Journal of Bangladesh Academy of Sciences, Vol. 36, No. 2, 187-197, 2012

APPROXIMATE SOLUTION OF FOURTH ORDER NEAR CRITICALLY DAMPED NONLINEAR SYSTEMS WITH SPECIAL CONDITIONS

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ABSTRACT

A perturbation technique has been developed based on the Krylov-Bogoliubov-Mitropolskii (KBM) method to investigate the solution of fourth order near critically damped nonlinear systems in the case of $\lambda_1 \to \lambda_2$, $\lambda_4 \approx \lambda_3 + 2\lambda_1$ but $\lambda_4 \langle 2\lambda_3$ among the eigenvalues $\lambda_1, \lambda_2, \lambda_3, \lambda_4$. The solutions obtained by this technique were compared with those obtained by numerical method. The method has been explained by an example.

Key words: Perturbation method, Weak nonlinearity, Near critically damped nonlinear system

INTRODUCTION

The Krylov-Bogoliubov-Mitropolskii method (Krylov and Bogoliubov 1947, Bogoliubov and Mitropolskii 1961) is one of the widely used techniques to obtain analytical approximate solution of weakly nonlinear systems and this method was originally developed for finding periodic solutions of nonlinear systems with small nonlinearities. The method was extended by Popov (1956) to damped oscillatory systems. Murty *et al.* (1969) investigated an over-damped nonlinear system using Bogoliubov's method. Murty (1971) presented a unified KBM method for solving second order nonlinear systems which cover the un-damped, damped and over-damped cases. Alam and Sattar (1996) extended the KBM method for third order critically damped nonlinear systems. Alam (2002) also investigated the solution of third order nonlinear systems when two of the eigenvalues are almost equal and the other is small. Haque *et al.* (2011) investigated the solution of fourth order critically damped oscillatory nonlinear systems when two of the eigenvalues are real and equal and the other two are complex conjugate. Akbar *et al.* (2007) extended the KBM method for solving fourth order more critically damped nonlinear systems. Recently, Rahman *et al.* (2009) developed a technique for solving of fourth order near critically damped nonlinear systems. For the relation $\lambda_4 \approx \lambda_3 + 2\lambda_1$, the solution obtained in Rahman *et al.* (2009) broke-down. The aim of this article was to fill this gap, that is, the authors were interested to investigate the solution when the relation $\lambda_4 \approx \lambda_3 + 2\lambda_1$ exists among the eigenvalues $\lambda_1, \lambda_2, \lambda_3, \lambda_4$. The solutions obtained by this technique showed good coincidence with those obtained by numerical method.

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MATERIALS AND METHOD

Let us consider the fourth order weakly nonlinear ordinary differential equation in the following form

$$\frac{d^4x}{dt^4}+c_1\frac{d^3x}{dt^3}+c_2\frac{d^2x}{dt^2}+c_3\frac{dx}{dt}+c_4x=-\varepsilon f(x), \tag{1}$$

where ε is a positive small parameter, c_1, c_2, c_3, c_4 are constants and $f(x)$ is the given nonlinear function. The constants are defined in terms of the eigenvalues $-\lambda_{i,}$ $i=1, 2, 3, 4$ of the unperturbed Eq. of Eq. (1) as

$$c_1=\sum_{i=1}^{4}\lambda_i,\quad c_2=\sum_{\substack{i,j=1\\ i\neq j}}^{4}\lambda_i\lambda_j,\quad c_3=\sum_{\substack{i,j,k=1\\ i\neq j\neq k}}^{4}\lambda_i\lambda_j\lambda_k$$

and $c_4=\prod_{i=1}^{4}\lambda_i$.

The Eq. (1) becomes linear when $\varepsilon=0$, and suppose the eigenvalues $-\lambda_1$ and $-\lambda_2$ are almost equal ($\lambda_1\approx\lambda_2$) and other two eigenvalues $-\lambda_3$ and $-\lambda_4$ are distinct. Therefore, the unperturbed solution is

$$x(t,0)=\frac{1}{2}a_{1,0}(e^{-\lambda_1 t}+e^{-\lambda_2 t})+a_{2,0}\left(\frac{e^{-\lambda_1 t}-e^{-\lambda_2 t}}{\lambda_1-\lambda_2}\right)+a_{3,0}\,e^{-\lambda_3 t}+a_{4,0}\,e^{-\lambda_4 t}, \tag{2}$$

where $a_{i,0}(i=1, 2, 3, 4)$ are arbitrary constants.

