ON $T_0$ FUZZY BITOPOLLOGICAL SPACES

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ABSTRACT

In this paper, the authors introduced two notions of fuzzy pairwise-$T_0$ bitopological spaces and compared them with other such concepts. The authors also studied some other properties of these spaces.

Key words: Fuzzy Bitopological spaces, Quasi-coincidence, Q-neighbourhood, Fuzzy pairwise-$T_0$ bitopological spaces

INTRODUCTION

Kelly (1963) introduced bitopological spaces first time in 1963. The concept of fuzzy sets was introduced by Zadeh (1965). Chang (1968) and Lowen (1976) developed the theory of fuzzy topological spaces based on Zadeh’s concept. Fuzzy pairwise-$T_0$ (in short FPT$_i$, $i = 0, 1, 2$) bitopological spaces have been introduced earlier by Kandil and El-Shafee (1991). Fuzzy pairwise-$T_0$ separation axioms have also been introduced by Abu Sufiya et al. (1994) and Nouh (1996). Here the present authors introduced two definitions of fuzzy pairwise-$T_0$ bitopological spaces and obtained several of their properties.

PRELIMINARIES

**Definition:** A function $\mu$ from $X$ into the closed unit interval $I$ is called a fuzzy set in $X$. For every $x \in X$, $\mu(x) \in I$ is called the grade of membership of $x$. The class of all fuzzy sets from $X$ into the closed unit interval $I$ will be denoted by $I^X$ (Zadeh 1965).

**Definition:** A fuzzy set $\mu$ in a set $X$ is called a fuzzy singleton iff $\mu(x) = r$, $(0 < r \leq 1)$ for a certain $x \in X$ and $\mu(y) = 0$ for all points $y$ of $X$ except $x$. The fuzzy singleton is denoted by $x_r$ and $x$ is its support. The class of all fuzzy singletons in $X$ will be denoted by $S(X)$. If $\mu \in I^X$ and $x_r \in S(X)$, then we say that $x_r \in \mu$ iff $r \leq \mu(x)$ (Pau-Ming and Ying Ming1980).

**Definition:** A fuzzy set $\mu$ in a set $X$ is called a fuzzy point iff $\mu(x) = r$, $(0 < r < 1)$ for a certain $x_r \in X$ and $\mu(y) = 0$ for all points $y$ of $X$ except $x$. The fuzzy point is denoted by $x_r$ and $x$ is its support (Wong 1974).

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Definition: Let $I = [0, 1]$, $x$ be a non-empty set, and $I^*$ be the collection of all fuzzy sets in $X$. A fuzzy topology on $X$ is defined as a family $t$ of members of $I^*$ satisfying the following conditions:

(i) $0, 1 \in t$;
(ii) If $\mu_i \in t$ for each $i \in A$, then $\bigcup_{i \in A} \mu_i \in t$;
(iii) If $\mu_1, \mu_2 \in t$ then $\mu_1 \cap \mu_2 \in t$.

The pair $(X, t)$ is called a fuzzy topological space (fts in short) and members of $t$ are called $t$-open (or simply open) fuzzy sets. A fuzzy set $\mu$ is called a $t$-closed (or simply closed) fuzzy set if $1 - \mu \in t$ (Chang 1968).

Definition: Let $\omega$ be a real valued function on a topological space. If $\{x : f(x) > a\}$ is open for every real $\alpha \in I_0$, then $f$ is called lower semicontinuous function (Rudin 1974).

Definition: Let $x$ be a non-empty set and $T$ be a topology on $X$. Let $t = \omega(T)$ be the set of all lower semicontinuous (lsc in short) functions from $(X, T)$ to $I$ (with usual topology). Thus $\omega(T) = \{\mu \in I^* : \mu^{-1}(\alpha, 1) \in T\}$ for each $\alpha \in I_0$. It can be shown that $\omega(T)$ is a fuzzy topology on $X$ (Lowen 1976).

Definition: A fuzzy singleton $x_r$ is said to be quasi-coincident with $\mu$, denoted by $x_r q \mu$ iff $r + \mu(x) > 1$. If $x_r$ is not quasi-coincident with $\mu$, we write (Kandil and El-Shafee 1991).

Definition: A fuzzy set $u$ of $(X, t)$ is called quasi-neighborhood (Q-nbd in short) of $x_r$ iff there exists $v \in t$ such that $x_r q v$ and $v \subset u$.

