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Research Article

A novel nonlinear stochastic differential equation model with quadratic and logarithmic drift

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ABSTRACT

This study introduces a new class of nonlinear stochastic differential equations (SDEs) with a logarithmic drift term, given by $dX_t = (\propto X_t + \beta X_t^2 + \gamma log X_t)dt + \sigma dW_t$. Assuming a lognormal distribution, we derive expressions for the mean and variance and also determine the stationary distribution. Furthermore, the proposed model is compared with existing SDE models using economic data, including exchange rates, inflation rates, and interest rates. Model performance is evaluated using the Akaike Information Criterion (AIC) and the Bayesian Information Criterion (BIC). The results show that the proposed SDE outperforms the competing models.

Introduction

Stochastic processes have become an important foundation in modern applied mathematics and quantitative finance due to their capability to model systems influenced by uncertainty and randomness. The use of Stochastic Differential Equations (SDEs) has expanded significantly over the years as researchers seek more realistic representations of phenomena with inherent randomness, structural complexity, and temporal dependence (Liao, 2024; Mahmoud et al., 2023). This expansion showcases a broader recognition that deterministic models, although analytically convenient, are often limited in capturing the dynamic variability observed in empirical data.

Modeling Financial data mostly requires the use of Stochastic Differential Equations (SDEs) to describe the dynamics of interest rates, asset prices, inflation rates, exchange rates, and other financial quantities. Most Stochastic Differential Equations (SDEs) are assumed to follow linear models. Nevertheless, financial data often exhibit nonlinear characteristics that linear models alone cannot capture.

Stochastic Differential Equations can be linear or nonlinear, depending on the nature of the system being modeled. Linear Models have the advantage of being analytically tractable, allowing for closed-form solutions. Nonlinear Models, on the other hand, require numerical methods for estimation and analysis. (McCamley et al., 2018). These models are particularly valuable for analyzing longitudinal data with unequally spaced measurement occasions (Hecht and Zitzmann, 2020).

One of the critical aspects of SDEs is their ability to incorporate randomness into deterministic models, which is essential for accurately representing real-world phenomena. For instance, stochastic models have been shown to provide more realistic insights into biological processes, such as the dynamics of infectious diseases, where traditional ordinary differential equations (ODEs) often fall short due to their deterministic nature (Marwa et al., 2019). The stochasticity introduced into these models can be achieved through various methods, including adding random noise or using stochastic perturbations to

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capture environmental fluctuations. (Marwa et al., 2019).

The theoretical foundation of continuous-time stochastic modelling traces back to Ito's development of stochastic calculus, which provided a mathematically coherent framework for integrating randomness into dynamic systems. Building upon these early contributions, contemporary research has extended SDEs to include path-dependence, memory effects, and non-standard noise structures, giving rise to Stochastic Delay Differential Equations (SDDEs) and related generalizations (Caraballo et al., 2024; de Feo et al., 2023). These models have proven valuable, particularly for capturing delayed responses, regime changes, and structural nonlinearities that occur in finance, economics, engineering, and biological systems.

In modelling financial data, SDEs have proven to be among the most important tools for characterizing the evolution of asset prices, interest rates, exchange rates and volatility. Recent studies have shown that classical linear drift formulations, such as Geometric Brownian Motion (GBM) and Ornstein – Uhlenbeck (OU) process, are limited in capturing key facts of financial markets, including heavy – tailed distributions, volatility clustering, asymmetric mean reversion, and structural breaks (Coulibaly et al., 2024; Nagarajan et al., 2025; Shen and Tang, 2025). These limitations have motivated the development of richer nonlinear SDE specifications, including regime-switching, fractional-order dynamics, and nonlinear drift structure (Chumpong et al., 2024; Oduselu-Hassan, 2025). Akintunde et al. (2020) introduced a Logarithmic mean reverting sine diffusion (LMRSD) process for modelling dynamics of nonlinear dynamical systems. The model was tested on two empirical datasets and compared with existing models; the study showed that it outperformed them, with the lowest values of the information criterion.

Motivated by these previous methodological and empirical considerations, this paper proposes a new class of nonlinear SDEs in which the drift term consists of a combination of linear, quadratic, and logarithmic terms. This proposed model captures a broader range of behaviors observed in financial markets, including volatility clustering, heavy tails, and skewness. This combination also introduces rich dynamics not present in traditional models such as the Geometric Brownian Motion (GBM) and the Ornstein-Uhlenbeck process and allows the model to capture diverse behaviors observed in empirical data. We derive the stationary distribution using the Fokker–Planck equation, while the mean and variance for this model are derived under the assumption that the process follows a lognormal distribution, which is common in financial applications.