When $\varepsilon\neq 0$, following Alam (2002) technique we choose the solution of Eq. (1) in the form

$$\begin{aligned}x(t,\varepsilon)=\frac{1}{2}a_1(t)(e^{-\lambda_1 t}+e^{-\lambda_2 t})+a_2(t)\left(\frac{e^{-\lambda_1 t}-e^{-\lambda_2 t}}{\lambda_1-\lambda_2}\right)+a_3(t)e^{-\lambda_3 t}\\ +a_4(t)e^{-\lambda_4 t}+\varepsilon u_1(a_1,a_2,a_3,a_4,t)+\varepsilon^2\ldots,\end{aligned} \tag{3}$$

where $a_i(i=1, 2, 3, 4)$ satisfy the following first order differential equations:

$$\frac{da_i(t)}{dt}=\varepsilon A_1(a_1,a_2,a_3,a_4,t)+\varepsilon^2\cdots,\qquad i=1,2,3,4. \tag{4}$$

Confining only to a first few terms $1, 2, 3,\ldots, n$ in the series expansions Eqs. (3) and (4), we calculate the functions u_1 and A_i, $i=1, 2, 3, 4$ such that $a_i(t)$, $i=1, 2, 3, 4$ appearing in Eqs. (3) and (4) satisfy the given differential Eq. (1) with an accuracy of order ε^{n+1}. To determine the unknown functions u_1, A_1, A_2, A_3, A_4, it is assumed (as customary in the KBM method) that the correction term u_1 does not contain secular-type term $t\,e^{-\lambda_i t}$, which make them large. Differentiating Eq. (3) four times with respect to *t*, substituting the derivatives $\frac{d^4x}{dt^4}, \frac{d^3x}{dt^3}, \frac{d^2x}{dt^2}, \frac{dx}{dt}$ and x in the original Eq. (1), utilizing the relations presented in Eq. (4) and finally equating the coefficients of ε, we obtain

$$\frac{1}{2}\Big(e^{-\lambda_1 t}(D-\lambda_1+\lambda_2)(D-\lambda_1+\lambda_3)(D-\lambda_1+\lambda_4)+e^{-\lambda_2 t}(D-\lambda_2+\lambda_1)(D-\lambda_2+\lambda_3)(D-\lambda_2+\lambda_4)\Big)A_1$$
$$+(D+\lambda_4)\left(e^{-\lambda_1 t}(\lambda_1-\lambda_3-\frac{3}{2}D)+e^{-\lambda_2 t}(\lambda_2-\lambda_3-\frac{3}{2}D)\right)A_2+e^{-\lambda_3 t}(D-\lambda_3+\lambda_1)(D-\lambda_3+\lambda_2)\times$$
$$(D-\lambda_3+\lambda_4)A_3+e^{-\lambda_4 t}(D-\lambda_4+\lambda_1)(D-\lambda_4+\lambda_2)(D-\lambda_4+\lambda_3)A_4+\left(\frac{e^{-\lambda_1 t}-e^{-\lambda_2 t}}{\lambda_1-\lambda_2}\right)(D+\lambda_4)D\times \quad (5)$$
$$\left(D+\lambda_3-\frac{\lambda_1+\lambda}{2}\right)A_2-\left(\frac{\lambda_1 e^{-\lambda_1 t}-\lambda_2 e^{-\lambda_2 t}}{\lambda_1-\lambda_2}\right)D\left(D+\lambda_3-\frac{\lambda_1+\lambda}{2}\right)A_2+(D+\lambda_1)(D+\lambda_2)(D+\lambda_3)\times$$
$$(D+\lambda_4)u_1=-f^{(0)},$$

where $f^{(0)}=f(x_0)$

and $x_0=\frac{1}{2}a_1(t)(e^{-\lambda_1 t}+e^{-\lambda_2 t})+a_2(t)\left(\frac{e^{-\lambda_1 t}-e^{-\lambda_2 t}}{\lambda_1-\lambda_2}\right)+a_3(t)\,e^{-\lambda_3 t}+a_4(t)\,e^{-\lambda_4 t}$.

It is assumed that the function $f^{(0)}$ can be expanded in power series (Taylor's series) in the form (Bogoliubov and Mitropolskii 1961 for details)

$$f^{(0)}=\sum_{r=0}^{n}F_r(a_3e^{-\lambda_3 t},a_4e^{-\lambda_4 t})\left\{\frac{1}{2}a_1(e^{-\lambda_1 t}+e^{-\lambda_2 t})+a_2\left(\frac{e^{-\lambda_1 t}-e^{-\lambda_2 t}}{\lambda_1-\lambda_2}\right)\right\}^r, \quad (6)$$

where n is the order of polynomial of the nonlinear function f. This assumption is certainly valid when f is a polynomial function of x. Such polynomial functions cover some special and important systems in mechanics. Following Alam's (2002), in this paper the authors assumed that u_1 does not contain the terms F_0 and F_1 of $f^{(0)}$, since the system is considered to near critically damped. Substituting the value of $f^{(0)}$ from Eq. (6) into Eq. (5) and equating the coefficients of like powers of $\left(\frac{e^{\lambda_1 t}-e^{\lambda_2 t}}{\lambda_1-\lambda_2}\right)$, the present authors obtained