If $x_r$ is a fuzzy point, then $N(x_r, t) = \{\mu \in t : x_r \in u\}$ is the family of all fuzzy $t$-open neighborhoods (t-nbds) of $x_r$. Similarly, if $x_r$ is a fuzzy point, then $N_q(x_r, t) = \{\mu \in t : x_r q u\}$ is the family of all Q-neighborhoods (Q-nbd) of $x_r$ (Nouh 1996).

Definition: A system $(X, t_1, t_2)$ consisting of a set $x$ with two fuzzy topologies $t_1$ and $t_2$ on $X$ is called a fuzzy bitopological space (fbts in short) (Kandil et al. 1999).

Fuzzy Pairwise $T_\alpha$-Spaces

Definition: An fbts $(X, t_1, t_2)$ is called

(a) $FPT_{\alpha}(i)$ iff for every pair of fuzzy singletons $x_{r_1}, y_{r_2} \in S(X)$ with $x \neq y$, there is a $t_1$-open fuzzy set or a $t_2$-open fuzzy set which contains one of the fuzzy singletons and not quasi-coincident with the other.

(b) $FPT_{\alpha}(ii)$ iff for every pair of fuzzy singletons $x_{r_1}, y_{r_2} \in S(X)$ with $x \neq y$, there exists a fuzzy set $\mu \in t_1 \cup t_2$ such that $(x_{r_1} q \mu, y_{r_2} \cap \mu = 0)$ or $(y_{r_2} q \mu, x_{r_1} \cap \mu = 0)$. 
(c) FPT$_d$(iii) iff for any two distinct fuzzy points $p, q$ in $X$ there exists a fuzzy set $\mu \in t_1 \cup t_2$ such that $(p \in \mu, q \cap \mu = 0)$ or $(q \in \mu, p \cap \mu = 0)$ (Abu et al. 1994).

(d) FPT$_d$(iv) space iff for every pair of fuzzy singletons $x_p, y_q \in S(X)$ such that $x \neq y$, there is a $t_1$-open fuzzy set or a $t_2$-open fuzzy set which is Q-nbd of one of the fuzzy singletons and not quasi-coincident with other (Nouh 1996).

(e) FPT$_d$(v) space iff for every pair of fuzzy singletons $x_p, y_q \in S(X)$ with $x \neq y$, there exists a fuzzy set $\mu \in t_1 \cup t_2$ such that $(x_p \in \mu, \mu \subseteq (y_q)^r)$ or $(y_q \in \mu, \mu \subseteq (y_q)^r)$ (Abu et al. 1994).

(f) FPT$_d$(vi) iff for any two distinct fuzzy points $p, q$ in $X$ there exists a fuzzy set $\mu \in t_1 \cup t_2$ such that $(p \in \mu, q \subseteq \mu^r)$ or $(q \in \mu, q \subseteq \mu^r)$ (Abu et al. 1994).

**Theorem:** Let $(X, t_1, t_2)$ be an fts. Then the workers have the following implications:

(a) $\iff$ (b) $\iff$ (c) $\iff$ (d) $\iff$ (e) and (d) $\implies$ (f).

**Proof:** (a) $\implies$ (b): Let $(X, t_1, t_2)$ be FPT$_d$(i)-space. Then by definition, for the singletons $x_p, y_q \in S(X)$ with $x \neq y$, choose $r^* \in (0, 1)$ such that $r^* > 1 - r$. Since $(X, t_1, t_2)$ is FPT$_d$(i), then there exists a fuzzy set $\mu \in t_1 \cup t_2$ such that $x_p \in \mu$ and $y_q \in \mu$.

Since $\mu(x) \geq r^*$ and $r^* > 1 - r$, then $\mu(x) > 1 - r$, that is, $r + \mu(x) > 1$ and so $x_q \mu$.

Also, $y_q \in \mu \implies 1 + \mu(y) \leq 1 \implies \mu(y) \leq 1 - 1 = 0$, that is, $\mu(y) = 0$. Now for any fuzzy singletons $x_p, y_q \in S(X)$, it is seen that, $x_q \mu$ and $y_q \in \mu$. Hence $(X, t_1, t_2)$ is FPT$_d$(ii)-space.