Materials and method

Model Description

In this chapter, we present the proposed methodological framework. We begin with the nonlinear SDE. The SDE was formulated by adding a logarithmic term $\gamma log X_t$ to the drift term.

$$dX_t = (\propto X_t + \beta X_t^2 + \gamma log X_t)dt + \sigma dW_t \qquad (1)$$

Where:

X_t is the process of interest.

$\propto$, β, γ, σ are constants that oversee the drift and volatility of the process.

W_t is a Weiner process (or Brownian motion).

The noise term is assumed to follow standard Brownian motion, with constant volatility σ.

Mean

We assumed the distribution of X_t is stationary, so the time derivative of the mean is zero:

$$\frac{d}{dt} E[X_t] = 0 \qquad (2)$$

$$E(\propto X_t + \beta X_t^2 + \gamma log X_t)dt = 0 \qquad (3)$$

This gives the equation:

$$\propto \mathrm{E}[\mathrm{X}] + \beta \mathrm{E}[X^2] + \gamma \mathrm{E}[\log \mathrm{X}] = 0 \qquad (4)$$

We assumed a log-normal moment distribution:

Assume $X_t \sim LogNormal(\mu, \sigma_X^2)$, then:

$$E[X] = e^{\mu + \frac{\sigma_X^2}{2}} \quad (5)$$

$$E[X^2] = e^{2\mu + 2\sigma_X^2} \quad (6)$$

$$E[\log X] = \mu \quad (7)$$

Substituting 5,6, and 7 into equation (3), we have

$$\propto e^{\mu + \frac{\sigma_X^2}{2}} + \beta e^{2\mu + 2\sigma_X^2} + \gamma\mu = 0 \quad (8)$$

The mean in terms of (μ, σ_X^2) is,

$$E[X] = e^{\mu + \frac{\sigma_X^2}{2}} \quad (9)$$

Where μ satisfies $\propto e^{\mu + \frac{\sigma_X^2}{2}} + \beta e^{2\mu + 2\sigma_X^2} + \gamma\mu = 0$

Variance

Assuming $X_t \sim LogNormal(\mu, \sigma_X^2)$ of equation 5 and 6, the variance is, $Var(X_t) = E[X^2] - (E[X])^2$

$$e^{2\mu + 2\sigma_X^2} - (e^{\mu + \frac{\sigma_X^2}{2}})^2 \quad (10)$$

$$Var(X_t) = e^{2\mu + 2\sigma_X^2} - e^{2\mu + \sigma_X^2} \quad (11)$$

Factor out $e^{2\mu + \sigma_X^2}$,$Var(X_t) = e^{2\mu + \sigma_X^2}(e^{\sigma_X^2} - 1)$(12)

Stationary Distribution

We used the Fokker–Planck equation to find the stationary distribution of our nonlinear SDE. We begin with the Fokker–Planck equation for general SDE,

$$dX_t = \mu X_t dt + \sigma dW_t \quad (13)$$

The Fokker – Planck equation for the probability density $\pi(x,t)$ is,$\frac{\partial \pi(x,t)}{\partial t} = -\frac{\partial}{\partial x}[\mu(x)\pi(x,t)] + \frac{1}{2}\sigma^2 \frac{\partial^2 \pi(x,t)}{\partial x^2}$ (14)

For a stationary distribution $\frac{\partial \pi}{\partial t} = 0$, the equation becomes,

$$\frac{d}{dx}\left[-\mu(x)\pi(x) + \frac{1}{2}\frac{d}{dx}\{\sigma^2\pi(x)\}\right] = 0 \quad (15)$$

Integrating once, and assuming probability flux goes to zero at boundaries (natural boundary conditions), we get,

$$\mu(x)\pi(x) = \frac{1}{2}\frac{d}{dx}\{\sigma^2\pi(x)\} \quad (16)$$

For constant diffusion $\sigma(x) = \sigma$, this simplifies to,

$$\mu(x)\pi(x) = \frac{\sigma^2}{2}\frac{d\pi(x)}{dx} \quad (17)$$

Now, plugging in the drift of our nonlinear SDE

$$\mu(x) = \propto x + \beta x^2 + \gamma \log x \quad (18)$$

So, we solve,

$$(\propto x + \beta x^2 + \gamma \log x)\pi(x) = \frac{\sigma^2}{2}\frac{d\pi(x)}{dx} \quad (19)$$

Rewriting,

$$\frac{d\pi(x)}{dx} = \frac{2}{\sigma^2}(\propto x + \beta x^2 + \gamma \log x)\pi(x) \quad (20)$$

