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Short Communication

The Reve’s Puzzle with single relaxation of the Divine rule

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ABSTRACT

This paper proposes a new version of the Reve’s puzzle, allowing at most one violation of the “divine rule”. Letting S(n) be the minimum number of moves required to solve the problem with n(≥ 1) discs, an scheme is given to find the dynamic programming equation satisfied by S(n). A closed- form expression of S(n) is derived.

Introduction

The Tower of Hanoi puzzle, due to the French mathematician Lucas (1883), is as follows: Given are three pegs. Initially, one peg contains $n(\geq 1)$ discs of varying sizes, in a tower (in increasing order, from top to bottom). The objective is to transfer the tower to another peg, in minimum number of moves, where each move shifts only one (topmost) disc from one peg to another peg, under the “divine rule” that, during the transfer process, no disc can ever be placed on top of a smaller disc. It is known that the total number of moves necessary is 2^n -1.

The 4-peg generalization, also called the Reve’s puzzle, is due to Dudeney (1958). In its general form, the Reve’s puzzle is as follows: There are $n(\geq 1)$ discs d_1, d_2, d_n of varying sizes, and four pegs, S, P_1, P_2 and D. In the initial configuration, the discs rest on the *source peg S*, in a tower. The objective is to transfer the tower to the *destination peg D*, in minimum number of moves, under the “divine rule”.

There are several generalizations of the Tower of Hanoi problem and the Reve`s puzzle, for which the reader is referred to Majumdar (2012, 2013) and Hinz et al. (2018).

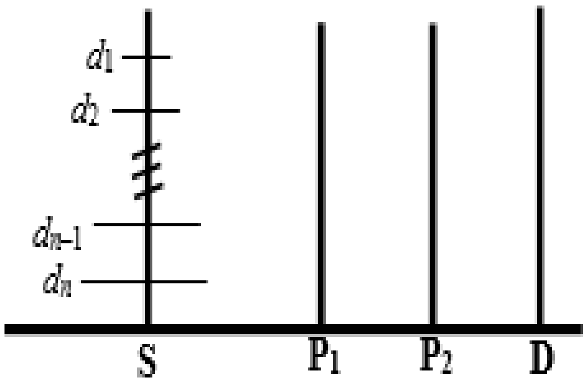

Fig.1. The initial state of the Reve’s puzzle

Chen et al. (2007) proposed a variant of the Tower of Hanoi problem which permits (at most) $r(\geq 1)$ violations of the “divine rule”. In the new generalization, the objective is to shift the tower from the peg S to the peg D in minimum number of moves, such that, for (at most) r moves, some disc may be put on top of a smaller disc.

A natural generalization of the problem of Chen et al. (2007) to the 4-peg case is the Reve’s puzzle which permits single relaxation of the “divine rule”. Thus, during the transfer process, at most once, a disc may be placed on a smaller disc.

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Let $S(n)$ be the minimum number of moves necessary to solve the Reve's puzzle with n discs and single relaxation of the "divine rule". The third section derives a closed-form expression of $S(n)$, considering all possible cases. The next section gives the background materials, while some remarks are given in the final section.

Background materials

Let the minimum number of moves necessary to solve the Reve's puzzle with $n(\geq 1)$ discs be denoted by $M(n)$. Then, $M(n)$ satisfies the dynamic programming equation below (see, for example, Roth (1974), Wood (1981), Hinz (1989), Chu et al. (1991), Majumdar (1994, 2012) and Hinz et al. (2018): For $n \geq 4$,

$$M(n) = \min_{1 \leq \ell \leq n-1} \left\{2M(\ell) + 2^{n-\ell} - 1\right\}, \quad (1a)$$

with

$$M(0) = 0, \quad (1b)$$

$$M(n) = 2n-1 \text{ for all } 1 \leq n \leq 3. \quad (1c)$$

Recently, Bousche (2014) claimed an analytical proof of optimality of the scheme leading to the equation (1a).

In what follows, the following results would be needed; for a proof, the reader is referred to Majumdar (1994, 2012).

