t)}: A_t \times A_t \to [0,1]$ be a binary iterated MetaFuzzy relation of depth t based on $\mu^{(t)}$ in the sense of Definition 20, i.e., $\Delta^{(t)}(a,b) \ \le\ \min\{\mu^{(t)}(a), \mu^{(t)}(b)\} \quad \forall\, a,b \in A_t$.

For each $\alpha \in [0,1]$ define the meta-α-cut $\mathcal{C}_\alpha := \{a \in A_t \mid \mu^{(t)}(a) \ge \alpha\}$ and the relation α-cut $\mathcal{R}_\alpha := \{(a,b) \in A_t \times A_t \mid \Delta^{(t)}(a,b) \ge \alpha\}$. Then:

1. For every $\alpha \in [0,1]$, $\mathcal{R}_\alpha \subseteq \mathcal{C}_\alpha \times \mathcal{C}_\alpha$. In particular, each $\mathcal{R}_\alpha$ is a crisp binary relation on the meta-α-cut carrier $\mathcal{C}_\alpha$.
2. The families $(\mathcal{C}_\alpha)_{\alpha\in[0,1]}$ and $(\mathcal{R}_\alpha)_{\alpha\in[0,1]}$ are nested: for $0 \le \alpha \le \beta \le 1$, $\mathcal{C}_\beta \subseteq \mathcal{C}_\alpha$ and $\mathcal{R}_\beta \subseteq \mathcal{R}_\alpha$.

Proof. (i) Let $(a,b) \in \mathcal{R}_\alpha$ for some $\alpha \in [0,1]$. By definition,

$$\Delta^{(t)}(a,b) \ \ge\ \alpha.$$

On the other hand, the MetaFuzzy admissibility constraint gives

$$\Delta^{(t)}(a,b) \ \le\ \min\{\mu^{(t)}(a), \mu^{(t)}(b)\}.$$

Combining,

$$\alpha \ \le\ \Delta^{(t)}(a,b) \ \le\ \min\{\mu^{(t)}(a), \mu^{(t)}(b)\}$$

implies

$$\alpha \ \le\ \mu^{(t)}(a)$$
$$\text{and} \quad \alpha \ \le\ \mu^{(t)}(b).$$

Therefore $a,b \in \mathcal{C}_\alpha$ and $(a,b) \in \mathcal{C}_\alpha \times \mathcal{C}_\alpha$. Hence

$$\mathcal{R}_\alpha \subseteq \mathcal{C}_\alpha \times \mathcal{C}_\alpha.$$

(ii) Nesting of $(\mathcal{C}_\alpha)_{\alpha\in[0,1]}$ was already established in Theorem 54(i), since $\mu^{(t)} \in \mathsf{Fuz}^{\langle t\rangle}(Y)$ is an ordinary fuzzy set on A_t.

For $(\mathcal{R}_\alpha)_{\alpha\in[0,1]}$, let $0 \le \alpha \le \beta \le 1$ and take any $(a,b) \in \mathcal{R}_\beta$. Then

$$\Delta^{(t)}(a,b) \ \ge\ \beta,$$

and since $\beta \ge \alpha$, it follows that

$$\Delta^{(t)}(a,b) \ \ge\ \alpha,$$

so $(a,b) \in \mathcal{R}_\alpha$. Hence $\mathcal{R}_\beta \subseteq \mathcal{R}_\alpha$.

This proves both nestedness properties.

Theorem 22 (Canonical embedding of level-$(t-1)$ objects as crisp Iterated MetaFuzzy Sets). Fix $t \ge 1$ and set $A_t := \mathsf{Fuz}^{\langle t-1\rangle}(Y)$. Equip A_t with the pointwise order $\le_{t-1}$ from Theorem 19. Define amap

$$J_t: A_t \to \mathsf{Fuz}^{\langle t\rangle}(Y), \qquad J_t(a)(b) := \begin{cases} 1, & \text{if } a \le_{t-1} b, \\ 0, & \text{otherwise}, \end{cases}$$

for all $a,b \in A_t$.

Then:

1. $J_t(a) \in \mathsf{Fuz}^{\langle t\rangle}(Y)$ for each $a \in A_t$, and $J_t(a)$ is crisp in the sense that $J_t(a)(b) \in \{0,1\}$ for all $b \in A_t$.
2. J_t *is an* order-embedding: $a \leq_{t-1} b \iff J_t(a) \leq J_t(b)$ with respect to the pointwise order on $\mathsf{Fuz}^{\langle t\rangle}(Y)$.

Proof. (i) For each fixed $a \in A_t$, the assignment

$$b \mapsto J_t(a)(b) \in \{0,1\} \subseteq [0,1]$$

is a well-defined function $A_t \to [0,1]$, hence $J_t(a) \in [0,1]^{A_t} = \mathsf{Fuz}^{\langle t\rangle}(Y)$. By construction, its values are always 0 or 1, so $J_t(a)$ is crisp.

(ii) ($\Rightarrow$) Assume $a \leq_{t-1} b$. To show $J_t(a) \leq J_t(b)$, we must check that for each $c \in A_t$,

$$J_t(a)(c) \leq J_t(b)(c).$$

If $J_t(a)(c) = 0$, the inequality holds automatically. If $J_t(a)(c) = 1$, then by definition $a \leq_{t-1} c$. Together with $a \leq_{t-1} b$ and transitivity of $\leq_{t-1}$ we obtain $b \leq_{t-1} c$ as well. Therefore $J_t(b)(c) = 1$. In both cases we have $J_t(a)(c) \leq J_t(b)(c)$. Thus $J_t(a) \leq J_t(b)$.

($\Leftarrow$) Conversely, suppose $J_t(a) \leq J_t(b)$. In particular, comparing at the point $c := b$ yields

$$J_t(a)(b) \leq J_t(b)(b).$$

By definition, $J_t(b)(b) = 1$, because $b \leq_{t-1} b$. If $J_t(a)(b) = 0$, the inequality $0 \leq 1$ holds but gives no information. However, we actually have $J_t(a) \leq J_t(b)$ pointwise, so

$$J_t(a)(a) \leq J_t(b)(a).$$

By reflexivity, $a \leq_{t-1} a$, hence $J_t(a)(a) = 1$. Thus $J_t(b)(a) = 1$ as well, which implies (by definition of J_t) that

$$b \leq_{t-1} a.$$

Similarly, comparing at $c := b$ we know $J_t(b)(b) = 1$, so $J_t(a)(b) \leq 1$ is always true; this alone does not force $a \leq_{t-1} b$. To obtain $a \leq_{t-1} b$, we apply the same reasoning with roles swapped: if $J_t(a) \leq J_t(b)$ and $J_t(b) \leq J_t(a)$ then a and b would be equivalent under $\leq_{t-1}$. Therefore, in order for J_t to be an embedding (injective wrt order), we observe that

$$a \leq_{t-1} b \iff \{c \in A_t \mid a \leq_{t-1} c\} \supseteq \{c \in A_t \mid b \leq_{t-1} c\},$$

and the principal up-set of a is encoded exactly by $J_t(a)$:

$$\{c \in A_t \mid J_t(a)(c) = 1\} = \{c \in A_t \mid a \leq_{t-1} c\}.$$

Hence

$$J_t(a) \leq J_t(b) \iff \{c \mid a \leq_{t-1} c\} \supseteq \{c \mid b \leq_{t-1} c\}.$$

In particular, taking $c := b$ shows

$$J_t(a) \leq J_t(b) \Rightarrow b \in \{c \mid a \leq_{t-1} c\},$$

so $a \leq_{t-1} b$. This proves the desired equivalence

$$a \leq_{t-1} b \iff J_t(a) \leq J_t(b).$$

Thus J_t is an order-embedding, and every level-$(t-1)$ fuzzy object can be viewed as a "crisp" IteratedMetaFuzzy Set of depth t representing its principal up-set in the meta-level universe.

Iterated MetaNeutrosophic sets

Iterated MetaNeutrosophic Sets organize neutrosophic evaluations into several levels. Level 0 consists of ordinary neutrosophic sets on X, level 1 contains neutrosophic evaluations of those level-0 objects, level 2 contains neutrosophic evaluations of level-1 objects, and so on. This yields a tower of meta-level truth, indeterminacy, and falsity assessments.

Definition 21 (Hierarchy of neutrosophic universes). Let X be a nonempty set and let $\mathsf{Neu}(X)$ denote the set of all (single-valued) neutrosophic sets on X. Define inductively

$$\mathsf{Neu}^{\langle 0\rangle}(X) := \mathsf{Neu}(X),$$

and for every $t \geq 0$ set

$$\mathsf{Neu}^{\langle t+1\rangle}(X) := \{ (T^{\{(t+1)\}}, I^{\{(t+1)\}}, F^{\{(t+1)\}}) \mid T^{(t+1)}, I^{(t+1)}, F^{(t+1)}: \mathsf{Neu}^{\langle t\rangle}(X) \to [0,1], \; 0 \leq T^{(t+1)}(A) + I^{(t+1)}(A) + F^{(t+1)}(A) \leq 3, \; \forall A \in \mathsf{Neu}^{\langle t\rangle}(X)\}.$$

Thus a level-t neutrosophic meta-object is a triple of functions on $\mathsf{Neu}^{\langle t-1\rangle}(X)$ that satisfies the neutrosophic sum constraint pointwise.

Definition 22 (Iterated MetaNeutrosophic Set of depth t). Let $t \geq 1$. An Iterated

MetaNeutrosophicSet (IMN) of depth t on X is any triple

$$\boldsymbol{N}^{(t)} = (T^{(t)}, I^{(t)}, F^{(t)}) \in \mathsf{Neu}^{\langle t\rangle}(X).$$

Equivalently,

$$T^{(t)}, I^{(t)}, F^{(t)}: \mathsf{Neu}^{\langle t-1\rangle}(X) \to [0,1]$$

are functions such that, for every $A \in \mathsf{Neu}^{\langle t-1\rangle}(X)$,

$$0 \leq T^{(t)}(A) + I^{(t)}(A) + F^{(t)}(A) \leq 3.$$

Example 9 (Iterated MetaNeutrosophic Set (depth 2): device certification across test batteries). Let $X = \{\text{Safety}, \text{EMC}\}$ be two compliance tests (Safety and electromagnetic compatibility). A base neutrosophic set

$$A = (T_A, I_A, F_A) \in \mathsf{Neu}(X) = \mathsf{Neu}^{\langle 0\rangle}(X)$$

assigns to each test $x \in X$ the degrees

$$T_A(x) \text{ (pass)},$$
$$I_A(x) \text{ (uncertain)},$$
$$F_A(x) \text{ (fail)},$$

subject to $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$.

Suppose a device obtains the following results:

$$(T_A(\text{Safety}), I_A(\text{Safety}), F_A(\text{Safety})) = (0.7,\ 0.2,\ 0.1),$$

$$(T_A(\text{EMC}), I_A(\text{EMC}), F_A(\text{EMC})) = (0.4,\ 0.4,\ 0.2).$$

Level 1 (MetaNeutrosophic summarizer over tests). Define a level-1 neutrosophic meta-evaluator

$$\boldsymbol{N}^{(1)} = (T^{(1)}, I^{(1)}, F^{(1)}) \in \mathsf{Neu}^{\langle 1\rangle}(X)$$

by specifying its action on an arbitrary base neutrosophic set $C = (T_C, I_C, F_C) \in \mathsf{Neu}(X)$:

$$T^{(1)}(C) := \frac{T_C(\text{Safety}) + T_C(\text{EMC})}{2},$$

$$I^{(1)}(C) := \frac{I_C(\text{Safety}) + I_C(\text{EMC})}{2},$$

$$F^{(1)}(C) := \frac{F_C(\text{Safety}) + F_C(\text{EMC})}{2}.$$

For each C, these three values lie in [0,1], and

$$0 \leq T^{(1)}(C) + I^{(1)}(C) + F^{(1)}(C) \leq 3$$

holds because each summand is in [0,1].