This is a first-order linear ODE in $\pi(x)$, and it can be solved via an integrating factor,

$$\frac{d\pi(x)}{dx} = f(x), \pi(x) \quad (21)$$

Where $f(x) = \frac{2}{\sigma^2}(\propto x + \beta x^2 + \gamma \log x)\pi(x)$ (22)

$$\pi(x) = C\exp(\int f(x)dx) \quad (23)$$

Now, computing the integral, we have,

$$\int f(x)dx = \frac{2}{\sigma^2}(\int \propto x dx + \int \beta x^2 dx \int \gamma \log x dx) \quad (24)$$

$$\frac{2}{\sigma^2}(\frac{\propto}{2}x^2 + \frac{\beta}{3}x^3 + \gamma \log x - \gamma x) \quad (25)$$

So, the stationary density is,

$$\pi(x) = C\exp\left[\frac{2}{\sigma^2}\left(\frac{\propto}{2}x^2 + \frac{\beta}{3}x^3 + \gamma \log x - \gamma x\right)\right] \quad (26)$$

Now we normalize the density to get the final form, we write,

$$\pi(x) = \frac{1}{Z}\exp\left[\frac{2}{\sigma^2}\left(\frac{\propto}{2}x^2 + \frac{\beta}{3}x^3 + \gamma \log x - \gamma x\right)\right] \quad (27)$$

Where Z is the normalizing constant,

$$Z = \int_0^\infty \exp\left[\frac{2}{\sigma^2}\left(\frac{\propto}{2}x^2 + \frac{\beta}{3}x^3 + \gamma \log x - \gamma x\right)\right] \quad (28)$$

In conclusion, the stationarity density is,

$$\pi(x) = \alpha\exp\left[\frac{2}{\sigma^2}\left(\frac{\propto}{2}x^2 + \frac{\beta}{3}x^3 + \gamma \log x - \gamma x\right)\right], x > 0 \quad (29)$$

Results and Discussion

The proposed model was applied, and compared with existing models, such as GBMP, LMRSL, MRP, and LMRP. Three data sets were used in this study. The data sets used are Nigerian Interest rate data (1970-2020), Nigerian Inflation rate data (1970-2020), and Nigerian Naira-to-US Dollar data (1970-2020). Figs. 1, 2, and 3 show the plots of the realizations of Nigerian Interest rate data, Nigerian Inflation rate data and Nigerian Naira-to-US Dollar data, respectively.

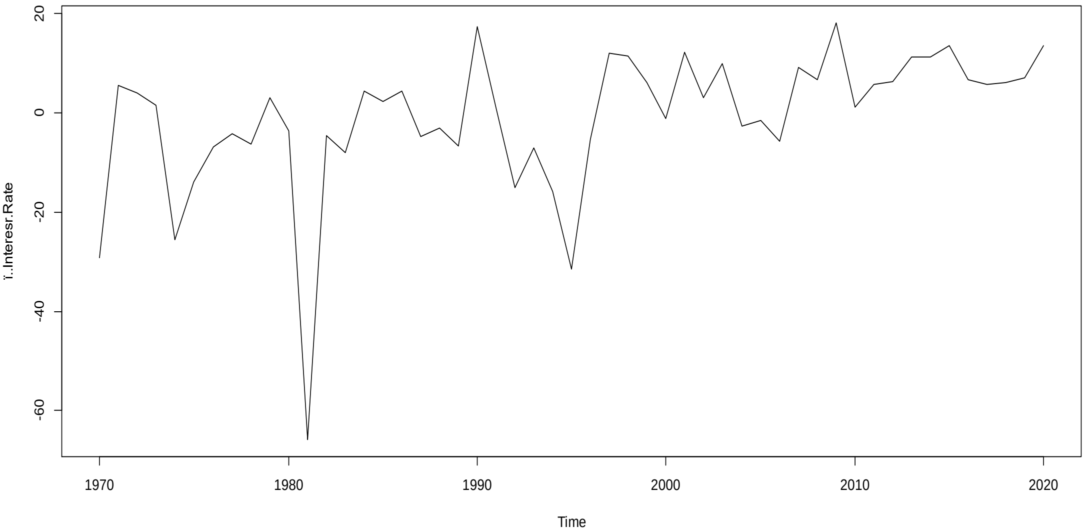

Fig. 1. Time plot of interest rate (1970-2020).

Table 1. Performance of the proposed model in comparison with the other existing models for interest rate.

