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Research Article

Proof of Fermat's Last Theorem when it is already proven for the exponent *n=3*

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ABSTRACT

Pythagorean triples have been known for a long time. Fermat's last theorem sets the limit of any two integers summing up to the third, all raised to the same power, which, according to the theorem, can never be more than 2. After centuries of effort and waiting, Andrew Wiles provided a large and complicated proof of this theorem; this article provides an elementary one. *Mathematical subject classification: 11D41, 11D45.*

Introduction

Among the works compiled by Euclid, the related to the Pythagorean triangle (Clapham and Nicholson, 2009; Fitzpatrick, 2008) was one of the most prominent and useful. Pythagoras stated that the sum of the squares on two arms adjacent to the correct angle of a right-angled triangle is equal to the square on the hypotenuse. The theorem is one of the most ancient mathematical results. Evidence shows that the Babylonians and the Chinese knew it 1000 years before Pythagoras. (Krantz, 2006) The Egyptians used at least one combination of Pythagorean triples, 3, 4, and 5, to construct corners of pyramids. For centuries, mathematicians were pondering whether this type of relationship was possible with powers higher than 2 over the triplet bases. Fermat's last theorem, acceptably conjectured for centuries, states its impossibility. Andrew Wiles provided his colossal and complicated proof just before the advent of the 21st century. (Wiles, 1995) We wonder whether Fermat himself worked out a proof that he could not fit into the margin of Bachet's translation of Diophantus's Arithmetica. (Krantz, 2006) We provide an elementary proof.

Andrew Wiles proved enough of the Taniyama-Shimura-Weil conjecture to prove Fermat's Last Theorem. The conjecture, which later mathematicians after Wiles proved as the modularity theorem, appeared around 1955. Gerhard Frey G, in 1984, noticed an apparent link between the conjecture and Fermat's Last Theorem (Frey G, 1986; Wikipedia contributors, 2019; Wiles AJ, 1995).

Proofs of Fermat's Last Theorem for specific exponents date back to the time of Fermat himself, who outline a proof for $n = 4$ by infinite descent, alternative proofs given by many, from Bessy (de Bessy F, 1676) to Dolan (2011), Euler not missing out. In 1770, Leonhard Euler gave a proof for $n = 3$, but his proof required a missing lemma, which was later proven and provided by other mathematicians. (Friedberg R, 1994) Independent and alternative proofs were published later. Independent and alternative proofs for $n = 5,6,7$, etc., were published over the years. (Wikipedia contributors, 2018).

Andrew Wiles's proof of Fermat's Last Theorem is massive in size; the mathematics used in it is inaccessible to the understanding of generally educated people and requires most modern mathematics developed only in the later part of the twentieth century. Earlier proofs for specific exponents should have given rise to the thought that the possibility of a solution to Fermat's equation has ceased from $n = 3$ onwards; we rearranged and

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modified the equation to show why that is so general. In contrast to Andrew Wiles' proof, our method uses elementary mathematics to explore and understand most of the general educated public for the first time, leaving no doubt about Fermat's Last Theorem.

Proof of Fermat's Last Theorem

Theorem 2.1. Fermat's last theorem (Krantz, 2006; Wikipedia contributors, 2018; 2019) states that if x; y; z; n are positive integers then

$x^n + y^n \neq z^n$ when $n > 2$.

Proof. Definitely if n = 0, LHS = 1 + 1 = 2 will always be greater than RHS = 1. If n = 1, LHS will be equal to RHS when the sum of integers on LHS is put on RHS.

Let us assume, n and $x < y < z$ are positive integers, and in certain conditions,

$$x^n + y^n = z^n \tag{2.1}$$

We know from the binomial theorem,

$(x+y)^n = x^n + {}^nC_1 x^{n-1} \ldots + {}^nC_r x^{n-r} \ldots + y^n$ So for $n > 1$, we have,

$$(x+y)^n > x^n + y^n$$

Or,

$(x+y)^n > z^n$ So,

$$x + y > z \tag{2.2}$$

$$2y > x + y > z \tag{2.3}$$

Also, for the positive integers,

$$z - x > z - y \geq 1 \tag{2.4}$$

$$z - x \geq 2 \tag{2.5}$$

As from (2.1), we have $x^n = z^n - y^n = (z - y)(z^{n-1} + z^{n-2}y + z^{n-3}y^2 \ldots + y^{n-1})$, for showing lower bound of the solution to (2.1), we get,

$$x \geq 1 \tag{2.6}$$

$$y \geq 2 \tag{2.7}$$

$$z \geq 3 \tag{2.8}$$

Again rearranging (2.1), we have,

$$y^n = z^n - x^n = (z - x)(z^{n-1} + z^{n-2}x + z^{n-3}x^2 \ldots + x^{n-1})$$

$= (z-x)z^{n-1}(1 + \frac{x}{z} + \cdots + \frac{x^{n-1}}{z^{n-1}})$Or,

$$\ln y^n = \ln((z-x)z^{n-1}\left(1 + \frac{x}{z} + \cdots + \frac{x^{n-1}}{z^{n-1}}\right))$$

$$= \ln(z-x) + \ln z^{n-1} + \ln\left(1 + \frac{x}{z} + \cdots + \frac{x^{n-1}}{z^{n-1}}\right)$$