(b) $\implies$ (c): Let $x_p, y_q$ be two distinct fuzzy points in $X$. Choose $r^* \in (0, 1)$ such that $r^* < 1 - r$. Since $(X, t_1, t_2)$ is FPT$_d$(ii) and $x_p, y_q$ are distinct fuzzy singletons, then

$$\exists \mu \in N_\theta(x_{p^r}, t_1). \mu \cup y_q = 0 \quad \text{or} \quad \exists \mu \in N_\theta(x_{p^r}, t_2). \mu \cup x_q = 0.$$

Now, let $\mu \in N_\theta(x_{p^r}, t_1)$ $(\mu \cap y_q = 0)$. That is, $r^* + \mu(x) > 1$ and $(\mu \cap y_q = 0)$. Since $r^* + \mu(x) > 1$ and $r^* < 1 - r$, then $\mu(x) > r$ and so $x_p \in \mu$. Hence $(X, t_1, t_2)$ is FPT$_d$(iii)-space.

(c) $\implies$ (d): Let $x_p, y_q \in S(X)$ with $x \neq y$. Choose $r^*, s^* \in (0, 1)$ such that $r^* > 1 - r$ and $s^* > 1 - s$. Since $(X, t_1, t_2)$ is FPT$_d$(iii) and $x_p, y_q$ are distinct fuzzy points, then there exists a fuzzy set $\mu \in t_1 \cup t_2$ such that $x_p \in \mu$ and $x_q \cap \mu = 0$. Since $\mu(x) > r^*$ and $s^* > 1 - r$, then we have $\mu(x) > 1 - r$, that is, $r + \mu(x) > 1$ and so $x_q \mu$.

Again, $y_q \cap \mu = 0$ implies $y_q \cap \mu = 0$, since $\mu(y) = 0$. So, $y_q \in \mu$. Hence $(X, t_1, t_2)$ is FPT$_d$(iv)-space.

(d) $\implies$ (e): Let $x_p, y_q$ be two distinct fuzzy singletons in $X$. Choose $r^* \in (0, 1)$ such that $r^* < 1 - r$. Since $(X, t_1, t_2)$ is FPT$_d$(iv), then there exists a $\mu \in t_1 \cup t_2$ such that $r^* q \mu$
and $y, \overline{\xi} \mu$.

Now, since $r^* + \mu(x) > 1$ and $r^* < 1 - r$, then $\mu(x) > r$. So, $x_i \in \mu$. Also, $y, \overline{\xi} \mu \Rightarrow s + \mu(y) \leq 1 \Rightarrow s \leq 1 - \mu(y)$. So, $\mu \subseteq (y)^c$. Hence $(X, t_1, t_2)$ is FPT$_d$$(v)$-space.

(e) $\Rightarrow$ (a): Let $x_i, y_i \in S(X)$ with $x \neq y$. Since $(X, t_1, t_2)$ is FPT$_d$$(v)$, then there exists a fuzzy set $\mu \in t_1 \cup t_2$ such that $x_i \in \mu$ and $\mu \subseteq (y)^c$.

Also, $\mu \subseteq (y)^c \Rightarrow \mu(y) \leq 1 - s \Rightarrow s + \mu(y) \leq 1$ and so, $y, \overline{\xi} \mu$. Hence $(X, t_1, t_2)$ is FPT$_d$$(i)$-space.

(d) $\Rightarrow$ (f): Let $x_i, y_i$ be two distinct fuzzy points in $X$. Choose $r^* \in (0, 1)$ such that $r^* < 1 - r$. Since $(X, t_1, t_2)$ is FPT$_d$$(iv)$ and $x_i, y_i$ are distinct fuzzy singletons, then there exists a fuzzy set $\mu \in t_1 \cup t_2$ such that $x_i, q \mu$ and $y_i, \overline{\xi} \mu$.

Since $r^* + \mu(x) > 1$ and $r^* < 1 - r$, then one has $\mu(x) > r$ and so $x_i \in \mu$. Also, $y, \overline{\xi} \mu \Rightarrow s + \mu(y) \leq 1 \Rightarrow s \leq 1 - \mu(y)$. So, $y_i \subseteq \mu$. Hence $(X, t_1, t_2)$ is FPT$_d$$(vi)$-space.

(f) $\Rightarrow$ (d): Example: Let $X = 1$, $t_1 = t_2 = \{0, \lambda : \lambda(x) > 0, \forall x \in X\}$. Let $x_i$ and $y_i$ be distinct fuzzy singletons in $X$ and $\gamma = \min\{1 - r, 1 - s\}$

Now, if $\gamma \neq 0$, the authors define $\mu$ as follows: $\mu(x) = 1$ and $\mu(y) = \frac{y}{8}$, if $y \neq x$.