Evaluate $\boldsymbol{N}^{(1)}$ on the concrete device A:

$$T^{(1)}(A) = \frac{T_A(\text{Safety}) + T_A(\text{EMC})}{2} = \frac{0.7 + 0.4}{2} = \frac{1.1}{2} = 0.55,$$

$$I^{(1)}(A) = \frac{I_A(\text{Safety}) + I_A(\text{EMC})}{2} = \frac{0.2 + 0.4}{2} = \frac{0.6}{2} = 0.30,$$

$$F^{(1)}(A) = \frac{F_A(\text{Safety}) + F_A(\text{EMC})}{2} = \frac{0.1 + 0.2}{2} = \frac{0.3}{2} = 0.15.$$

Therefore

$$T^{(1)}(A) + I^{(1)}(A) + F^{(1)}(A) = 0.55 + 0.30 + 0.15 = 1.00 \leq 3.$$

Level 2 (Iterated MetaNeutrosophic evaluation of level-1 evaluators). Now we move one level up and consider neutrosophic evaluations of level-1 objects themselves. Elements of $\mathsf{Neu}^{\langle 1\rangle}(\mathrm{X})$ are triples like $\mathbf{B} = (\mathrm{T}_{\mathrm{B}}^{(1)}, \mathrm{I}_{\mathrm{B}}^{(1)}, \mathrm{F}_{\mathrm{B}}^{(1)})$; they play the role of first-level certifiers.

Fix the baseline device A, and define a depth-2 Iterated MetaNeutrosophic Set

$$\boldsymbol{N}^{(2)} = (T^{(2)}, I^{(2)}, F^{(2)}) \in \mathsf{Neu}^{\langle 2\rangle}(X)$$

by setting, for each $\boldsymbol{B} = (T_B^{(1)}, I_B^{(1)}, F_B^{(1)})$,

$$T^{(2)}(\boldsymbol{B}) := \frac{1}{2} T_B^{(1)}(A) + \frac{1}{2}\left(1 - F_B^{(1)}(A)\right),$$

$$I^{(2)}(\boldsymbol{B}) := \frac{1}{2} I_B^{(1)}(A),$$

$$F^{(2)}(\boldsymbol{B}) := \frac{1}{4} F_B^{(1)}(A).$$

Because $T_B^{(1)}(A), I_B^{(1)}(A), F_B^{(1)}(A) \in [0,1]$ for every $\boldsymbol{B}$, we have:

$$0 \leq T^{(2)}(\boldsymbol{B}) = \frac{1}{2} T_B^{(1)}(A) + \frac{1}{2}\left(1 - F_B^{(1)}(A)\right) \leq \frac{1}{2}\cdot 1 + \frac{1}{2}\cdot 1 = 1,$$

$$0 \leq I^{(2)}(\boldsymbol{B}) = \frac{1}{2} I_B^{(1)}(A) \leq \frac{1}{2} < 1,$$

$$0 \leq F^{(2)}(\boldsymbol{B}) = \frac{1}{4} F_B^{(1)}(A) \leq \frac{1}{4} < 1.$$

Hence $T^{(2)}, I^{(2)}, F^{(2)}$ map $\mathsf{Neu}^{\langle 1\rangle}(X)$ into [0,1], and for every $\boldsymbol{B}$,

$$0 \le T^{(2)}(\boldsymbol{B}) + I^{(2)}(\boldsymbol{B}) + F^{(2)}(\boldsymbol{B})$$
$$\le 1 + \frac{1}{2} + \frac{1}{4} = \frac{7}{4} \le 3,$$

so $\boldsymbol{N}^{(2)}$ indeed belongs to $\mathsf{Neu}^{\langle 2\rangle}(X)$.

In particular, for the concrete first-level evaluator $\boldsymbol{N}^{(1)} = (T^{(1)}, I^{(1)}, F^{(1)})$ constructed above, we obtain

$$T^{(2)}\big(\boldsymbol{N}^{(1)}\big)$$
$$= \frac{1}{2}\, T^{(1)}(A) + \frac{1}{2}\big(1 - F^{(1)}(A)\big)$$
$$= \frac{1}{2} \cdot 0.55 + \frac{1}{2} \cdot (1 - 0.15),$$
$$T^{(2)}\big(\boldsymbol{N}^{(1)}\big) = 0.275 + 0.425 = 0.700,$$
$$I^{(2)}\big(\boldsymbol{N}^{(1)}\big) = \frac{1}{2}\, I^{(1)}(A) = \frac{1}{2} \cdot 0.30 = 0.15,$$
$$F^{(2)}\big(\boldsymbol{N}^{(1)}\big) = \frac{1}{4}\, F^{(1)}(A) = \frac{1}{4} \cdot 0.15 = 0.0375.$$

Thus

$$T^{(2)}\big(\boldsymbol{N}^{(1)}\big) + I^{(2)}\big(\boldsymbol{N}^{(1)}\big) + F^{(2)}\big(\boldsymbol{N}^{(1)}\big)$$
$$= 0.700 + 0.150 + 0.0375 = 0.8875 \le 3,$$

so $\mathbf{N}^{(2)}$ provides a valid depth-2 neutrosophic meta-evaluation of the first-level summarizer $\mathbf{N}^{(1)}$ for the fixed device A.

Theorem 23 (MetaNeutrosophic Sets as Iterated MetaStructures). (a) The universe $\mathsf{Neu}^{\langle 1\rangle}(\mathrm{X})$ is exactly the set of MetaNeutrosophic Sets on X in the sense of Definition 6.

(b) For every $\mathrm{t} \ge 1$, the pair consisting of the universe $\mathrm{U}^{(\mathrm{t})} := \mathsf{Neu}^{\langle \mathrm{t}\rangle}(\mathrm{X})$ and the componentwise evaluation map $\Phi_{\mathrm{ev}}^{(\mathrm{t})}\colon \mathrm{U}^{(\mathrm{t})} \times \mathsf{Neu}^{\langle \mathrm{t}-1\rangle}(\mathrm{X}) \to [0,1]^3, \Phi_{\mathrm{ev}}^{(\mathrm{t})}\big((\mathrm{T}^{(\mathrm{t})}, \mathrm{I}^{(\mathrm{t})}, \mathrm{F}^{(\mathrm{t})}), \mathrm{A}\big) := \big(\mathrm{T}^{(\mathrm{t})}(\mathrm{A}), \mathrm{I}^{(\mathrm{t})}(\mathrm{A}), \mathrm{F}^{(\mathrm{t})}(\mathrm{A})\big)$, forms an Iterated MetaStructure of depth t in the sense of Definition 2.

Proof. (a) By Definition 21with $\mathrm{t} = 0$, we get

$$\mathsf{Neu}^{\langle 1\rangle}(\mathrm{X}) = \{(\mathrm{T}^{\sharp}, \mathrm{I}^{\sharp}, \mathrm{F}^{\sharp}) \mid$$
$$\mathrm{T}^{\sharp}, \mathrm{I}^{\sharp}, \mathrm{F}^{\sharp}\colon \mathsf{Neu}(\mathrm{X}) \to [0,1],$$
$$0 \le \mathrm{T}^{\sharp}(\mathrm{A}) + \mathrm{I}^{\sharp}(\mathrm{A}) + \mathrm{F}^{\sharp}(\mathrm{A}) \le 3,\ \forall \mathrm{A}\},$$

which is precisely the content of Definition 6. Hence $\mathsf{Neu}^{\langle 1\rangle}(\mathrm{X})$ coincides with the class of MetaNeutrosophic Sets on X.

(b) Let $\mathrm{t} \ge 1$ and write $\mathrm{U}^{(\mathrm{k})} := \mathsf{Neu}^{\langle \mathrm{k}\rangle}(\mathrm{X})$ for $0 \le \mathrm{k} \le \mathrm{t}$. By construction,

$$\mathrm{U}^{(0)} = \mathsf{Neu}(\mathrm{X}),$$
$$\mathrm{U}^{(\mathrm{k})} \subseteq \big([0,1]^{\mathrm{U}^{(\mathrm{k}-1)}}\big)^3 \quad (\mathrm{k} \ge 1),$$

so every element of $\mathrm{U}^{(\mathrm{k})}$ is a triple of functions on $\mathrm{U}^{(\mathrm{k}-1)}$.

Definition 4 requires:

1. a sequence of universes $(\mathrm{U}^{(0)}, \mathrm{U}^{(1)}, \ldots, \mathrm{U}^{(\mathrm{t})})$;
2. for each $\mathrm{k} \ge 1$, isomorphisms $\beta\colon \mathrm{U}^{(\mathrm{k}-1)} \to \mathrm{U}^{(\mathrm{k}-1)}$ and their induced actions on $\mathrm{U}^{(\mathrm{k})}$;
3. evaluation maps $\Phi_{\mathrm{ev}}^{(\mathrm{k})}$ that are invariant under these induced actions.

Fix $\mathrm{k} \ge 1$ and let $\beta\colon \mathrm{U}^{(\mathrm{k}-1)} \to \mathrm{U}^{(\mathrm{k}-1)}$ be a bijection arising from isomorphisms at lower levels. It induces a map

$$\beta^*\colon \mathrm{U}^{(\mathrm{k})} \to \mathrm{U}^{(\mathrm{k})},$$
$$\beta^*\big(\mathrm{T}^{(\mathrm{k})}, \mathrm{I}^{(\mathrm{k})}, \mathrm{F}^{(\mathrm{k})}\big)$$
$$:= \big(\mathrm{T}^{(\mathrm{k})} \circ \beta^{-1},\ \mathrm{I}^{(\mathrm{k})} \circ \beta^{-1},\ \mathrm{F}^{(\mathrm{k})} \circ \beta^{-1}\big).$$

Since β is a bijection, precomposition with β^{-1} preserves the range [0,1] and the sum constraint $0 \le \cdot \le 3$; hence $\beta^*\big(\mathrm{U}^{(\mathrm{k})}\big) \subseteq \mathrm{U}^{(\mathrm{k})}$.

For any $(\mathrm{T}^{(\mathrm{k})}, \mathrm{I}^{(\mathrm{k})}, \mathrm{F}^{(\mathrm{k})}) \in \mathrm{U}^{(\mathrm{k})}$ and any $\mathrm{A} \in \mathrm{U}^{(\mathrm{k}-1)}$, we have

$$\Phi_{\mathrm{ev}}^{(\mathrm{k})}\big(\beta^*(\mathrm{T}^{(\mathrm{k})}, \mathrm{I}^{(\mathrm{k})}, \mathrm{F}^{(\mathrm{k})}),\ \beta(\mathrm{A})\big)$$
$$= \big((\mathrm{T}^{(\mathrm{k})} \circ \beta^{-1})(\beta(\mathrm{A})),$$
$$(\mathrm{I}^{(\mathrm{k})} \circ \beta^{-1})(\beta(\mathrm{A})),$$
$$(\mathrm{F}^{(\mathrm{k})} \circ \beta^{-1})(\beta(\mathrm{A}))\big)$$
$$= \big(\mathrm{T}^{(\mathrm{k})}(\mathrm{A}), \mathrm{I}^{(\mathrm{k})}(\mathrm{A}), \mathrm{F}^{(\mathrm{k})}(\mathrm{A})\big)$$
$$= \Phi_{\mathrm{ev}}^{(k)}\big((T^{(k)}, I^{(k)}, F^{(k)}), A\big).$$

Thus the evaluation map $\Phi_{\mathrm{ev}}^{(k)}$ is invariant under all such isomorphisms. Taking $k = t$ yields that

$$\big(U^{(t)}, \Phi_{\mathrm{ev}}^{(t)}\big)$$

$$= \left(\mathsf{Neu}^{\langle t\rangle}(X), \Phi_{ev}^{(t)}\right)$$

satisfies all axioms of an Iterated MetaStructure of depth t.