***Lemma* 1:** For any $n \geq 1$,

$$M(n+1) - M(n) \geq M(n) - M(n-1).$$

***Corollary* 1:** $M(n+1) - M(n) > 4$ for all $n \geq 6$.

***Proof*:** Since for $n \geq 6$ (see Table 1),

$$M(n+1) - M(n) \geq M(7) - M(6)$$

$$= 8 > 4 = M(6) - M(5),$$

the result follows (by virtue of Lemma 1).

The solution of the optimality equation (1) is given below for reference later (for a proof, the reader is referred to Majumdar (1994, 2012), Hinz et al. (2018) and Majumdar (2021)).

***Theorem* 1:** $M(n)$ is given as below:

(1) for $s = 1, 2, \ldots$, $M\left(\frac{s(s+1)}{2}\right)$ is attained at the unique point $\ell = \frac{s(s-1)}{2}$, with

$$M\left(\frac{s(s+1)}{2}\right) = 2^s(s-1)+1,$$

(2) for $\frac{s(s+1)}{2} \leq n < \frac{(s+1)(s+2)}{2}$, $M(n)$ is attained at $\ell = n-s-1, n-s$, with

$$M(n) = 2^s\left\{n - \frac{s(s-1)}{2} - 1\right\} + 1.$$

In Theorem 1 above, ℓ is the value at which $2M(\ell) + 2^{n-\ell} - 1$ in equation (1a) is minimized. The values of $M(n)$ for some small n are given in Table 1.

Let $S_3(n)$ be the minimum number of moves necessary to solve the problem of Chen et al. (2007) with $n(\geq 1)$ discs and one relaxation of the "divine rule". Then, $S_3(n)$ is given as follows.

***Lemma* 2:** For any $n \geq 1$,

$$S_3(n) = \begin{cases} 2n-1, & if\, 1 \leq n \leq 3 \\ 2^{n-2}+5, & if\, n \geq 4 \end{cases}$$

The problem and its solution

The problem considered in this paper is as follows: Given is a tower of $n(\geq 1)$ discs (of different sizes) resting on the peg S, with the smallest disc at the top. The objective is to shift this tower to the peg D, using the two auxiliary pegs P_1 and P_2, in minimum number of moves, under the condition that each move transfers the topmost disc from one peg to another, such that only once, some disc may be put on a smaller one, and in any of the other moves, no disc can be placed on a smaller disc.

Let $S(n)$ denote the minimum number of moves necessry to solve the problem above. The theorem below gives a closed form expression of $S(n)$.

***Theorem* 2:** For $n \geq 1$,

$$S(n) = \begin{cases} 2n-1, & if\ 1 \leq n \leq 4 \\ 4n-9, & if\ 4 \leq n \leq 8 \\ M(n-3)+10, & if\ n \geq 8 \end{cases}$$

***Proof*:** The case $1 \leq n \leq 4$ is trivial. For example, when $n = 4$, an optimal strategy is as follows: Move the smallest disc, d_1, from S to P_1, then shift the disc d_2 (from S) to P_2, next move the disc d_3 (from S) to P_1 (on top of d_1, thereby violating the "divine rule" once). Now, move the largest disc (from S) to D. After shifting the disc d_3 (from P_1) to D, move the discs d_2 and d_1, in this order, to D, to complete the tower on D. This scheme involves 7 moves.

So, let $n \geq 5$. There are three possible schemes, which are described below.

Case 1: The first scheme is as follows:

1. shift the topmost $k(\geq 1)$ smallest discs, $d_1, d_2, \ldots, d_k$, from S to P_1 (using all the available four pegs), in (minimum) $M(k)$ moves,
2. move the disc d_{k+1} from S to P_1, violating the "divine rule",
3. shift the tower of remaining $n-k-1$ discs (from S) to D, (using the three pegs available) in (minimum) $2^{n-k-1}-1$ moves,
4. transfer the disc d_{k+1} from P_1 to D,
5. finally, move the tower (of k discs) on P_1 to D, to complete the tower on it.