Again, if $\gamma = 0$, one can define $\mu$ as follows: $\eta(x) = 1$ and $\eta(y) = 0.1$, if $y \neq x$.

Then $\mu \in t_1 = t_2$. For any pair of distinct fuzzy points $x_i$, $y_i$ is $X$, it is seen that $x_i \in \mu$ and $y_i \subseteq \mu$. Therefore, $(X, t_1, t_2)$ is FPT$_d$$(vi)$-space. But if one takes $x_i, y_i \in S(X)$, then it can be seen that $x_i, q \eta$ and $y_i, \overline{\xi} \eta$. Hence $(X, t_1, t_2)$ is FPT$_d$$(iv)$-space.

**Theorem:** Let $(X, s, t)$ be a fuzzy topological space, $A \subset X$, and $S_A = \left\{ \frac{u}{A} : u \in s \right\}$, $t_A = \left\{ \frac{v}{A} : v \in t \right\}$. Then

(a) $(X, s, t)$ is FPT$_d$$(i)$ $\Rightarrow$ $(A, S_A, t_A)$ is FPT$_d$$(i)$.
(b) $(X, s, t)$ is FPT$_d$$(ii)$ $\Rightarrow$ $(A, S_A, t_A)$ is FPT$_d$$(ii)$.
(c) $(X, s, t)$ is FPT$_d$$(iii)$ $\Rightarrow$ $(A, S_A, t_A)$ is FPT$_d$$(iii)$.
(d) $(X, s, t)$ is FPT$_d$$(iv)$ $\Rightarrow$ $(A, S_A, t_A)$ is FPT$_d$$(iv)$.
(e) $(X, s, t)$ is FPT$_d$$(v)$ $\Rightarrow$ $(A, S_A, t_A)$ is FPT$_d$$(v)$.
(f) $(X, s, t)$ is FPT$_d$$(vi)$ $\Rightarrow$ $(A, S_A, t_A)$ is FPT$_d$$(vi)$.

**Proof:** (a) Suppose $(X, s, t)$ is FPT$_d$$(i)$. One has to show that $(A, S_A, t_A)$ is FPT$_d$$(i)$. Let $a_u, b_v \in S(A)$ with $a \neq b$. Since $(X, s, t)$ is FPT$_d$$(i)$, there exists a $u \in s \cup t$ such that $a_u \in u$ and $b_v, \overline{\xi} u$, that is, $u(a) \geq r$ and $u(b) + p \leq 1$. 


Now \( \frac{u}{A} \in S_{A} \cup t_{A} \) and \( (\frac{u}{A}) \cdot (a) = u(a) \). Then \( (uA) \cdot (a) \geq r \). So, \( a_{r} \in \frac{u}{A} \).

Also \( (\frac{u}{A}) \cdot (b) + p \leq 1 \), since \( (\frac{u}{A}) \cdot (b) = u(b) \). Hence \( (A, S_{A}, t_{A}) \) is \( \text{FPT}_{d}(i) \).

(f) Suppose \( (X, s, t) \) is \( \text{FPT}_{d}(vi) \). The authors have to show that \( (A, S_{A}, t_{A}) \) is \( \text{FPT}_{d}(vi) \).

Let \( a_{r}, b_{p} \) be two distinct fuzzy points in \( A \). Then \( a_{r}, b_{p} \) are two distinct fuzzy points in \( X \). Since \( (X, s, t) \) is \( \text{FPT}_{d}(vi) \), there exists a \( u \in s \cup t \) such that \( a_{r} \in u \) and \( b_{p} \subseteq u' \), that is, \( u(a) > r \) and \( 1 - u(b) \geq p \).

Now \( \frac{u}{A} \in S_{A} \cup t_{A} \) and \( (\frac{u}{A}) \cdot (a) = u(a) \). Then \( (uA) \cdot (a) > r \). So, \( a_{r} \in \frac{u}{A} \).

Also \( 1 - (\frac{u}{A}) \cdot (b) \geq p \), since \( (\frac{u}{A}) \cdot (b) = u(b) \) and so \( b_{p} \subseteq (\frac{u}{A}) \cdot c \). Hence \( (A, S_{A}, t_{A}) \) is \( \text{FPT}_{d}(vi) \).

The proofs of (b), (c), (d) and (e) are similar.

**Theorem:** Let \( (X, T_{1}, T_{2}) \) be a bitopological space. Then

(a) \( (X, T_{1}, T_{2}) \) is \( \text{PT}_{d} \) \( \Rightarrow \) \( (X, \omega(T_{1}), \omega(T_{2}) \) is \( \text{FPT}_{d}(i) \).