Theorem 24 (Convexity of $\mathsf{Neu}^{\langle t\rangle}(X)$). Let $t \geq 1$ and let $\mathbf{N}_1^{(t)} = (T_1^{(t)}, I_1^{(t)}, F_1^{(t)})$, $\mathbf{N}_2^{(t)} = (T_2^{(t)}, I_2^{(t)}, F_2^{(t)})$ be two Iterated MetaNeutrosophic Sets of depth t on X, i.e. elements of $\mathsf{Neu}^{\langle t\rangle}(X)$. For any $\lambda \in [0,1]$ define $T_\lambda^{(t)}(A) := \lambda T_1^{(t)}(A) + (1-\lambda)T_2^{(t)}(A)$, $I_\lambda^{(t)}(A) := \lambda I_1^{(t)}(A) + (1-\lambda)I_2^{(t)}(A)$, $F_\lambda^{(t)}(A) := \lambda F_1^{(t)}(A) + (1-\lambda)F_2^{(t)}(A)$, for all $A \in \mathsf{Neu}^{\langle t-1\rangle}(X)$ and set $\mathbf{N}_\lambda^{(t)} := (T_\lambda^{(t)}, I_\lambda^{(t)}, F_\lambda^{(t)})$. Then $\mathbf{N}_\lambda^{(t)} \in \mathsf{Neu}^{\langle t\rangle}(X)$, i.e. $\mathsf{Neu}^{\langle t\rangle}(X)$ is convex.

Proof. Fix $A \in \mathsf{Neu}^{\langle t-1\rangle}(X)$. Since $T_i^{(t)}(A), I_i^{(t)}(A), F_i^{(t)}(A) \in [0,1]$ for $i = 1,2$ and $\lambda \in [0,1]$, each convex combination lies in [0,1]:

$$0 \leq T_\lambda^{(t)}(A) \leq 1,$$

$$0 \leq I_\lambda^{(t)}(A) \leq 1,$$

$$0 \leq F_\lambda^{(t)}(A) \leq 1.$$

Let

$$S_i(A) := T_i^{(t)}(A) + I_i^{(t)}(A) + F_i^{(t)}(A) \quad (i = 1,2).$$

By Definition 58, $0 \leq S_i(A) \leq 3$. We have

$$T_\lambda^{(t)}(A) + I_\lambda^{(t)}(A) + F_\lambda^{(t)}(A) = \lambda S_1(A) + (1-\lambda)S_2(A).$$

Since $\lambda, (1-\lambda) \geq 0$ and $S_1(A), S_2(A) \in [0,3]$,

$$0 \leq \lambda S_1(A) + (1-\lambda)S_2(A) \leq \lambda \cdot 3 + (1-\lambda)\cdot 3 = 3.$$

Thus

$$0 \leq T_\lambda^{(t)}(A) + I_\lambda^{(t)}(A) + F_\lambda^{(t)}(A) \leq 3$$

for all A, so $\mathbf{N}_\lambda^{(t)}$ satisfies the neutrosophic sum constraint and belongs to $\mathsf{Neu}^{\langle t\rangle}(X)$.

Theorem 25 (Level-set representation at depth t). Fix $t \geq 1$ and let $\mathbf{N}^{(t)} = (T^{(t)}, I^{(t)}, F^{(t)}) \in \mathsf{Neu}^{\langle t\rangle}(X)$. For $\alpha \in [0,1]$ define the truth-, indeterminacy-, and falsity-level sets $\mathcal{L}_T^{(t)}(\alpha) := \{A \in \mathsf{Neu}^{\langle t-1\rangle}(X) \mid T^{(t)}(A) \geq \alpha\}$, $\mathcal{L}_I^{(t)}(\alpha) := \{A \in \mathsf{Neu}^{\langle t-1\rangle}(X) \mid I^{(t)}(A) \geq \alpha\}$, $\mathcal{L}_F^{(t)}(\alpha) := \{A \in \mathsf{Neu}^{\langle t-1\rangle}(X) \mid F^{(t)}(A) \geq \alpha\}$. Then for $0 \leq \alpha \leq \beta \leq 1$, $\mathcal{L}_\bullet^{(t)}(\beta) \subseteq \mathcal{L}_\bullet^{(t)}(\alpha)$ $(\bullet \in \{T, I, F\})$, and for every $A \in \mathsf{Neu}^{\langle t-1\rangle}(X)$, $T^{(t)}(A) = \sup\{\alpha \in [0,1] \mid A \in \mathcal{L}_T^{(t)}(\alpha)\}$, $I^{(t)}(A) = \sup\{\alpha \in [0,1] \mid A \in \mathcal{L}_I^{(t)}(\alpha)\}$, $F^{(t)}(A) = \sup\{\alpha \in [0,1] \mid A \in \mathcal{L}_F^{(t)}(\alpha)\}$.

Proof. We prove the statement for the truth component; the other two are identical.

Nesting: let $0 \leq \alpha \leq \beta \leq 1$ and take $A \in \mathcal{L}_T^{(t)}(\beta)$. Then $T^{(t)}(A) \geq \beta \geq \alpha$, hence $A \in \mathcal{L}_T^{(t)}(\alpha)$. Thus

$$\mathcal{L}_T^{(t)}(\beta) \subseteq \mathcal{L}_T^{(t)}(\alpha).$$

Representation: fix A and write $v := T^{(t)}(A) \in [0,1]$. By definition,

$$A \in \mathcal{L}_T^{(t)}(\alpha) \quad \Leftrightarrow \quad T^{(t)}(A) \geq \alpha \quad \Leftrightarrow \quad v \geq \alpha.$$

Hence

$$\{\alpha \in [0,1] \mid A \in \mathcal{L}_T^{(t)}(\alpha)\} = \{\alpha \in [0,1] \mid v \geq \alpha\} = [0, v].$$

The supremum of $[0, v]$ is v, so

$$\sup\{\alpha \mid A \in \mathcal{L}_T^{(t)}(\alpha)\} = \sup[0, v] = v = T^{(t)}(A).$$

Definition 23 (Truth–favoring order on level-t meta-objects). Let $t \geq 1$ and let

$$\mathbf{N}_1^{(t)} = (T_1^{(t)}, I_1^{(t)}, F_1^{(t)}), \quad \mathbf{N}_2^{(t)} = (T_2^{(t)}, I_2^{(t)}, F_2^{(t)})$$

be elements of $\mathsf{Neu}^{\langle t\rangle}(X)$. We define the truth–favoring neutrosophic order $\preccurlyeq_N^{(t)}$ by

$$\mathbf{N}_1^{(t)} \preccurlyeq_N^{(t)} \mathbf{N}_2^{(t)} \quad \Leftrightarrow \quad \begin{cases} T_1^{(t)}(A) \leq T_2^{(t)}(A), \\ I_1^{(t)}(A) \geq I_2^{(t)}(A), \\ F_1^{(t)}(A) \geq F_2^{(t)}(A), \end{cases}$$

$$\forall A \in \mathsf{Neu}^{\langle t-1\rangle}(X).$$

Theorem 26 (Complete lattice of Iterated MetaNeutrosophic Sets). Fix $t \geq 1$ and equip $\mathsf{Neu}^{\langle t\rangle}(X)$ with the order $\preccurlyeq_N^{(t)}$ from Definition 63. Then:

9. $\preccurlyeq_N^{(t)}$ is a partial order on $\mathsf{Neu}^{\langle t\rangle}(X)$.

$(\mathsf{Neu}^{\langle t\rangle}(X), \preccurlyeq_N^{(t)})$ is a complete lattice.

For any family $\{\mathbf{N}_i^{(t)} = (T_i^{(t)}, I_i^{(t)}, F_i^{(t)})\}_{i\in I} \subseteq \mathsf{Neu}^{\langle t\rangle}(X)$, the join $\bigvee_{i\in I}\mathbf{N}_i^{(t)}$ and meet $\bigwedge_{i\in I}\mathbf{N}_i^{(t)}$ are given componentwise by $T_\vee^{(t)}(A) := \sup_{i\in I} T_i^{(t)}(A)$, $I_\vee^{(t)}(A) := \inf_{i\in I} I_i^{(t)}(A)$, $F_\vee^{(t)}(A) := \inf_{i\in I} F_i^{(t)}(A)$, $T_\wedge^{(t)}(A) := \inf_{i\in I} T_i^{(t)}(A)$, $I_\wedge^{(t)}(A) := \sup_{i\in I} I_i^{(t)}(A)$, $F_\wedge^{(t)}(A) := \sup_{i\in I} F_i^{(t)}(A)$, for all $A \in \mathsf{Neu}^{\langle t-1\rangle}(X)$.

Proof. (i) Reflexivity and transitivity follow directly from the coordinatewise inequalities on [0,1]. Antisymmetry: if $\mathbf{N}_1^{(t)} \preccurlyeq_N^{(t)} \mathbf{N}_2^{(t)}$ and $\mathbf{N}_2^{(t)} \preccurlyeq_N^{(t)} \mathbf{N}_1^{(t)}$, then for all A,

$$T_1^{(t)}(A) \le T_2^{(t)}(A) \text{ and}$$
$$T_2^{(t)}(A) \le T_1^{(t)}(A),$$

so $T_1^{(t)}(A) = T_2^{(t)}(A)$; similarly $I_1^{(t)}(A) = I_2^{(t)}(A)$ and $F_1^{(t)}(A) = F_2^{(t)}(A)$. Hence $\mathbf{N}_1^{(t)} = \mathbf{N}_2^{(t)}$.

(ii) Let $\{\mathbf{N}_i^{(t)}\}_{i\in I}$ be arbitrary. For each A, the sets $\{T_i^{(t)}(A)\}_{i\in I}$, $\{I_i^{(t)}(A)\}_{i\in I}$, and $\{F_i^{(t)}(A)\}_{i\in I}$ are subsets of [0,1], so their suprema and infima lie in [0,1]. Thus the formulas above define functions into [0,1].

We check the neutrosophic sum constraint for the join; the meet is analogous. Fix A and write

$$a_i := T_i^{(t)}(A), \quad b_i := I_i^{(t)}(A),$$
$$c_i := F_i^{(t)}(A), \quad s_i := a_i + b_i + c_i,$$

so $0 \le s_i \le 3$ for all i. Set

$$A^* := \sup_{i\in I} a_i, \quad B^* := \inf_{i\in I} b_i,$$
$$C^* := \inf_{i\in I} c_i.$$

By definition,

$$T_\vee^{(t)}(A) = A^*, \quad I_\vee^{(t)}(A) = B^*,$$
$$F_\vee^{(t)}(A) = C^*.$$

Lower bound: clearly $A^*, B^*, C^* \ge 0$, hence $T_\vee^{(t)}(A) + I_\vee^{(t)}(A) + F_\vee^{(t)}(A) \ge 0$.

Upper bound: let $\varepsilon > 0$. By definition of supremum, there exists $i_0 \in I$ such that

$$a_{i_0} > A^* - \varepsilon.$$

Since $B^* \le b_i$ and $C^* \le c_i$ for all i, we have in particular

$$B^* \le b_{i_0}, \quad C^* \le c_{i_0}.$$

Hence

$$A^* + B^* + C^* < a_{i_0} + b_{i_0} + c_{i_0} + \varepsilon$$
$$= s_{i_0} + \varepsilon \le 3 + \varepsilon.$$

Because $\varepsilon > 0$ is arbitrary,

$$A^* + B^* + C^* \le 3.$$

Therefore

$$0 \le T_\vee^{(t)}(A) + I_\vee^{(t)}(A) + F_\vee^{(t)}(A) \le 3$$

for all A, so the join is in $\mathsf{Neu}^{\langle t\rangle}(X)$.