The above scheme requires minimum

$$\min_{1 \leq k \leq n-1} 2\{M(k)+1\} + 2^{n-k-1} - 1$$
$$= M(n-1)+2, \qquad (2)$$

moves, where the expression in equation (2) follows from equation (1). Note that, in equation (2), $M(n-1)$ is attained at a point k with $k \leq n-2 < n-1$.

Case 2: The second scheme to follow is as below:

1. shift the topmost $k(\geq 1)$ smallest discs from S to P_1, in (minimum) $M(k)$ moves.
2. move the $n-k$ discs (remaining on S) to D, using the three available pegs, in (minimum) $S_3(n-k)$ moves.
3. finally, shift the tower of k discs from P_1 to D, in (minimum) $M(k)$ moves, thereby completing the tower on D.

The above scheme requires (minimum)

$$2M(k) + S_3(n-k) = 2M(k) + 2^{n-k-2} + 5$$

number of moves, where k is chosen so as to minimize the total number of moves. Thus, the minimum number of moves involved under this scheme is

$$\min_{1 \leq k \leq n-1} \{2M(k) + 2^{n-k-2}\} + 5$$
$$= M(n-2)+6, \qquad (3)$$

where in getting equation (3), equation (1) has been used. Recall that, $M(n-2)$ is attained at a point k with $k \leq n-3 < n-1$. Thus, the value of $M(n-2)$ is not affected if the range of k is extended (to $n-1$) in equation (3).

Now, by Corollary 1, for $n \geq 8$

$$M(n-1)+2 > M(n-2)+6.$$

Thus, the second scheme is better than the first one for $n \geq 8$.

Case 3: The third scheme is as below:

1. shift the topmost $k(\geq 1)$ smallest discs from S to P_1, (in (minimum) $M(k)$ moves),
2. transfer the disc d_{k+1} (from S) to D,
3. shift the disc d_{k+2} (from S) to P_2,
4. move the disc d_{k+3} (from S) to P_1, on top of the tower of k smallest discs, violating the "divine rule" once,
5. transfer the disc d_{k+2} (from P_2) to P_1, on the disc d_{k+3},
6. move the disc d_{k+1} (from D) to P_1, on the disc d_{k+2}.

After Step 6, there are two towers on the peg P_1, namely, the tower of three discs, d_{k+1}, d_{k+2} and d_{k+3}, on top of the tower of the smallest k discs. Next, follow the steps below.

7. transfer the tower of $n-k-3$ discs, still lying on S, to D, in (minimum) $2^{n-k-3}-1$ moves,
8. move the disc d_{k+1} (from P_1) to S,

9. transfer the disc d_{k+2} (from P_1) to P_2,
10. shift the disc d_{k+3} (from P_1) to D,
11. move the disc d_{k+2} (from P_2) to D,
12. finally,shift the disc d_{k+1} (from S) to D, to complete the tower on D.

The minimum number of moves involved in the above scheme is

$$\min_{1 \le k \le n-1} 2\{M(k)+5\} + 2^{n-k-3} - 1$$
$$= M(n-3)+10. \quad (4)$$

Note that, in equation (4), $M(n-3)$ is attained at a point k with $k \le n-3 < n$.

Now, by Corollary 1, for all $n \ge 9$,

$$M(n-2)+6 > M(n-3)+10.$$

Hence, the third scheme is better than the second one when $n \ge 9$. Since for $n \ge 8$, the second scheme is better than the first one, it follows that the third scheme is the only optimal scheme when $n \ge 9$. It now remains to compare the values of $M(n-1)+2$, $M(n-2)+6$ and $M(n-3)+10$ when $5 \le n \le 8$.

Now, since

$$M(4)+2 = 11 = M(3)+6,$$

it follows that the first and the second schemes both are optimal when $n = 5$; again, since

$$M(5)+2 = M(4)+6 = M(3)+10 = 15,$$
$$M(6)+2 = M(5)+6 = M(4)+10 = 19,$$

it follows that, for $n = 6, 7$, all the three schemes are optimal; and finally, since

$$M(6)+6 = 23 = M(5)+10,$$

it follows that, for $n = 8$, the second and the third schemes are optimal. Thus, so far as the number of moves is concerned, the second scheme may be disregarded.