(b) \( (X, T_{1}, T_{2}) \) is \( \text{PT}_{d} \) \( \Rightarrow \) \( (X, \omega(T_{1}), \omega(T_{2}) \) is \( \text{FPT}_{d}(ii) \).

(c) \( (X, T_{1}, T_{2}) \) is \( \text{PT}_{d} \) \( \Rightarrow \) \( (X, \omega(T_{1}), \omega(T_{2}) \) is \( \text{FPT}_{d}(iii) \).

(d) \( (X, T_{1}, T_{2}) \) is \( \text{PT}_{d} \) \( \Rightarrow \) \( (X, \omega(T_{1}), \omega(T_{2}) \) is \( \text{FPT}_{d}(iv) \).

(e) \( (X, T_{1}, T_{2}) \) is \( \text{PT}_{d} \) \( \Rightarrow \) \( (X, \omega(T_{1}), \omega(T_{2}) \) is \( \text{FPT}_{d}(v) \).

(f) \( (X, T_{1}, T_{2}) \) is \( \text{PT}_{d} \) \( \Rightarrow \) \( (X, \omega(T_{1}), \omega(T_{2}) \) is \( \text{FPT}_{d}(vi) \).

**Proof:** (a) Suppose that \( (X, T_{1}, T_{2}) \) is \( \text{PT}_{d} \). Then the authors are to show that \( (X, \omega(T_{1}), \omega(T_{2}) \) is \( \text{FPT}_{d}(i) \). Let \( x_{p}, y_{p} \in S(X) \) with \( x \neq y \). Since \( (X, T_{1}, T_{2}) \) is \( \text{PT}_{d} \) then

\[
(\exists U \in N(x, T_{1}), (y \not\in U) \text{ or } (\exists V \in N(x, T_{2}), (y \not\in V) \text{ Then } 1_{p} \in N(x_{p}, \omega(T_{1})), y, \overline{\mu} u), \text{ or } 1_{p} \in N(x_{p}, \omega(T_{2})), y, \overline{\mu} v). \text{ Thus } (X, \omega(T_{1}), \omega(T_{2}) \) is \( \text{FPT}_{d}(i) \).

Conversely, suppose that \( (X, \omega(T_{1}), \omega(T_{2}) \) is \( \text{FPT}_{d}(ii) \), then one has to show that \( (X, T_{1}, T_{2}) \) is \( \text{PT}_{d} \). Let \( x, y \in X \) such that \( x \neq y \). Since \( (X, \omega(T_{1}), \omega(T_{2}) \) is \( \text{FPT}_{d}(i) \), then

\[
(\exists \mu \in N(x, \omega(T_{1})), y, \overline{\mu} u) \text{ or } (\exists \eta \in N(x, \omega(T_{2})), \mu, \overline{\eta} u).
\]

Now, let \( \mu \in N(x, \omega(T_{1})), y, \overline{\mu} u \) i.e. \( \mu(x) = 1 \) and \( \mu(y) = 0 \). Hence \( x \in \mu^{-1}(0,1) \in T_{1} \) and \( y \not\in \mu^{-1}(0,1) \in T_{1} \). Hence \( (X, T_{1}, T_{2}) \) is \( \text{PT}_{d} \).

(f) Suppose that \( (X, T_{1}, T_{2}) \) is \( \text{PT}_{d} \). One has to show that \( (X, \omega(T_{1}), \omega(T_{2}) \) is
Let \( x_p, y \) be two distinct fuzzy points in \( X \). Since \((X, T_i, T_j)\) is \( PT_0 \)
then \( (\exists U \in N(x, T_i), (y, U)) \) or \( (\exists V \in N(x, T_j), (y \neq V)), \)

Then \( l_x \in N(x_p, \omega(T_i)), l_y \in N(y_p, \omega(T_j)) \) or \( (l_x \cap \omega(T_i) = 0) \) or \( l_y \in N(x_p, \omega(T_j)), (l_y \cap \omega(T_j) = 0) \) which implies \( l_x \in N(x_p, \omega(T_i)), y \subseteq (l_y) \) or \( l_y \in N(x_p, \omega(T_j)), y \subseteq (l_x) \). Thus \((X, \omega(T_i), \omega(T_j))\) is \( FPT_{(vi)} \).