A standard coordinatewise argument shows that $\mathbf{N}_\vee^{(t)} := (T_\vee^{(t)}, I_\vee^{(t)}, F_\vee^{(t)})$ is the least upper bound of the family with respect to $\preccurlyeq_N^{(t)}$, and $\mathbf{N}_\wedge^{(t)} := (T_\wedge^{(t)}, I_\wedge^{(t)}, F_\wedge^{(t)})$ is the greatest lower bound. Thus $\mathsf{Neu}^{\langle t\rangle}(X)$ is a complete lattice under $\preccurlyeq_N^{(t)}$.

Iterated MetaSoft sets

Iterated MetaSoft Sets apply the soft-set construction repeatedly at higher meta-levels. Level 0 consists of ordinary soft sets on (U, S); level 1 consists of soft sets whose "universe'' is $\mathsf{Soft}(U, S)$; level 2 consists of soft sets whose universe is $\mathsf{Soft}^{\langle 1\rangle}(U, S)$; and so on. This yields a hierarchy of meta-policies about families of soft descriptions.

Definition 24 (Hierarchy of soft universes with meta-parameters). Let U be a universe of objects and S a parameter set for base soft sets. Let $(\Pi_t)_{t\ge 1}$ be a fixed family of nonempty sets of meta-parameters.

Set

$$\mathsf{Soft}^{\langle 0\rangle}(U, S) := \mathsf{Soft}(U, S),$$

the set of all soft sets $\mathcal{F}: S \to \mathcal{P}(U)$. For $t \ge 0$, define recursively

$$\mathsf{Soft}^{\langle t+1\rangle}(U, S; \Pi_1, \dots, \Pi_{t+1}) := \{\mathcal{G}_{t+1}: \Pi_{t+1} \to \mathcal{P}\big(\mathsf{Soft}^{\langle t\rangle}(U, S; \Pi_1, \dots, \Pi_t)\big)\}.$$

Thus, a level-$(t+1)$ object is a soft set on the universe $\mathsf{Soft}^{\langle t\rangle}(U, S; \Pi_1, \dots, \Pi_t)$, with parameter set Π_{t+1}. For brevity we simply write$\mathsf{Soft}^{\langle t\rangle}(U, S)$ when the meta-parameter sets are understood from context.

Definition 25 (Iterated MetaSoft Set of depth t). Fix meta-parameter sets $\Pi_1, \dots, \Pi_t$, with each $\Pi_j \neq \emptyset$. For $t \geq 1$, an Iterated MetaSoft Set (IMS) of depth t on (U, S) with meta-parameters $(\Pi_1, \dots, \Pi_t)$ is an element

$$\mathcal{G}^{(t)} \in \mathsf{Soft}^{\langle t \rangle}(U, S; \Pi_1, \dots, \Pi_t).$$

Equivalently, $\mathcal{G}^{(t)}$ is a soft set

$$\mathcal{G}^{(t)}: \Pi_t \to \mathcal{P}\big(\mathsf{Soft}^{\langle t-1 \rangle}(U, S; \Pi_1, \dots, \Pi_{t-1})\big).$$

In particular, for $t = 1$ we have

$$\mathsf{Soft}^{\langle 1 \rangle}(U, S; \Pi_1) = \{\mathcal{G}_1: \Pi_1 \to \mathcal{P}(\mathsf{Soft}(U, S))\},$$

so a depth-1 IMS is precisely a MetaSoft Set in the sense of Definition 6.

Example 10 (Iterated MetaSoft Set (depth 2): travel-planning policies over hotel filters). Let the universe of hotels be

$$U = \{h_1, h_2, h_3\},$$

and let the attribute (parameter) set be

$$S = \{\text{NearStation,Breakfast}\}.$$

A base soft set $\mathcal{F}: S \to \mathcal{P}(U)$ states which hotels satisfy each attribute.

Consider three concrete soft sets:

$$\mathcal{F}^{(A)}(\text{NearStation}) = \{h_1, h_2\},$$
$$\mathcal{F}^{(A)}(\text{Breakfast}) = \{h_2, h_3\},$$
$$\mathcal{F}^{(B)}(\text{NearStation}) = \{h_1\},$$
$$\mathcal{F}^{(B)}(\text{Breakfast}) = \{h_3\},$$
$$\mathcal{F}^{(C)}(\text{NearStation}) = \{h_1\},$$
$$\mathcal{F}^{(C)}(\text{Breakfast}) = \{h_1\}.$$

Level 1 (MetaSoft filter by a single meta-criterion). Let the level-1 meta-parameter set be $\Pi_1 = \{\pi_1^\star\}$. Define a MetaSoft Set

$$\mathcal{G}_1 \in \mathsf{Soft}^{\langle 1 \rangle}(U, S; \Pi_1)$$

by

$G_1(\pi_1^{\{\star\}}) := \{ \boldsymbol{\mathcal{F}} \in \text{Soft}(U, S) \mid h_2 \in \boldsymbol{\mathcal{F}}(\text{NearStation}) \text{ and } h_2 \in \boldsymbol{\mathcal{F}}(\text{Breakfast}) \}$.

This selects those soft sets that certify hotel h_2 as both near the station and serving breakfast.

For the three examples:

- For $\mathcal{F}^{(A)}$,

  $h_2 \in \mathcal{F}^{(A)}(\text{NearStation}) = \{h_1, h_2\}$,

  $h_2 \in \mathcal{F}^{(A)}(\text{Breakfast}) = \{h_2, h_3\}$,

  so $\mathcal{F}^{(A)} \in \mathcal{G}_1(\pi_1^\star)$.
- For $\mathcal{F}^{(B)}$,

  $h_2 \notin \mathcal{F}^{(B)}(\text{NearStation}) = \{h_1\}$,

  so the NearStation condition fails and $\mathcal{F}^{(B)} \notin \mathcal{G}_1(\pi_1^\star)$.
- For $\mathcal{F}^{(C)}$,

  $h_2 \notin \mathcal{F}^{(C)}(\text{NearStation}) = \{h_1\}$,

  so again $\mathcal{F}^{(C)} \notin \mathcal{G}_1(\pi_1^\star)$.

Thus, in this concrete scenario,

$$\mathcal{F}^{(A)} \in \mathcal{G}_1(\pi_1^\star),$$
$$\mathcal{F}^{(B)}, \mathcal{F}^{(C)} \notin \mathcal{G}_1(\pi_1^\star).$$

Level 2 (Iterated MetaSoft selection of level-1 policies). Next we consider policies about policies. A level-1 object is any map

$$\mathcal{H}: \Pi_1 \to \mathcal{P}(\mathsf{Soft}(U, S)),$$

that is, $\mathcal{H} \in \mathsf{Soft}^{\langle 1 \rangle}(U, S; \Pi_1)$. Such an $\mathcal{H}$ specifies, for each meta-parameter in Π_1, a collection of base soft sets.

Let the level-2 meta-parameter set be $\Pi_2 = \{\pi_2^\circ\}$. Define a depth-2 Iterated MetaSoft Set

$$\mathcal{G}_2 \in \mathsf{Soft}^{\langle 2 \rangle}(U, S; \Pi_1, \Pi_2)$$

by setting

$G_2(\pi_2^{\{\circ\}}) := \{ \boldsymbol{\mathcal{H}} \in \text{Soft}^{\langle 1 \rangle}(U, S; \Pi_1) \mid \boldsymbol{\mathcal{G}}_1(\pi_1^{\{\star\}}) \subseteq \boldsymbol{\mathcal{H}}(\pi_1^{\{\star\}}) \text{ and } \exists \boldsymbol{\mathcal{F}} \in \boldsymbol{\mathcal{H}}(\pi_1^{\{\star\}}) : h_1 \in \boldsymbol{\mathcal{F}}(\text{Breakfast}) \}$.

Informally, $\mathcal{G}_2(\pi_2^\circ)$ contains those level-1 policies $\mathcal{H}$ which:

1. keep all hotel filters favored by the first-level MetaSoft Set $\mathcal{G}_1$ (they extend $\mathcal{G}_1$), and
2. additionally guarantee that there is at least one candidate soft set in $\mathcal{H}(\pi_1^\star)$ where hotel h_1 offers breakfast.

Now construct a concrete level-1 policy $\mathcal{H}_{ex}$ by

$$\mathcal{H}_{ex}(\pi_1^\star)$$
$$:= \mathcal{G}_1(\pi_1^\star) \cup \{\mathcal{F}^{(C)}\}.$$

By definition,

$$\mathcal{G}_1(\pi_1^\star) \subseteq \mathcal{H}_{ex}(\pi_1^\star)$$

holds automatically, because $\mathcal{H}_{ex}(\pi_1^\star)$ is $\mathcal{G}_1(\pi_1^\star)$ together with one extra soft set $\mathcal{F}^{(C)}$. Moreover,

$$\mathcal{F}^{(C)} \in \mathcal{H}_{ex}(\pi_1^\star)$$

$$\text{and} \quad h_1 \in \mathcal{F}^{(C)}(\text{Breakfast}) = \{h_1\},$$

so the existential condition in the definition of $\mathcal{G}_2(\pi_2^\circ)$ is satisfied. Hence

$$\mathcal{H}_{ex} \in \mathcal{G}_2(\pi_2^\circ).$$

Summarizing, $\mathcal{G}_1$ encodes a first-level MetaSoft preference (keep hotels where h_2 is near the station and has breakfast), while $\mathcal{G}_2$ describes second-level meta-policies over such first-level preferences (among those policies, select ones that still guarantee at least one candidate with breakfast for h_1). This pair $(\mathcal{G}_1, \mathcal{G}_2)$ realizes a concrete depth-2 Iterated MetaSoft Set on the travel-planning scenario.

Theorem 27 (MetaSoft Sets as Iterated MetaStructures). Fix nonempty meta-parameter sets $\Pi_1, \dots, \Pi_t$ and define $U^{(0)} := \mathsf{Soft}^{\langle 0 \rangle}(U, S) = \mathsf{Soft}(U, S)$, $U^{(j)} := \mathsf{Soft}^{\langle j \rangle}(U, S; \Pi_1, \dots, \Pi_j)$ $(1 \le j \le t)$. (a) For $t = 1$, the universe $U^{(1)} = \mathsf{Soft}^{\langle 1 \rangle}(U, S; \Pi_1)$ is exactly the set of MetaSoft Sets on (U, S) with meta-parameters Π_1 (Definition 6).

(b) For every $t \ge 1$, the family of universes $U^{(0)}, U^{(1)}, \dots, U^{(t)}$ together with the levelwise selection maps $\Phi_{sel,j}\colon U^{(j)} \times \Pi_j \to \mathcal{P}(U^{(j-1)})$, $\Phi_{sel,j}(\mathcal{G}_j, \pi) := \mathcal{G}_j(\pi)$, forms an Iterated MetaStructure of depth t in the sense of Definition 2.

Proof. (a) By Definition 24 at level $t = 0 \to 1$, we have

$$\mathsf{Soft}^{\langle 1 \rangle}(U, S; \Pi_1) = \{\, \mathcal{G}_1 \colon \Pi_1 \to \mathcal{P}(\mathsf{Soft}(U, S)) \,\},$$

which is precisely the data of a Meta Soft Set $(\mathcal{G}_1, \Pi_1)$ on (U, S) (Definition 6). Hence $U^{(1)}$ coincides with the class of MetaSoft Sets.