To complete the proof, note that, using Table 1, it may readily be verified that, for $4 \le n \le 7$,

$$M(n-1) = 4n-11,$$

so that the minimum number of moves under the first scheme is simply $4n–9$. Finally, note that, this number remains valid when $n = 8$ as well.

Thus, the theorem is established.

Table 1. $M(n)$ and $S(n)$ for $4 \le n \le 10$

n	4	5	6	7	8	9	10
$M(n)$	9	13	17	25	33	41	49
$S(n)$	7	11	15	19	23	27	35

Remarks

The purpose of the paper is to initiate the study on a new variant of the Reve's puzzle, which permits relaxation of the "divine rule". Interestingly, for the new version, the optimal value function $S(n)$ can be expressed in terms of $M(n)$ only. Therefore, Theorem 1 may be exploited to find the properties and closed-form expressions of $S(n)$. An immediate generalization is the Reve's puzzle with $r(\ge 2)$ relaxations of the "divine rule". Another problem of interest is the bottleneck Reve's puzzle, introduced by Majumdar (1996) and Majumdar et al. (1996), which is still open, though a conjecture about the solution is given in Majumdar (2013). A third generalization of the Reve's puzzle has been treated recently by Majumdar (2016).

Conflicts of Interest

The author declares that there is no conflict of interest regarding the publication of this article.

References

Bouche T. La Quartrieme Tour de Hanoi. *Bulletin Belgian Math. Soc.* 2014; 21: 895-912.

Chen X, Tian B and Wang L. Santa Claus' Towers of Hanoi. *Graphs and Combinatorics*, 2007; 23 (Supplement): 153-167.

Chu XPing and Johnsonbaugh R. The Four-peg Tower of Hanoi puzzle. *SIGCSE Bulletin*, 1991; 23(3): 2-4.

Claus N. (=Lucas, E.) La Tour d Hanoi. *Jeu de Calcul. Sci. Nat.*, 1883; 1(8): 127-128.

Dudeney HE *The Canterbury Puzzles*. Dover, 4th Edition (1958).

Hinz AM, Klavzar S and Petr C. *The Tower of Hanoi -Maths & Myths.* Springer, 2nd Edition, 2018.

Hinz AM. An iterative algorithm for Tower of Hanoi problem with four pegs. *Computing*, 1989; 42: 133-140.

Majumdar AAK and Halder A. A recursive algorithm for the Bottleneck Reve's puzzle. *Surikaisekikenkyusho Kokyuroku*, 1996; 947: 150-161.

Majumdar AAK *The Classical Tower of Hanoi Problem and Its Generalizations, Vol.* 1: *Multi-Peg Generalizations*. Lambert Academic Publishing, U.S.A., 2012.

Majumdar AAK. Some local-value relationships for the recurrence relation related to the Tower of Hanoi problem. *Proceedings Pakistan Acad. Sci.* 2016; 53(2): 187-201.

Majumdar AAK. The Bottleneck Tower of Hanoi problem with four pegs. *Proceedings Pakistan Acad. Sci.* 1996; 33(1-2): 126-127 (Short Communication).

Majumdar AAK. *The Classical Tower of Hanoi Problem and Its Generalizations, Vol.* 2: *Other Generalizations*. Lambert Academic Publishing, U.S.A., 2013.

Majumdar AAK. The generalized four-peg Tower of Hanoi problem. *Optimization*, 1994; 29:349-360.

Majumdar AAK. The Reve's puzzle revisited. *Proceedings Pakistan Acad. Sci.* 2021; 58(2): 11-18.

Roth T. The Tower of Brahma revisited. *J. Recreational Math.* 1974; 7(2): 116-119.

Wood D. The Towers of Brahma and Hanoi revisited. *J. Recreational Math.* 1981; 14: 17- 24.