Conversely, suppose that \((X, \omega(T_i), \omega(T_j))\) is \( FPT_{(vi)} \). One has to show that \((X, T_i, T_j)\) is \( PT_0 \). Let \( x, y \in X \) such that \( x \neq y \). Take \( r \) such that \( 0.5 < r < 1 \). Since \((X, \omega(T_i), \omega(T_j))\) is \( FPT_{(vi)} \), then \( (\exists \mu \in N(x_p, \omega(T_i)), y \subseteq \mu) \) or \( (\exists \eta \in N(x_p, \omega(T_j)), y \subseteq \eta) \).

Now, let \( \mu \omega N(x_p, \omega(T_i)), y \subseteq \mu \). i.e. \( \mu(x) > r \) and \( \mu(y) \leq 1 - r \).

Hence \( x \in \mu^{-1}(0.5, 1) \) and \( y \not\in \mu^{-1}(0.5, 1) \). Hence \((X, T_i, T_j)\) is \( PT_0 \).

Proofs of (b), (c) and (e) are similar and for the proof of (d) (Nouh 1996).

**Theorem:** Product of any two \( FPT_{(j)} \)-spaces is \( FPT_{(j)} \)-space where \( j = i, ii, iii, iv, v, vi \).

**Proof:** Suppose \((X_1, s_1, t_1)\) and \((X_2, s_2, t_2)\) are \( FPT_{(i)} \), then one has to show that \((X_1 \times X_2, s_1 \times s_2, t_1 \times t_2)\) is \( FPT_{(i)} \). Let \((x, y)_p, (x_1, y_1)_q \in X_1 \times X_2\) with \((x, y) \neq (x_1, y_1)\).

It can be assumed without loss of generality that \( x \neq x_1 \).

Since \( x_p, (x_1)_q \in S(X_1) \) with \( x \neq x_1 \) and \((X_1, s_1, t_1)\) is \( FPT_{(i)} \), then \( (\exists \mu \in N(x_p, s_1), ((x_1)_q \not\in \mu)) \) or \( (\exists \eta \in N(x_p, t_1), ((x_1)_q \not\in \eta)). \)

Now, let \( \mu \in N(x_p, s_1), ((x_1)_q \not\in \mu) \). That is, \( \mu(x) \geq p \) and \( \mu(x_1) + q \leq 1 \).

Since \( \mu \times X_2 \subseteq s_1 \times s_2 \) and \((\mu \times X_2) (x, y) = \min \{ \mu(x), X_2(y) \} \) then one has \((x, y)_p \in \mu \times X_2 \).

Also, \((\mu \times X_2) (x, y)_p + q = \mu(x_1) + q \leq 1 \). Hence \((X_1 \times X_2, s_1 \times s_2, t_1 \times t_2)\) is \( FPT_{(i)} \).

Suppose \((X_1, s_1, t_1)\) and \((X_2, s_2, t_2)\) are \( FPT_{(vi)} \), then one has to show that \((X_1 \times X_2, s_1 \times s_2, t_1 \times t_2)\) is \( FPT_{(vi)} \). Let \((x, y)_p, (x_1, y_1)_q \) be two distinct fuzzy points in \( X_1 \times X_2 \). One assumes without loss of generality that \( x \neq x_1 \). Since \( x_p, (x_1)_q \) are two distinct fuzzy points in \( X_1 \) and \((X_1, s_1, t_1)\) is \( FPT_{(vi)} \), then \( (\exists \mu \in N(x_p, s_1), ((x_1)_q \not\in \mu)) \) or \( (\exists \eta \in N(x_p, t_1), ((x_1)_q \not\in \eta)). \)

Now, let \( \mu \in N(x_p, s_1), (x_1)_q \not\in \mu \). That is, \( \mu(x) > p \) and \( q \leq 1 - \mu(x)_p \).

Since \( \mu \times X_2 \subseteq s_1 \times s_2 \) and \((\mu \times X_2) (x, y) = \min \{ \mu(x), X_2(y) \} \) then one has \((x, y)_p \in \mu \times X_2 \).
Also, for all \((x, y) \in X_1 \times X_2\), we have \(1 - (\mu \times \mu) (x, y) = 1 - \mu(x) \geq q\).

So, \((s_1, y_1) \subseteq (\mu \times \mu)\). Hence \((X_1 \times X_2, s_1 \times s_2, t_1 \times t_2)\) is FPT\(_d\)(vi).

Proofs of the other claims are similar.