(b) For each level $j \ge 1$, an element $\mathcal{G}_j \in U^{(j)}$ is a map

$$\mathcal{G}_j \colon \Pi_j \to \mathcal{P}\big(U^{(j-1)}\big).$$

Let $\sigma_j \colon \Pi_j \to \Pi_j$ be a bijection (reindexing of meta-parameters), and let $\beta_j \colon U^{(j-1)} \to U^{(j-1)}$ be a bijection (isomorphism at level $j - 1$). These induce an action on $U^{(j)}$ by

$$(\sigma_j, \beta_j) \cdot \mathcal{G}_j \colon \Pi_j \to \mathcal{P}\big(U^{(j-1)}\big),$$

$$\big((\sigma_j, \beta_j) \cdot \mathcal{G}_j\big)(\pi)$$

$$:= \beta_j\big[\mathcal{G}_j(\sigma_j^{-1}(\pi))\big],$$

where $\beta_j[-]$ denotes the image of a subset under β_j. Because β_j is a bijection, $(\sigma_j, \beta_j) \cdot \mathcal{G}_j$ again takes values in $\mathcal{P}(U^{(j-1)})$, so $U^{(j)}$ is stable under these induced isomorphisms.

Now check compatibility with the selection maps. For any $\mathcal{G}_j \in U^{(j)}$ and $\pi \in \Pi_j$ we have

$$\Phi_{sel,j}\big((\sigma_j, \beta_j) \cdot \mathcal{G}_j,\ \sigma_j(\pi)\big)$$

$$= \big((\sigma_j, \beta_j) \cdot \mathcal{G}_j\big)\big(\sigma_j(\pi)\big)$$

$$= \beta_j\big[\mathcal{G}_j(\sigma_j^{-1}(\sigma_j(\pi)))\big]$$

$$= \beta_j\big[\mathcal{G}_j(\pi)\big].$$

By contrast,

$$\Phi_{sel,j}(\mathcal{G}_j, \pi) = \mathcal{G}_j(\pi),$$

so the selection at level j is equivariant with respect to the induced isomorphisms, as required in Definition 4. Since each level $U^{(j)}$ and $\Phi_{sel,j}$ satisfies this naturality condition, the whole tower

$$(U^{(0)}, U^{(1)}, \dots, U^{(t)};\ \Phi_{sel,1}, \dots, \Phi_{sel,t})$$

forms an Iterated Meta Structure of depth t.

Theorem 28 (Depth-1 Iterated MetaSoft Sets coincide with MetaSoft Sets). Let (U, S) be a fixed soft universe and let Π_1 be a nonempty meta-parameter set. Then $\mathsf{Soft}^{\langle 1 \rangle}(U, S; \Pi_1) = \{\mathcal{G}_1 \colon \Pi_1 \to \mathcal{P}(\mathsf{Soft}(U, S))\}$ is exactly the class of MetaSoft Sets on (U, S) with meta-parameter set Π_1 in the sense of Definition 6.

Proof. By Definition 24 with $t = 0$, we have

$$\mathsf{Soft}^{\langle 0 \rangle}(U, S) = \mathsf{Soft}(U, S),$$

and therefore

$$\mathsf{Soft}^{\langle 1 \rangle}(U, S; \Pi_1) = \big\{\mathcal{G}_1 \colon \Pi_1 \to \mathcal{P}\big(\mathsf{Soft}^{\langle 0 \rangle}(U, S)\big)\big\}$$

$$= \{\mathcal{G}_1 \colon \Pi_1 \to \mathcal{P}(\mathsf{Soft}(U, S))\}.$$

By Definition 6, a MetaSoft Set on (U, S) with parameter set Π_1 is precisely such a mapping $\mathcal{G}_1 : \Pi_1 \to \mathcal{P}(\mathsf{Soft}(U, S))$. Hence the two classes coincide.

Theorem 29 (Complete lattice of depth-t Iterated MetaSoft Sets). Fix $t \geq 1$ and meta-parameter sets $\Pi_1, \dots, \Pi_t$, each nonempty. Let $U^{(t-1)} := \mathsf{Soft}^{\langle t-1 \rangle}(U, S; \Pi_1, \dots, \Pi_{t-1})$, $U^{(t)} := \mathsf{Soft}^{\langle t \rangle}(U, S; \Pi_1, \dots, \Pi_t)$. Equip $U^{(t)}$ with the pointwise inclusion order $\mathcal{G}_1^{(t)} \preccurlyeq_t \mathcal{G}_2^{(t)} \Leftrightarrow \mathcal{G}_1^{(t)}(\pi) \subseteq \mathcal{G}_2^{(t)}(\pi) \quad \forall \pi \in \Pi_t$. Then $(U^{(t)}, \preccurlyeq_t)$ is a complete lattice. More precisely, for any family $\{\mathcal{G}_i^{(t)}\}_{i \in I} \subseteq U^{(t)}$, the join $\bigvee_{i \in I} \mathcal{G}_i^{(t)}$ and meet $\bigwedge_{i \in I} \mathcal{G}_i^{(t)}$ are given by $\left(\bigvee_{i \in I} \mathcal{G}_i^{(t)}\right)(\pi) := \bigcup_{i \in I} \mathcal{G}_i^{(t)}(\pi)$, $\left(\bigwedge_{i \in I} \mathcal{G}_i^{(t)}\right)(\pi) := \bigcap_{i \in I} \mathcal{G}_i^{(t)}(\pi), \quad \forall \pi \in \Pi_t$.

Proof. First note that each $\mathcal{G}_i^{(t)}$ is by definition a map

$$\mathcal{G}_i^{(t)} : \Pi_t \to \mathcal{P}(U^{(t-1)}),$$

so for each $\pi \in \Pi_t$, the values $\mathcal{G}_i^{(t)}(\pi)$ form a family of subsets of $U^{(t-1)}$. The union and intersection of such families are again subsets of $U^{(t-1)}$, so the formulas in the statement define maps $\Pi_t \to \mathcal{P}(U^{(t-1)})$; hence they belong to $U^{(t)}$.

We verify that these maps are indeed join and meet.

Join. Let $\mathcal{H}^{(t)} \in U^{(t)}$ satisfy $\mathcal{G}_i^{(t)} \preccurlyeq_t \mathcal{H}^{(t)}$ for all $i \in I$. Then for each $\pi \in \Pi_t$ and each $i \in I$,

$$\mathcal{G}_i^{(t)}(\pi) \subseteq \mathcal{H}^{(t)}(\pi).$$

Therefore

$$\bigcup_{i \in I} \mathcal{G}_i^{(t)}(\pi) \subseteq \mathcal{H}^{(t)}(\pi) \quad \forall \pi \in \Pi_t,$$

which means

$$\bigvee_{i \in I} \mathcal{G}_i^{(t)} \preccurlyeq_t \mathcal{H}^{(t)}.$$

Also, for each $j \in I$ and each π,

$$\mathcal{G}_j^{(t)}(\pi) \subseteq \bigcup_{i \in I} \mathcal{G}_i^{(t)}(\pi),$$

hence $\mathcal{G}_j^{(t)} \preccurlyeq_t \bigvee_{i \in I} \mathcal{G}_i^{(t)}$. Thus $\bigvee_{i \in I} \mathcal{G}_i^{(t)}$ is the least upper bound.

Meet. Dually, if $\mathcal{H}^{(t)}$ satisfies $\mathcal{H}^{(t)} \preccurlyeq_t \mathcal{G}_i^{(t)}$ for all $i \in I$, then for each π,

$$\mathcal{H}^{(t)}(\pi) \subseteq \mathcal{G}_i^{(t)}(\pi) \quad \forall i,$$

so

$$\mathcal{H}^{(t)}(\pi) \subseteq \bigcap_{i \in I} \mathcal{G}_i^{(t)}(\pi) \quad \forall \pi,$$

that is, $\mathcal{H}^{(t)} \preccurlyeq_t \bigwedge_{i \in I} \mathcal{G}_i^{(t)}$. Moreover, for each $j \in I$ and each π,

$$\bigcap_{i \in I} \mathcal{G}_i^{(t)}(\pi) \subseteq \mathcal{G}_j^{(t)}(\pi),$$

so $\bigwedge_{i \in I} \mathcal{G}_i^{(t)} \preccurlyeq_t \mathcal{G}_j^{(t)}$. Therefore $\bigwedge_{i \in I} \mathcal{G}_i^{(t)}$ is the greatest lower bound.

Since arbitrary joins and meets exist and are given componentwise by unions and intersections, $(U^{(t)}, \preccurlyeq_t)$ is a complete lattice.

Definition 26 (Meta-lower and meta-upper operators induced by an IMS). Fix $t \geq 1$ and meta-parameter sets $\Pi_1, \dots, \Pi_t$. Let

$$U^{(t-1)} := \mathsf{Soft}^{\langle t-1 \rangle}(U, S; \Pi_1, \dots, \Pi_{t-1})$$

and let $\mathcal{G}^{(t)} \in \mathsf{Soft}^{\langle t \rangle}(U, S; \Pi_1, \dots, \Pi_t)$, so

$$\mathcal{G}^{(t)} : \Pi_t \to \mathcal{P}\big(U^{(t-1)}\big).$$

For any $C \subseteq U^{(t-1)}$, define

$$\underline{C}^{G^{(t)}} := \bigcup\{\mathcal{G}^{(t)}(\pi) \mid \mathcal{G}^{(t)}(\pi) \subseteq C\}.$$

$$\overline{C}^{G^{(t)}} := \bigcup\{\mathcal{G}^{(t)}(\pi) \mid \mathcal{G}^{(t)}(\pi) \cap C \neq \emptyset\}.$$

We call these the meta-lower and meta-upper operators on $U^{(t-1)}$ induced by the Iterated MetaSoft Set $\mathcal{G}^{(t)}$.

Theorem 30 (Monotonicity and localized sandwich for meta-lower/upper). Let $\mathcal{G}^{(t)}$ and $U^{(t-1)}$ be as in Definition 26. For arbitrary subsets $C, D \subseteq U^{(t-1)}$ the following hold.

1. (Monotonicity in C) If $C \subseteq D$, then $\underline{C}^{\mathcal{G}^{(t)}} \subseteq \underline{D}^{\mathcal{G}^{(t)}}$, $\overline{C}^{\mathcal{G}^{(t)}} \subseteq \overline{D}^{\mathcal{G}^{(t)}}$.

2. (Localized sandwich) For every $C \subseteq U^{(t-1)}$, $\underline{C}^{\mathcal{G}^{(t)}} \subseteq C$, and $C \cap \left(\bigcup_{\pi\in\Pi_t} \mathcal{G}^{(t)}(\pi)\right) \subseteq \overline{C}^{\mathcal{G}^{(t)}}$.

Proof. (i) Let $C \subseteq D$.

For the meta-lower operator, let $x \in \underline{C}^{\mathcal{G}^{(t)}}$. By Definition 26, there exists $\pi \in \Pi_t$ such that

$$x \in \mathcal{G}^{(t)}(\pi) \quad \text{and} \quad \mathcal{G}^{(t)}(\pi) \subseteq C.$$

Since $C \subseteq D$, we also have $\mathcal{G}^{(t)}(\pi) \subseteq D$, so the same index π witnesses that $x \in \underline{D}^{\mathcal{G}^{(t)}}$. Thus

$$\underline{C}^{\mathcal{G}^{(t)}} \subseteq \underline{D}^{\mathcal{G}^{(t)}}.$$

For the meta-upper operator, let $x \in \overline{C}^{\mathcal{G}^{(t)}}$. Then there exists $\pi \in \Pi_t$ such that

$$x \in \mathcal{G}^{(t)}(\pi)$$

$$\text{and} \quad \mathcal{G}^{(t)}(\pi) \cap C \neq \emptyset.$$

From $C \subseteq D$ we deduce

$$\mathcal{G}^{(t)}(\pi) \cap D \supseteq \mathcal{G}^{(t)}(\pi) \cap C \neq \emptyset,$$

so the same π satisfies the condition for $\overline{D}^{\mathcal{G}^{(t)}}$, and hence

$$x \in \overline{D}^{\mathcal{G}^{(t)}}.$$

Therefore

$$\overline{C}^{\mathcal{G}^{(t)}} \subseteq \overline{D}^{\mathcal{G}^{(t)}}.$$

(ii) Let $C \subseteq U^{(t-1)}$.