Models	Theta 1	Theta 2	Theta 3	Theta 4	AIC	BIC
Geometric Brownian Motion Process (GBMP)	18.708535	4.307083	-	-	264.1226	267.9863
Vasicek Model	0.7334604	-0.0523150	13.1486770	-	405.5259	407.3895
Hull-White (extended Vasicek) Model	0.1525242	0.3283890	4.6961937	-	599.5245	601.3882
Mean Reverting Process (MRP)	-5.374235	4.114894	-	-	261.5663	265.4299
Mean Reverting Square Root Process (MRSRP)	-0.9374107	5.6694366	-	-	230.0693	233.9329
Mean Reverting Logarithmic Process (MRLP)	-7.006559	3.968649	-	-	259.5399	263.4036
Logarithmic Mean Reverting Processs (LMRP)	0.8416127	17.8034951	-	-	260.5464	264.41
Sine Diffusion Process	18.708535	4.307083	3.000000	-	266.1226	267.9863
Quadratic Loglinear Drift	-3.2325344	2.4831770	0.7823936	-1.2508919	8	7.863651

AIC: Akaike Information Criterion; BIC: Bayesian Information Criterion

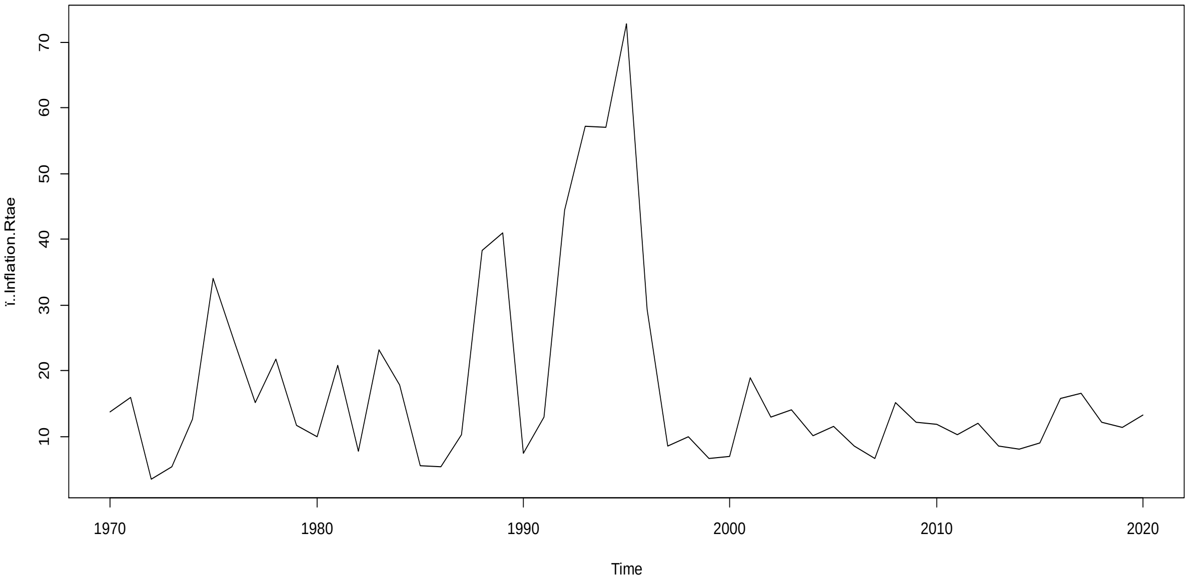

Fig. 2. Time plot of inflation rate (1970-2020).

Table 2. Performance of the proposed model in comparison with the other existing models for inflation rate.

Models	Theta 1	Theta 2	Theta 3	Theta 4	AIC	BIC
Geometric Brownian Motion Process (GBMP)	-1.3432274	0.8292275	-	-	381.5209	385.3449
Vasicek Model	0.3665149	17.4512875	11.3959955	-	383.5145	385.3386
Hull-White (extended Vasicek) Model	0.02879194	3.14479405	2.05850401	-	504.9414	506.7655
Mean Reverting Process (MRP)	45.117354	-1.080304	-	-	4	7.824046
Mean Reverting Square Root Process (MRSRP)	13.89840	2.99128	-	-	378.8435	382.6675
Mean Reverting Logarithmic Process (MRLP)	5.6024125	0.7526916	-	-	372.0306	375.8547
Logarithmic Mean Reverting Process (LMRP)	15.579575	3.904813	-	-	367.8736	371.6976
Sine Diffusion Model	-1.3432274	0.8292275	3.0000000	-	383.5209	385.3449
Quadratic Loglinear Drift	-0.4050182	0.7480697	0.6592802	-15.3437646	8	7.824046