**Theorem:** A bijective mapping from an fts \((X, t)\) to an fts \((Y, s)\) preserves the value of a fuzzy singleton (fuzzy point).

Proof: Let \(c\) be a fuzzy singleton in \(X\). So, there exist a point \(a \in Y\) such that \(f(c) = a\). Now \(f(c) = f(c) = \sup c(c) = c(c) = r\), since \(f\) is bijective. Hence \(a\) has same value as \(c\).

Note: Preimage of any fuzzy singleton (fuzzy point) under bijective mapping preserves its value.

Theorem: Let \((X, s, t)\) and \((Y, s_1, t_1)\) and be two fuzzy bitopological spaces and let \(f: X \rightarrow Y\) be bijective and FP-open. Then

\[(a) \quad (X, s, t) \text{ is } \text{FPT}_d(i) \Rightarrow (Y, s_1, t_1) \text{ is } \text{FPT}_d(i).\]

\[(b) \quad (X, s, t) \text{ is } \text{FPT}_d(ii) \Rightarrow (Y, s_1, t_1) \text{ is } \text{FPT}_d(ii).\]

\[(c) \quad (X, s, t) \text{ is } \text{FPT}_d(iii) \Rightarrow (Y, s_1, t_1) \text{ is } \text{FPT}_d(iii).\]

\[(d) \quad (X, s, t) \text{ is } \text{FPT}_d(iv) \Rightarrow (Y, s_1, t_1) \text{ is } \text{FPT}_d(iv).\]

\[(e) \quad (X, s, t) \text{ is } \text{FPT}_d(v) \Rightarrow (Y, s_1, t_1) \text{ is } \text{FPT}_d(v).\]

\[(f) \quad (X, s, t) \text{ is } \text{FPT}_d(vi) \Rightarrow (Y, s_1, t_1) \text{ is } \text{FPT}_d(vi).\]

Proof: (a) Suppose \((X, s, t)\) is FPT\(_d\)(i). The authors shall now show that \((Y, s_1, t_1)\) is FPT\(_d\)(i). Let \(a_\mu, b_\eta \in S(Y)\) with \(a \neq b\). Since \(f\) is bijective, then there exists \(c_\mu, d_\eta \in S(Y)\) such that \(f(c) = b, f(d) = b\) and \(c \neq d\). Again since \((X, s, t)\) is FPT\(_d\)(i), then

\[\exists \mu \in N(c, s), (d_\eta \not\leq \mu) \quad \text{or} \quad \exists \mu \in N(c, s), (d_\eta \not\leq \mu).\]

Now, \(\mu \in N(c, s), (d_\eta \not\leq \mu)\). That is, \(\mu(c) \geq r\) and \(\mu(d) + q \leq 1\).

Now \(f(\mu) = f(\mu), f(c) = \sup \mu(c) = \mu(c) \geq r\). So, \(a_\mu \in f(\mu)\).

Also \(f(\mu) = f(\mu), f(d) = q = \mu(d) + q \leq 1\). So, \(b_\eta \not\leq f(\mu)\).

Since \(f\) is FP-open, then \((\mu) \in s_1\). Hence \((Y, s_1, t_1)\) is FPT\(_d\)(i).

(f) Suppose \((Y, s, t)\) is FPT\(_d\)(vi). We shall show that \((Y, s_1, t_1)\) is FPT\(_d\)(vi). Let \(a_\mu, b_\eta \) be two distinct fuzzy points in \(Y\). Since \(f\) is bijective, then there exists \(c_\mu, d_\eta \in S(X)\) such that \(f(c) = a, f(d) = b\) and \(c \neq d\). Again since \((Y, s, t)\) is FPT\(_d\)(vi), then

\[\exists \mu \in N(c, s), (d_\eta \not\leq \mu) \quad \text{or} \quad \exists \mu \in N(c, s), (d_\eta \not\leq \mu).\]
Now, let \( \mu \in N(c, s) \), \( (d_q \subseteq \mu^i) \). That is, \( \mu(c) > r \) and \( q \leq 1 - \mu(d) \).

Now \( f(\mu) (a) = f(\mu) (f(c)) = \text{sup}_b \mu(c) = \mu(c) > r \). So, \( a_i \in f(\mu) \).

Also, \( 1 - f(\mu) (b) = 1 - f(\mu) (f(d)) = 1 - \mu(d) \geq q \). So, \( b_q \subseteq (f(\mu))^C \). Since \( c \) is FP-open, then \( (\mu) \in s_i \). Hence \( (Y, s_i, t_i) \) is \( \text{FPT}_d(vi) \).