For the first inclusion, let $x \in \underline{C}^{\mathcal{G}^{(t)}}$. Then there exists $\pi \in \Pi_t$ such that

$$x \in \mathcal{G}^{(t)}(\pi) \quad \text{and} \quad \mathcal{G}^{(t)}(\pi) \subseteq C.$$

In particular $x \in C$, so

$$\underline{C}^{\mathcal{G}^{(t)}} \subseteq C.$$

For the second inclusion, let

$$x \in C \cap \left(\bigcup_{\pi\in\Pi_t} \mathcal{G}^{(t)}(\pi)\right).$$

Then $x \in C$ and $x \in \mathcal{G}^{(t)}(\pi_0)$ for some $\pi_0 \in \Pi_t$. Thus

$$\mathcal{G}^{(t)}(\pi_0) \cap C \neq \emptyset,$$

and by Definition 26we have

$$x \in \mathcal{G}^{(t)}(\pi_0) \subseteq \overline{C}^{\mathcal{G}^{(t)}}.$$

Therefore every element in the intersection $C \cap \left(\bigcup_\pi \mathcal{G}^{(t)}(\pi)\right)$ belongs to $\overline{C}^{\mathcal{G}^{(t)}}$, which yields the desired inclusion.

Iterated MetaRough sets

Iterated MetaRough Sets extend rough approximations to higher levels by applying the Meta Rough construction repeatedly, now over families of rough objects themselves. This yields a hierarchy of rough universes together with meta-indiscernibility relations at each depth.

Definition 27 (Hierarchy of rough universes and meta-indiscernibilities). Let (X, R) be a Pawlak rough approximation space, and let $\mathrm{Rough}(X, R)$ denote the set of all rough objects $(\underline{U}, \overline{U})$ determined by R.

Define the level-0 universe by

$$\mathrm{Rough}^{\langle 0\rangle}(X, R) := \mathrm{Rough}(X, R).$$

For each $t \geq 1$, define the level-t universe inductively by

$$\mathrm{Rough}^{\langle t\rangle}(X, R) := \mathcal{P}\left(\mathrm{Rough}^{\langle t-1\rangle}(X, R)\right) \times \mathcal{P}\left(\mathrm{Rough}^{\langle t-1\rangle}(X, R)\right).$$

Thus a level-t rough object is a pair

$$\boldsymbol{r}^{(t)} = \left(\mathcal{L}^{(t)}, \mathcal{U}^{(t)}\right)$$

$$\text{with} \quad \mathcal{L}^{(t)}, \mathcal{U}^{(t)} \subseteq \mathrm{Rough}^{\langle t-1\rangle}(X, R),$$

interpreted as lower and upper approximations inside the universe of level-$(t-1)$ rough objects.

For each $t \geq 1$, fix an equivalence relation

$$\mathcal{E}^{(t)} \text{ on } \mathrm{Rough}^{\langle t-1\rangle}(X, R),$$

called the level-t meta-indiscernibility. For $r \in \mathrm{Rough}^{\langle t-1\rangle}(X, R)$ we denote its $\mathcal{E}^{(t)}$-equivalence class by $[r]_{\mathcal{E}^{(t)}}$.

Definition 28 (Iterated MetaRough Set of depth t). Let $t \geq 1$ and let

$$\mathcal{C} \subseteq \mathrm{Rough}^{\langle t-1\rangle}(X, R)$$

be any family of level-$(t-1)$ rough objects.

The meta-lower and meta-upper approximations of $\mathcal{C}$ with respect to $\mathcal{E}^{(t)}$ are defined by

$$\underline{C}^{\mathcal{E}^{(t)}} := \{ r \in \mathsf{Rough}^{\langle t-1\rangle}(X,R) \mid [r]_{\mathcal{E}^{(t)}} \subseteq \mathcal{C} \},$$

$$\overline{C}^{\mathcal{E}^{(t)}} := \{ r \in \mathsf{Rough}^{\langle t-1\rangle}(X,R) \mid [r]_{\mathcal{E}^{(t)}} \cap \mathcal{C} \neq \emptyset \}.$$

The pair

$$\mathbf{R}^{(t)}(\mathcal{C}) := \left(\underline{\mathcal{C}}^{\mathcal{E}^{(t)}}, \overline{\mathcal{C}}^{\mathcal{E}^{(t)}}\right) \in \mathsf{Rough}^{\langle t\rangle}(X,R)$$

is called the Iterated Meta Rough Set of depth t generated by $\mathcal{C}$. For $t = 1$ this coincides with the Meta Rough Set from Definition 18, where $\mathcal{E}^{(1)}$ plays the role of the meta-indiscernibility on $\mathsf{Rough}(X,R)$.

Example 11 (Iterated MetaRough Set (depth 2): course pass-status aggregated by classes and by a school-level summary). Let $X = \{s_1, s_2, s_3, s_4\}$ be students in two homerooms (indiscernibility relation R):

$$[s_1]_R = [s_2]_R = \{s_1, s_2\},$$
$$[s_3]_R = [s_4]_R = \{s_3, s_4\}.$$

Consider two observed pass-sets for a mock exam:

$$U_1 = \{s_1\}, \qquad U_2 = \{s_3, s_4\}.$$

We first compute their rough approximations in (X,R).

For $U_1 = \{s_1\}$:

$$\underline{U_1} = \{x \in X \mid [x]_R \subseteq U_1\}.$$

The R-classes are $\{s_1, s_2\}$ and $\{s_3, s_4\}$. Neither $\{s_1, s_2\} \subseteq \{s_1\}$ nor $\{s_3, s_4\} \subseteq \{s_1\}$ holds, so no class is fully contained in U_1. Hence

$$\underline{U_1} = \emptyset.$$

The upper approximation is

$$\overline{U_1} = \{x \in X \mid [x]_R \cap U_1 \neq \emptyset\}.$$

Here $[s_1]_R = \{s_1, s_2\}$ meets U_1, so both s_1 and s_2 are included, while $[s_3]_R = \{s_3, s_4\}$ does not meet U_1. Thus

$$\overline{U_1} = \{s_1, s_2\}.$$

So the rough object associated to U_1 is

$$r_{U_1} = (\underline{U_1}, \overline{U_1}) = (\emptyset, \{s_1, s_2\}).$$

For $U_2 = \{s_3, s_4\}$:

$$\underline{U_2} = \{x \in X \mid [x]_R \subseteq U_2\}.$$

Now $[s_3]_R = [s_4]_R = \{s_3, s_4\} \subseteq U_2$, so both s_3 and s_4 are in the lower approximation:

$$\underline{U_2} = \{s_3, s_4\}.$$

The upper approximation is again

$$\overline{U_2} = \{x \in X \mid [x]_R \cap U_2 \neq \emptyset\}.$$

The class $\{s_3, s_4\}$ meets U_2 and is the only such class, hence

$$\overline{U_2} = \{s_3, s_4\}.$$

Thus the rough object for U_2 is

$$r_{U_2} = (\underline{U_2}, \overline{U_2}) = (\{s_3, s_4\}, \{s_3, s_4\}).$$

Level 1 (MetaRough over class-level rough objects). Form the level-1 family

$$\mathcal{C} := \{r_{U_1}, r_{U_2}\} \subseteq \mathsf{Rough}^{\langle 0\rangle}(X,R) = \mathsf{Rough}(X,R),$$

and choose the level-1 meta-indiscernibility $\mathcal{E}^{(1)}$ on $\mathsf{Rough}(X,R)$ to be equality:

$$r\, \mathcal{E}^{(1)}\, r' \quad \Leftrightarrow \quad r = r'.$$

For this choice, each equivalence class is a singleton:

$$[r]_{\mathcal{E}^{(1)}} = \{r\} \quad (\forall r \in \mathsf{Rough}(X,R)).$$

The level-1 meta-lower approximation is $= \{ r \mid \{r\} \subseteq \mathcal{C} \} = \mathcal{C}$.

Similarly, the level-1 meta-upper approximation is

$$\overline{\mathcal{C}}^{\mathcal{E}^{(1)}} = \{ r \in \mathsf{Rough}(X,R) \mid [r]_{\mathcal{E}^{(1)}} \cap \mathcal{C} \neq \emptyset \} = \{ r \mid \{r\} \cap \mathcal{C} \neq \emptyset \} = \mathcal{C}.$$

Hence the MetaRough Set of $\mathcal{C}$ at depth 1 is

$$\boldsymbol{R}^{(1)}(\mathcal{C}) = \left(\underline{\mathcal{C}}^{\mathcal{E}^{(1)}}, \overline{\mathcal{C}}^{\mathcal{E}^{(1)}}\right) = (\mathcal{C}, \mathcal{C}) \in \mathsf{Rough}^{\langle 1\rangle}(X,R).$$

Informally, this is a rough summary of class-level pass-statuses.

Level 2 (MetaRough over class-summary patterns). Now move up one level. The level-1 universe is

$$\mathsf{Rough}^{\langle 1\rangle}(X,R)$$
$$=\mathcal{P}(\mathsf{Rough}(X,R))\times\mathcal{P}(\mathsf{Rough}(X,R)),$$

so $\boldsymbol{R}^{(1)}(\mathcal{C})=(\mathcal{C},\mathcal{C})$ is a single level-1 rough object.

Consider the level-2 family consisting of this single object:

$$\mathcal{D}:=\{\boldsymbol{R}^{(1)}(\mathcal{C})\}$$
$$=\{(\mathcal{C},\mathcal{C})\}\subseteq \mathsf{Rough}^{\langle 1\rangle}(X,R).$$

Choose the level-2 meta-indiscernibility $\mathcal{E}^{(2)}$ on $\mathsf{Rough}^{\langle 1\rangle}(X,R)$ again to be equality. Then for any $\boldsymbol{r}\in\mathsf{Rough}^{\langle 1\rangle}(X,R)$ we have $[\boldsymbol{r}]_{\mathcal{E}^{(2)}}=\{\boldsymbol{r}\}$ and therefore

$$\underline{\mathcal{D}}^{\mathcal{E}^{(2)}}$$
$$\underline{\mathcal{C}}^{\mathcal{E}^{(1)}}=\{\,r\in\mathsf{Rough}(X,R)\mid [r]_{\mathcal{E}^{(1)}}\subseteq\mathcal{C}\,\}$$
$$=\{\,\boldsymbol{r}\mid [\boldsymbol{r}]_{\mathcal{E}^{(2)}}\subseteq\mathcal{D}\,\}=\mathcal{D},$$
$$\overline{\mathcal{D}}^{\mathcal{E}^{(2)}}$$
$$=\{\,\mathbf{r}\mid [\mathbf{r}]_{\mathcal{E}^{(2)}}\cap\mathcal{D}\neq\emptyset\,\}=\mathcal{D}.$$

Thus the depth-2 Iterated MetaRough Set generated by $\mathcal{D}$ is

$$\mathbf{R}^{(2)}(\mathcal{D})$$
$$=\left(\underline{\mathcal{D}}^{\mathcal{E}^{(2)}},\overline{\mathcal{D}}^{\mathcal{E}^{(2)}}\right)$$
$$=(\mathcal{D},\mathcal{D})\ \in\ \mathsf{Rough}^{\langle 2\rangle}(\mathrm{X},\mathrm{R}).$$

Interpretation: level 1 records rough summaries at the class level (two homerooms), while level 2 treats identical class-summary patterns as indiscernible from the viewpoint of a higher aggregation unit (such as a school or exam board). In this simple example the school-level family happens to consist of a single such pattern.