AIC: Akaike Information Criterion; BIC: Bayesian Information Criterion

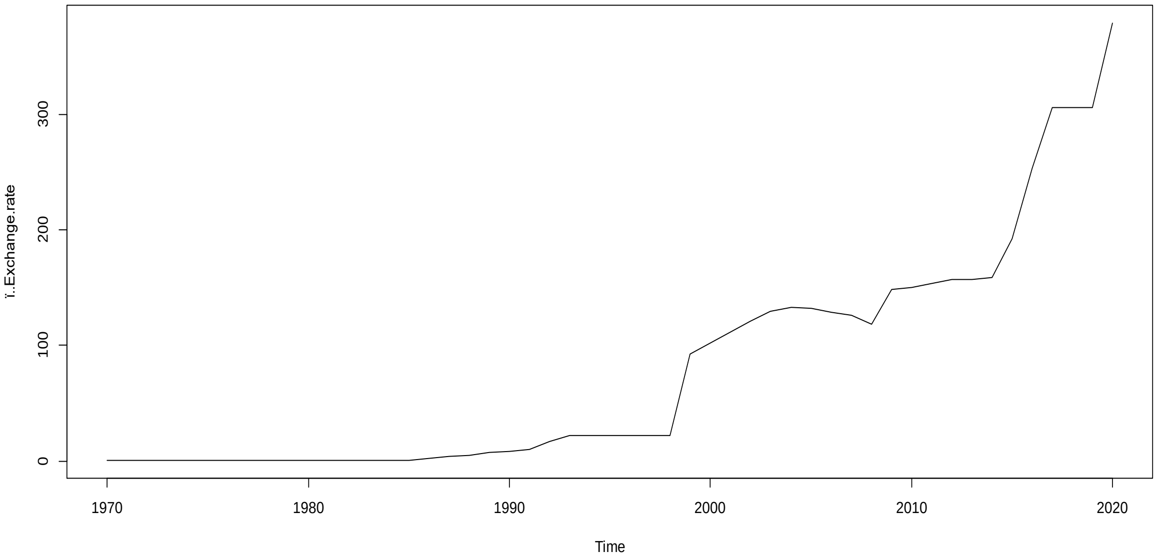

Fig. 3. Time plot of exchange rate.

Table 3. Performance of the proposed model in comparison with the other existing models for interest rate.

Exchange rate

Models	Theta 1	Theta 2	Theta 3	Theta 4	AIC	BIC
Geometric Brownian Motion Process (GBMP)	-0.5878979	0.5414701	-	-	345.2912	349.1153
Vasicek Model	-0.08521805	-15.58572705	16.66794661	-	420.7778	422.6018
Hull-White (extended Vasicek) Model	-0.02714862	1.04255587	-0.09606881	-	6	7.824046
Mean Reverting Process (MRP)	0.773846	1.118084	-	-	416.3496	420.1737
Mean Reverting Square Root Process (MRSRP)	2.379130	9.697673	-	-	496.878	500.7021
Mean Reverting Logarithmic Process (MRLP)	-0.5835662	-0.1563665	-	-	4	7.824046
Logarithmic Mean reverting Processs (LMRP)	-0.607859	4.895924	-	-	297.3123	301.1363
Logarithmic Mean Reverting Sine Diffusion (LMRSD)	-0.5878979	0.5414701	3.0000000	-	347.2912	349.1153
Quadratic Loglinear Drift	0.567964995	0.001940176	1.006402410	1.639346889	8	7.824046

AIC: Akaike Information Criterion; BIC: Bayesian Information Criterion

Conclusion

This study proposes a new class of nonlinear SDEs characterized by a drift function that incorporates linear, quadratic, and logarithmic terms. In extension beyond traditional linear models such as the Geometric Brownian Motion and the Ornstein-Uhlenbeck process, the proposed framework captures a richer set of dynamic behaviors commonly observed in financial and economic time series, including nonlinear mean reversion, heavy tails, asymmetry, and volatility clustering. The derivation of the Fokker-Planck equation, along with analytical expressions for the mean and variance under lognormality assumptions, provides deeper theoretical insight into the long-run properties of the model.

Future research may investigate parameter estimation using Bayesian, likelihood-based, or filtering methods; assess the model's empirical performance across diverse financial datasets; and extend the formulation to incorporate jumps, stochastic volatility or structural breaks.

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Authors contribution

Ampitan K.R. conducted all experiments and was responsible for drafting, revising, and approving the final manuscript. Agwuegbo S.O.N. supervised the research work. Akintunde A.A. performed the data analysis. Basirat A. contributed to data collection.

Conflict of interest

The authors declare that there is no conflict of interest regarding the publication of this article.

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