Proofs of (b), (c), (d) and (e) are similar.

**Theorem:** Let \( (Y, s, t) \) and \( (Y, s_i, t_i) \) be two fuzzy bitopological spaces and \( f : X \to Y \) be FP-continuous and bijective. Then

(a) \( (Y, s_i, t_i) \) is \( \text{FPT}_d(i) \) \( \Rightarrow \) \( (Y, s, t) \) is \( \text{FPT}_d(i) \).

(b) \( (Y, s_i, t_i) \) is \( \text{FPT}_d(ii) \) \( \Rightarrow \) \( (Y, s, t) \) is \( \text{FPT}_d(ii) \).

(c) \( (Y, s_i, t_i) \) is \( \text{FPT}_d(iii) \) \( \Rightarrow \) \( (Y, s, t) \) is \( \text{FPT}_d(iii) \).

(d) \( (Y, s_i, t_i) \) is \( \text{FPT}_d(iv) \) \( \Rightarrow \) \( (Y, s, t) \) is \( \text{FPT}_d(iv) \).

(e) \( (Y, s_i, t_i) \) is \( \text{FPT}_d(v) \) \( \Rightarrow \) \( (Y, s, t) \) is \( \text{FPT}_d(v) \).

(f) \( (Y, s_i, t_i) \) is \( \text{FPT}_d(vi) \) \( \Rightarrow \) \( (Y, s, t) \) is \( \text{FPT}_d(vi) \).

**Proof:** The authors shall prove (a) and (f) only.

(a) Suppose \( (Y, s, t) \) is \( \text{FPT}_d(i) \). One has to show that \( (Y, s, t) \) is \( \text{FPT}_d(i) \). Let \( c_r, d_q \in S(Y) \) such that \( q \neq d \). Then there exist \( a_i, b_q \in S(Y) \) such that \( f(c) = a, f(d) = b \) and \( a \neq b \), since \( f \) is one-one. Again since \( (Y, s_i, t_i) \) is \( \text{FPT}_d(i) \), then \( (\exists \mu N(a_i, s_i), (b_q \mu^i)) \) or \( (\exists \eta N(a_i, t_i), (b_q \eta^i)) \).

Now, let \( \mu \in N(a_i, s_i), (b_q \mu^i) \). That is, \( \mu(a) \succeq r \) and \( \mu(b) + q \leq 1 \).

Since \( f^{-1}(\mu) (c) = f^{-1}(\mu(c)) = f^{-1}(\mu) (a) \succeq r \), then \( c_i \in f^{-1}(\mu) \). So, \( a_i \eta f^{-1}(\mu) \).

Also, \( f^{-1}(\mu) (d) + q = f(d) + q = \mu(b) + q \leq 1 \). So, \( b_q \mu^i \).

Since \( f \) is FP-continuous, then \( f^{-1}(\mu) \in s \). Hence \( (Y, s, t) \) is \( \text{FPT}_d(i) \).

(f) Suppose \( (Y, s_i, t_i) \) is \( \text{FPT}_d(vi) \). Then one has to show that \( (Y, s, t) \) is \( \text{FPT}_d(vi) \). Let \( c_r, d_q \) be two distinct fuzzy points in \( X \). Then there exists distinct fuzzy points \( a_i, b_q \) in \( Y \) such that \( f(c) = a, f(d) = b \) and \( a \neq b \), since \( f \) is one-one. Again since \( (Y, s_i, t_i) \) is \( \text{FPT}_d(vi) \), then \( (\exists \mu N(a_i, s_i), (b_q \mu^i)) \) or \( (\exists \eta N(a_i, t_i), (b_q \eta^i)) \).

Now, let \( \mu \in N(a_i, s_i), (b_q \mu^i) \), that is, \( \mu(a) \succeq r \) and \( 1 - \mu(b) \geq q \).

Since \( f^{-1}(\mu) (c) = f^{-1}(\mu(c)) = f^{-1}(\mu) (a) \succeq r \), then \( c_i \in f^{-1}(\mu) \).
Also, $1 - f^{-1}(\mu)(d) = 1 - \mu(f(d)) = 1 - \mu(b) \geq q$. So, $d_q \in (f^{-1}(\mu))^c$.

Since $f$ is FP-continuous, then $f^{-1}(\mu) \in s$. Hence $(Y, s, t)$ is $\text{FPT}_0(vi)$.

REFERENCES


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