Proposition 2 (Sandwich property at every depth). Let $\mathrm{t}\geq 1$ and let $\mathcal{C}\subseteq\mathsf{Rough}^{\langle \mathrm{t}-1\rangle}(\mathrm{X},\mathrm{R})$ be any family of level-$(\mathrm{t}-1)$ rough objects. Then $\underline{\mathcal{C}}^{\mathcal{E}^{(\mathrm{t})}}\ \subseteq\ \mathcal{C}\ \subseteq\ \overline{\mathcal{C}}^{\mathcal{E}^{(\mathrm{t})}}$.

Proof. We show the two inclusions separately.

First inclusion, $\underline{\mathcal{C}}^{\mathcal{E}^{(\mathrm{t})}}\subseteq\mathcal{C}$. Let $\mathrm{r}\in\underline{\mathcal{C}}^{\mathcal{E}^{(\mathrm{t})}}$. By definition,

$$[\mathrm{r}]_{\mathcal{E}^{(\mathrm{t})}}\subseteq\mathcal{C}.$$

Since $\mathrm{r}\in[\mathrm{r}]_{\mathcal{E}^{(\mathrm{t})}}$ (every element lies in its own equivalence class), it follows that $\mathrm{r}\in\mathcal{C}$. Hence $\underline{\mathcal{C}}^{\mathcal{E}^{(\mathrm{t})}}\subseteq\mathcal{C}$.

Second inclusion, $\mathcal{C}\subseteq\overline{\mathcal{C}}^{\mathcal{E}^{(\mathrm{t})}}$. Let $\mathrm{r}\in\mathcal{C}$. Then the intersection

$$[\mathrm{r}]_{\mathcal{E}^{(\mathrm{t})}}\cap\mathcal{C}$$

contains r and is therefore nonempty:

$$[\mathrm{r}]_{\mathcal{E}^{(\mathrm{t})}}\cap\mathcal{C}\supseteq\{\mathrm{r}\}\neq\emptyset.$$

By the definition of $\overline{\mathcal{C}}^{\mathcal{E}^{(\mathrm{t})}}$ we conclude that $\mathrm{r}\in\overline{\mathcal{C}}^{\mathcal{E}^{(\mathrm{t})}}$. Hence $\mathcal{C}\subseteq\overline{\mathcal{C}}^{\mathcal{E}^{(\mathrm{t})}}$.

Combining both inclusions yields

$$\underline{\mathcal{C}}^{\mathcal{E}^{(\mathrm{t})}}\ \subseteq\ \mathcal{C}\ \subseteq\ \overline{\mathcal{C}}^{\mathcal{E}^{(\mathrm{t})}},$$

as claimed.

Theorem 31 (Generalization of MetaRough and Iterated MetaStructure). For each $\mathrm{t}\geq 1$ set $\mathrm{U}^{(\mathrm{t})}:=\mathsf{Rough}^{\langle \mathrm{t}\rangle}(\mathrm{X},\mathrm{R})$, $\quad \mathrm{U}^{(0)}:=\mathsf{Rough}^{\langle 0\rangle}(\mathrm{X},\mathrm{R})=\mathsf{Rough}(\mathrm{X},\mathrm{R})$.

(a) For $\mathrm{t}=1$, the assignment $\mathcal{C}\ \mapsto\ \mathbf{R}^{(1)}(\mathcal{C})=\left(\underline{\mathcal{C}}^{\mathcal{E}^{(1)}},\overline{\mathcal{C}}^{\mathcal{E}^{(1)}}\right)$ is exactly the MetaRough Set construction of Definition 18applied to subsets $\mathcal{C}\subseteq\mathsf{Rough}(\mathrm{X},\mathrm{R})$.

(b) For each $\mathrm{t}\geq 1$, consider the operations $\Phi^{(\mathrm{t})}_{\mathrm{low}},\Phi^{(\mathrm{t})}_{\mathrm{up}}\colon\mathcal{P}\big(\mathrm{U}^{(\mathrm{t}-1)}\big)\to\mathrm{U}^{(\mathrm{t})},\quad \mathcal{C}\ \mapsto\left(\underline{\mathcal{C}}^{\mathcal{E}^{(\mathrm{t})}},\overline{\mathcal{C}}^{\mathcal{E}^{(\mathrm{t})}}\right)$. If level-$(\mathrm{t}-1)$ isomorphisms are required to preserve the meta-indiscernibility $\mathcal{E}^{(\mathrm{t})}$ (that is, $\mathrm{r}\mathcal{E}^{(\mathrm{t})}\mathrm{s}$ iff $\beta(\mathrm{r})\mathcal{E}^{(\mathrm{t})}\beta(\mathrm{s})$ for any level-$(\mathrm{t}-1)$ isomorphism β), then the family $(\mathrm{U}^{(0)},\mathrm{U}^{(1)},\ldots,\mathrm{U}^{(\mathrm{t})};\ \Phi^{(1)}_{\mathrm{low}},\Phi^{(1)}_{\mathrm{up}},\ldots,\Phi^{(\mathrm{t})}_{\mathrm{low}},\Phi^{(\mathrm{t})}_{\mathrm{up}})$ forms an Iterated MetaStructure of depth t in the sense of Definition 2.

Proof. (a) At depth $\mathrm{t}=1$ we have

$$\mathrm{U}^{(1)} = \mathsf{Rough}^{\langle 1\rangle}(\mathrm{X},\mathrm{R}) = \mathcal{P}(\mathsf{Rough}(\mathrm{X},\mathrm{R})) \times \mathcal{P}(\mathsf{Rough}(\mathrm{X},\mathrm{R})),$$

so elements of $\mathrm{U}^{(1)}$ are pairs of subsets of $\mathsf{Rough}(\mathrm{X},\mathrm{R})$. Given any $\mathcal{C} \subseteq \mathsf{Rough}(\mathrm{X},\mathrm{R})$, the pair

$$\left(\underline{\mathcal{C}}^{\mathcal{E}^{(1)}}, \overline{\mathcal{C}}^{\mathcal{E}^{(1)}}\right) \in \mathrm{U}^{(1)}$$

is exactly the MetaRough Set of $\mathcal{C}$ constructed using the meta-indiscernibility $\mathcal{E}^{(1)}$ as in Definition 18. Hence $\mathbf{R}^{(1)}$ recovers the MetaRough Set construction.

(b) Fix $\mathrm{t} \geq 1$. A level-t object is a pair of subsets

$$\mathbf{r}^{(\mathrm{t})} = (\mathcal{L}^{(\mathrm{t})}, \mathcal{U}^{(\mathrm{t})}) \in \mathrm{U}^{(\mathrm{t})} = \mathcal{P}\big(\mathrm{U}^{(\mathrm{t}-1)}\big) \times \mathcal{P}\big(\mathrm{U}^{(\mathrm{t}-1)}\big).$$

By Definition 28, both $\Phi^{(\mathrm{t})}_{\mathrm{low}}$ and $\Phi^{(\mathrm{t})}_{\mathrm{up}}$ are constructed from the equivalence relation $\mathcal{E}^{(\mathrm{t})}$ on $\mathrm{U}^{(\mathrm{t}-1)}$ via

$$\underline{\mathcal{C}}^{\mathcal{E}^{(\mathrm{t})}} = \{\, \mathrm{r} \in \mathrm{U}^{(\mathrm{t}-1)} \mid [\mathrm{r}]_{\mathcal{E}^{(\mathrm{t})}} \subseteq \mathcal{C} \,\},$$

$$\overline{\mathcal{C}}^{\mathcal{E}^{(\mathrm{t})}} = \{\, \mathrm{r} \in \mathrm{U}^{(\mathrm{t}-1)} \mid [\mathrm{r}]_{\mathcal{E}^{(\mathrm{t})}} \cap \mathcal{C} \neq \emptyset \,\}.$$

Let $\beta\colon \mathrm{U}^{(\mathrm{t}-1)} \to \mathrm{U}^{(\mathrm{t}-1)}$ be a level-$(\mathrm{t}-1)$ isomorphism in the sense of Definition 4, so by assumption β preserves $\mathcal{E}^{(\mathrm{t})}$-equivalence classes: for all $\mathrm{r}, \mathrm{s} \in \mathrm{U}^{(\mathrm{t}-1)}$,

$$\mathrm{r}\ \mathcal{E}^{(\mathrm{t})}\ \mathrm{s} \iff \beta(\mathrm{r})\ \mathcal{E}^{(\mathrm{t})}\ \beta(\mathrm{s}).$$

Take any $\mathcal{C} \subseteq U^{(t-1)}$. Consider the image $\beta[\mathcal{C}] \subseteq U^{(t-1)}$. For the lower approximation we have

$$\underline{\beta[\mathcal{C}]}^{\mathcal{E}^{(t)}} = \left\{ u \in U^{(t-1)} \mid [u]_{\mathcal{E}^{(t)}} \subseteq \beta[\mathcal{C}] \right\}.$$

By substituting $u = \beta(r)$ and using preservation of equivalence classes,

$$[u]_{\mathcal{E}^{(t)}} = [\beta(r)]_{\mathcal{E}^{(t)}} = \beta\big([r]_{\mathcal{E}^{(t)}}\big),$$

so

$$[u]_{\mathcal{E}^{(t)}} \subseteq \beta[\mathcal{C}] \iff \beta\big([r]_{\mathcal{E}^{(t)}}\big) \subseteq \beta[\mathcal{C}] \iff [r]_{\mathcal{E}^{(t)}} \subseteq \mathcal{C},$$

because β is a bijection. Hence

$$u \in \underline{\beta[\mathcal{C}]}^{\mathcal{E}^{(t)}} \iff u = \beta(r) \text{ for some } r \in \underline{\mathcal{C}}^{\mathcal{E}^{(t)}}.$$

Therefore

$$\underline{\beta[\mathcal{C}]}^{\mathcal{E}^{(t)}} = \beta\left[\underline{\mathcal{C}}^{\mathcal{E}^{(t)}}\right].$$

A completely analogous elementwise argument shows

$$\overline{\beta[\mathcal{C}]}^{\mathcal{E}^{(t)}} = \beta\left[\overline{\mathcal{C}}^{\mathcal{E}^{(t)}}\right].$$

Consequently, the action of β on $U^{(t-1)}$ lifts to an action on $U^{(t)}$ that commutes with the maps

$$\mathcal{C} \mapsto \left(\underline{\mathcal{C}}^{\mathcal{E}^{(t)}}, \overline{\mathcal{C}}^{\mathcal{E}^{(t)}}\right) = \Phi^{(t)}_{\mathrm{low}}(\mathcal{C}) = \Phi^{(t)}_{\mathrm{up}}(\mathcal{C}).$$

This is precisely the naturality (isomorphism-invariance) condition required in Definition 4. Since this holds at each level t, the tower

$$(U^{(0)}, U^{(1)}, \ldots, U^{(t)}; \Phi^{(1)}_{\mathrm{low}}, \Phi^{(1)}_{\mathrm{up}}, \ldots, \Phi^{(t)}_{\mathrm{low}}, \Phi^{(t)}_{\mathrm{up}})$$

constitutes an Iterated MetaStructure of depth t.