Or,

$$1 = \frac{\ln(z-x)}{n\ln y} + \frac{n-1}{n} \cdot \frac{\ln z}{\ln y} + \frac{\ln\left(1+\frac{x}{z}+\cdots+\frac{x^{n-1}}{z^{n-1}}\right)}{n\ln y} \tag{2.9}$$

Applying the findings from the previous inequalities in (2.9), we get $\ln(z-x) \geq \ln 2 \geq 0.693$ and $\ln\left(1 + \frac{x}{z} + \cdots + \frac{x^{n-1}}{z^{n-1}}\right) > \ln 1 > 0$. So, $\frac{\ln(z-x)}{n\ln y} > 0$ and $\frac{\ln\left(1+\frac{x}{z}+\cdots+\frac{x^{n-1}}{z^{n-1}}\right)}{n\ln y} > 0$. Also, $1 < \frac{\ln z}{\ln y} < \frac{\ln 2y}{\ln y} = 1 + \frac{\ln 2}{\ln y}$.

As we stated at the beginning of this proof section, when $n = 0$, the equation is not possible; when $n = 1$, we get $\frac{n-1}{n} = 0$; when $n = 2$, we get $\frac{n-1}{n} = \frac{1}{2}$; with the increase of n, we get an increasing $\frac{n-1}{n} \to 1$. But Fermat's Last Theorem was proven for $n = 3$ first by Euler. (Friedberg, 1994; Ribenboim, 2009; Hellegouarch, 2001) So (2.9) is not possible for $n = 3$; consequently, with increasing $\frac{n-1}{n}$, equation (2.9) is not possible for $n > 3$. This should generally prove the Fermat's Last Theorem for any $n > 2$.

But we would like to make it look clearer. It would be better for understanding the equation if we restrict n to one place only, and show other terms falling within ranges. For clarity and brevity, we introduce a few more notations. If,

$$1 < \frac{\ln z}{\ln y} = 1 + \delta < \frac{\ln 2y}{\ln y} = 1 + \frac{\ln 2}{\ln y}$$

We have

$0 < \delta < \frac{\ln 2}{\ln y} < \frac{0.693}{\ln y}$ Also, when

$$1 > \alpha > 0 > -\alpha > \frac{\ln\left(1 - \frac{1}{2}\right)}{\ln y} = -\frac{0.693}{\ln y}$$

So that,

$$\frac{\ln(2y - x)}{\ln y} = \frac{\ln y}{\ln y} + \frac{\ln 2}{\ln y} + \frac{\ln\left(1 - \frac{x}{2y}\right)}{\ln y}$$

$$= 1 + \frac{0.693}{\ln y} - \alpha > \frac{\ln(z-x)}{\ln y} = \eta$$

$$\geq \frac{\ln 2}{\ln y} > \delta > 0$$

And,

$$1 = \frac{\ln y}{\ln y} = \frac{\ln \frac{2y}{2}}{\ln y} > \frac{\ln \frac{z}{2}}{\ln y} \geq \frac{\ln \frac{z}{z-x}}{\ln y} = \frac{\ln \frac{1}{1-\frac{x}{z}}}{\ln y}$$
$$> \frac{\ln\left(1 + \frac{x}{z} + \cdots + \frac{x^{n-1}}{z^{n-1}}\right)}{\ln y} = \epsilon > 0$$

where we have a natural logarithm of partial geometric series in the numerator, which is less than its infinite series. Rewriting (2.9), we get,

$$1 = \frac{n-1}{n} \cdot (1 + \delta) + \frac{1}{n} \cdot (\eta + \epsilon)$$

Or,

$$1 = 1 + \delta + \frac{1}{n} \cdot (\eta + \epsilon - 1 - \delta) \qquad (2.10)$$

To satisfy equation (2.10), we must have $\frac{1}{n} \cdot (\eta + \epsilon - 1 - \delta)$ negative to cancel out δ.

With the increase in n, we have to decrease $\frac{1}{n} \to 0$, gradually, rendering its effect negligible in equation (2.10) for the canceling out purpose. When Fermat's Last Theorem had already proven for $n = 3$, the equation (2.10) is not possible for $n = 3$; consequently, with $\frac{1}{n} \to 0$, equation (2.10) is not further possible for $n > 3$. This proves the Fermat's Last Theorem generally for any $n > 3$. □

Conclusion

For millennia, Pythagoras's theorem has remained one of the most useful propositions of geometry in practical life, e.g., for making corners, measuring distances on earth and space, and graphs of analytic geometry helping its progression. For centuries, Fermat's last theorem remained an enigma until Andrew Wiles declared its large, complicated, fascinating proof. This article provides an elementary proof of Fermat's last theorem.

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Conflict of Interest

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Data Availability Statements

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