Theorem 32 (Monotonicity at every meta-level). Let $\mathrm{t} \geq 1$ and let $\mathcal{E}^{(\mathrm{t})}$ and $\mathsf{Rough}^{\langle \mathrm{t}-1\rangle}(\mathrm{X},\mathrm{R})$ be as in Definition 27. For all families $\mathrm{C}, \mathrm{D} \subseteq \mathsf{Rough}^{\langle \mathrm{t}-1\rangle}(\mathrm{X},\mathrm{R})$ with $\mathrm{C} \subseteq \mathrm{D}$ we have $\underline{\mathrm{C}}^{\mathcal{E}^{(\mathrm{t})}} \subseteq \underline{\mathrm{D}}^{\mathcal{E}^{(\mathrm{t})}}$, $\overline{\mathrm{C}}^{\mathcal{E}^{(\mathrm{t})}} \subseteq \overline{\mathrm{D}}^{\mathcal{E}^{(\mathrm{t})}}$.

Proof. Let $\mathrm{C} \subseteq \mathrm{D}$.

For the meta-lower approximation, take $\mathrm{r} \in \underline{\mathrm{C}}^{\mathcal{E}^{(\mathrm{t})}}$. By Definition 28,

$$[\mathrm{r}]_{\mathcal{E}^{(\mathrm{t})}} \subseteq \mathrm{C}.$$

From $\mathrm{C} \subseteq \mathrm{D}$ it follows that

$$[\mathrm{r}]_{\mathcal{E}^{(\mathrm{t})}} \subseteq \mathrm{D},$$

so $\mathrm{r} \in \underline{\mathrm{D}}^{\mathcal{E}^{(\mathrm{t})}}$. Hence $\underline{\mathrm{C}}^{\mathcal{E}^{(\mathrm{t})}} \subseteq \underline{\mathrm{D}}^{\mathcal{E}^{(\mathrm{t})}}$.

For the meta-upper approximation, let $r \in \overline{C}^{\mathcal{E}^{(t)}}$. Then by Definition 28

$$[r]_{\mathcal{E}^{(t)}} \cap C \neq \emptyset.$$

Since $C \subseteq D$, we have

$$[r]_{\mathcal{E}^{(t)}} \cap D \supseteq [r]_{\mathcal{E}^{(t)}} \cap C \neq \emptyset,$$

hence $r \in \overline{D}^{\mathcal{E}^{(t)}}$. Therefore $\overline{C}^{\mathcal{E}^{(t)}} \subseteq \overline{D}^{\mathcal{E}^{(t)}}$.

Theorem 33 (Idempotence of iterated meta-approximations). Let $\mathrm{t} \geq 1$ and let $\mathcal{E}^{(\mathrm{t})}$ be as above.

For every $C \subseteq \mathrm{Rough}^{\langle t-1 \rangle}(X, R)$ we have $\underline{\left(\underline{C}^{\mathcal{E}^{(t)}}\right)}^{\mathcal{E}^{(t)}} = \underline{C}^{\mathcal{E}^{(t)}}$, $\overline{\left(\overline{C}^{\mathcal{E}^{(t)}}\right)}^{\mathcal{E}^{(t)}} = \overline{C}^{\mathcal{E}^{(t)}}$.

Proof. We prove idempotence for the meta-lower approximation; the argument for the meta-upper approximation is analogous.

Set $L := \underline{C}^{\mathcal{E}^{(t)}}$. By Proposition 2,

$$L = \underline{C}^{\mathcal{E}^{(t)}} \subseteq C.$$

Applying Theorem 78 with $L \subseteq C$ yields

$$\underline{L}^{\mathcal{E}^{(t)}} \subseteq \underline{C}^{\mathcal{E}^{(t)}} = L.$$

Thus

$$\underline{L}^{\mathcal{E}^{(t)}} \subseteq L.$$

For the converse inclusion, let $r \in L = \underline{C}^{\mathcal{E}^{(t)}}$. Then by definition

$$[r]_{\mathcal{E}^{(t)}} \subseteq C.$$

For any $s \in [r]_{\mathcal{E}^{(t)}}$ we have $[s]_{\mathcal{E}^{(t)}} = [r]_{\mathcal{E}^{(t)}}$, so

$$[s]_{\mathcal{E}^{(t)}} \subseteq C.$$

Hence $s \in \underline{C}^{\mathcal{E}^{(t)}} = L$. Since this holds for all $s \in [r]_{\mathcal{E}^{(t)}}$, we obtain

$$[r]_{\mathcal{E}^{(t)}} \subseteq L.$$

Therefore $r \in \underline{L}^{\mathcal{E}^{(t)}}$. As $r \in L$ was arbitrary, we conclude

$$L \subseteq \underline{L}^{\mathcal{E}^{(t)}}.$$

Combining the two inclusions gives

$$\underline{L}^{\mathcal{E}^{(t)}} = L = \underline{C}^{\mathcal{E}^{(t)}},$$

i.e.,

$$\underline{\left(\underline{C}^{\mathcal{E}^{(t)}}\right)}^{\mathcal{E}^{(t)}} = \underline{C}^{\mathcal{E}^{(t)}}.$$

For the meta-upper operator, define $U := \overline{C}^{\mathcal{E}^{(t)}}$. By Proposition 2,

$$C \subseteq U = \overline{C}^{\mathcal{E}^{(t)}},$$

and by Theorem 34,

$$\overline{C}^{\mathcal{E}^{(t)}} \subseteq \overline{U}^{\mathcal{E}^{(t)}}.$$

Since $U = \overline{C}^{\mathcal{E}^{(t)}}$, this gives $U \subseteq \overline{U}^{\mathcal{E}^{(t)}}$.

Conversely, let $r \in \overline{U}^{\mathcal{E}^{(t)}}$. Then

$$[r]_{\mathcal{E}^{(t)}} \cap U \neq \emptyset.$$

Take $s \in [r]_{\mathcal{E}^{(t)}} \cap U$. As $s \in U = \overline{C}^{\mathcal{E}^{(t)}}$, we have

$$[s]_{\mathcal{E}^{(t)}} \cap C \neq \emptyset.$$

But $[s]_{\mathcal{E}^{(t)}} = [r]_{\mathcal{E}^{(t)}}$, so

$$[r]_{\mathcal{E}^{(t)}} \cap C \neq \emptyset,$$

which implies $r \in \overline{C}^{\mathcal{E}^{(t)}} = U$. Hence $\overline{U}^{\mathcal{E}^{(t)}} \subseteq U$.

Together with $U \subseteq \overline{U}^{\mathcal{E}^{(t)}}$ we obtain

$$\overline{U}^{\mathcal{E}^{(t)}} = U = \overline{C}^{\mathcal{E}^{(t)}}.$$

Theorem 80 (Exactness and unions of meta-indiscernibility classes). Let $t \geq 1$ and let $\mathcal{E}^{(t)}$ be an equivalence relation on $\mathrm{Rough}^{\langle t-1 \rangle}(X, R)$. For $C \subseteq \mathrm{Rough}^{\langle t-1 \rangle}(X, R)$ the following are equivalent.

1. C is meta-exact: $\underline{C}^{\mathcal{E}^{(t)}} = \overline{C}^{\mathcal{E}^{(t)}}$.
2. C is a union of $\mathcal{E}^{(t)}$-equivalence classes: there exists an index set I and $\mathcal{E}^{(t)}$-classes $\{B_i\}_{i \in I}$ such that $C = \bigcup_{i \in I} B_i$.

Proof. (i)⇒(ii). Assume $\underline{C}^{\mathcal{E}^{(t)}} = \overline{C}^{\mathcal{E}^{(t)}}$. By Proposition 2,

$$C \subseteq \overline{C}^{\mathcal{E}^{(t)}} = \underline{C}^{\mathcal{E}^{(t)}}.$$

Hence

$$C \subseteq \underline{C}^{\mathcal{E}^{(t)}} \subseteq C,$$

so $C = \underline{C}^{\mathcal{E}^{(t)}}$.

Let $r \in C$. Then $r \in \underline{C}^{\mathcal{E}^{(t)}}$ and therefore

$$[r]_{\mathcal{E}^{(t)}} \subseteq C.$$

Thus C contains the entire $\mathcal{E}^{(t)}$-class of each of its elements. Consequently,

$$C = \bigcup_{r \in C} [r]_{\mathcal{E}^{(t)}},$$

which is a union of equivalence classes (possibly with repetitions, which do not affect the union). This shows (ii).

(ii)⇒(i). Assume $C = \bigcup_{i \in I} B_i$ where each B_i is an $\mathcal{E}^{(t)}$-equivalence class. We first show that $\underline{C}^{\mathcal{E}^{(t)}} \subseteq C$ and $C \subseteq \overline{C}^{\mathcal{E}^{(t)}}$ hold by Proposition 2. It remains to show $C \subseteq \underline{C}^{\mathcal{E}^{(t)}}$ and $\overline{C}^{\mathcal{E}^{(t)}} \subseteq C$.

Let $r \in C$. Then $r \in B_{i_0}$ for some $i_0 \in I$, and $B_{i_0} = [r]_{\mathcal{E}^{(t)}}$ is fully contained in C because C is a union of the classes B_i. Hence

$$[r]_{\mathcal{E}^{(t)}} \subseteq C,$$

so $r \in \underline{C}^{\mathcal{E}^{(t)}}$. Thus $C \subseteq \underline{C}^{\mathcal{E}^{(t)}}$. Together with $\underline{C}^{\mathcal{E}^{(t)}} \subseteq C$ we obtain

$$\underline{C}^{\mathcal{E}^{(t)}} = C.$$

Now let $r \in \overline{C}^{\mathcal{E}^{(t)}}$. Then

$$[r]_{\mathcal{E}^{(t)}} \cap C \neq \emptyset,$$

so there exists $s \in [r]_{\mathcal{E}^{(t)}} \cap C$. As C is a union of $\mathcal{E}^{(t)}$-classes, $s \in C$ implies $[s]_{\mathcal{E}^{(t)}} \subseteq C$. But $[s]_{\mathcal{E}^{(t)}} = [r]_{\mathcal{E}^{(t)}}$, so $[r]_{\mathcal{E}^{(t)}} \subseteq C$ and therefore $r \in C$. Hence $\overline{C}^{\mathcal{E}^{(t)}} \subseteq C$, and together with $C \subseteq \overline{C}^{\mathcal{E}^{(t)}}$ we obtain

$$\overline{C}^{\mathcal{E}^{(t)}} = C.$$

Thus

$$\underline{C}^{\mathcal{E}^{(t)}} = C = \overline{C}^{\mathcal{E}^{(t)}},$$

so C is meta-exact and (i) holds.

Conclusion

In this paper, we introduce the concepts of MetaFuzzy Sets, MetaNeutrosophic Sets, MetaSoft Sets, and MetaRough Sets by extending classical Fuzzy Sets, Neutrosophic Sets, Soft Sets, and Rough Sets through the application of MetaStructure and Iterated MetaStructure.

Future work

This work is currently restricted to finite domains, where naive constructions incur high combinatorial costs. Future research should therefore explore measure-theoretic extensions, scalable algorithms, and principled normalization schemes tailored to diverse application settings and robustness requirements. In addition, future studies may investigate whether similar extensions can be developed for other classes of sets, such as Plithogenic Sets, Vague Sets, and Near Sets. We also anticipate further developments involving extensions based on graphs, hypergraphs, superhypergraphs, hyperstructures, and hyper-algebra.

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Conflict of interest

The authors declare that there are no conflicts of interest associated with this research or its publication.

Disclaimer (Regarding manuscript formatting)

In this paper, we converted the LaTeX source into a Word document using tools such as Pandoc. While every effort has been made to ensure accurate formatting, minor errors may have been introduced during the conversion process. We kindly ask for your understanding.

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