$$e^{-\lambda_3 t}(D-\lambda_3+\lambda_1)(D-\lambda_3+\lambda_2)(D-\lambda_3+\lambda_4)A_3+e^{-\lambda_4 t}(D-\lambda_4+\lambda_1)(D-\lambda_4+\lambda_2)(D-\lambda_4+\lambda_3)A_4$$
$$+\frac{1}{2}\{e^{-\lambda_1 t}(D-\lambda_1+\lambda_2)(D-\lambda_1+\lambda_3)(D-\lambda_1+\lambda_4)+e^{-\lambda_2 t}(D-\lambda_2+\lambda_1)(D-\lambda_2+\lambda_3)\times$$
$$(D-\lambda_2+\lambda_4)\}A_1+(D+\lambda_4)\{e^{-\lambda_1 t}(\lambda_1-\lambda_3-\frac{3}{2}D)+e^{-\lambda_2 t}(\lambda_2-\lambda_3-\frac{3}{2}D)\}A_2-(\frac{\lambda_1 e^{-\lambda_1 t}-\lambda_2 e^{-\lambda_2 t}}{\lambda_1-\lambda_2}) \quad (7)$$
$$\times D\left(D+\lambda_3-\frac{\lambda_1+\lambda_2}{2}\right)A_2=-F_0-\frac{1}{2}a_1(e^{-\lambda_1 t}+e^{-\lambda_2 t})F_1,$$

$$D(D+\lambda_4)\left(D+\lambda_3-\frac{\lambda_1+\lambda_2}{2}\right)A_2=-a_2F_1, \tag{8}$$

and

$$(D+\lambda_1)(D+\lambda_2)(D+\lambda_3)(D+\lambda_4)u_1$$
$$=-\sum_{r=2}^{n}F_r(a_3e^{-\lambda_3 t},a_4e^{-\lambda_4 t})\times\left\{\frac{1}{2}a_1(e^{-\lambda_1 t}+e^{-\lambda_2 t})+a_2\left(\frac{e^{-\lambda_1 t}-e^{-\lambda_2 t}}{\lambda_1-\lambda_2}\right)\right\}^r. \tag{9}$$

KBM (Krylov and Bogoliubov 1947, Bogoliubov and Mitropolskii 1961), Alam and Sattar (1996) and Alam (2002) have imposed the condition that u_1 does not contain the fundamental terms (the solution presented in Eq. (2) is called generating solution and its terms are called fundamental terms) of $f^{(0)}$. The solution of Eq. (8) gives value of the unknown function A_2. If the nonlinear function f and the eigenvalues $-\lambda_1,-\lambda_2,-\lambda_3,-\lambda_4$ of the corresponding linear equation of Eq. (1) are not specified then it is not easy to solve the Eq. (7) for the unknown functions A_1, A_3 and A_4. When these are specified, the values of A_1, A_3 and A_4 can be found subject to the condition that the coefficients in the solutions of A_1, A_3 and A_4 do not become large (Akbar *et al*. 2007, Alam and Sttar 1996, Alam 2002 for details), as if A_1, A_3 and A_4 do not contain terms involving $t\,e^{-t}$. In this article, we have imposed the conditions that the relation $\lambda_4 \approx \lambda_3 + 2\lambda_1$ but $\lambda_4 \langle 2\lambda_3$ exists among the eigenvalues $\lambda_1, \lambda_3, \lambda_4$ (also $\lambda_1 \to \lambda_2$ since the system is near critically damped). These relations are important, because under these relations the coefficients in the solutions of A_1, A_3 and A_4 do not become large. Under these imposed conditions, the authors obtained the values of A_1, A_3 and A_4 from Eq. (7). Substituting the values of A_1, A_2, A_3 and A_4 in the Eq. (4), the authors obtained the solutions of $\frac{da_i}{dt}$ $(i=1,\ 2,\ 3,\ 4)$, which are proportional to the small parameter ε.

So these are slowly varying functions of time t, that is, these are almost constants and by integrating the values of a_i $(i=1,\ 2,\ 3, 4)$ are obtained. It is laborious work to solve the Eq. (9) for u_1. However, as $\lambda_1 \to \lambda_2$ it takes the following simple form

$$(D+\lambda_1)^2(D+\lambda_3)(D+\lambda_4)u_1=-\sum_{r=2}^{n}F_r(a_3e^{-\lambda_3 t},a_4e^{-\lambda_4 t})\left\{e^{-\lambda_1 t}(a_1-a_2t)\right\}^r. \tag{10}$$

Solving Eq. (10), we obtain the value of u_1. Finally, substituting the values of a_i $(i=1,\ 2,\ 3,\ 4)$ and u_1 in the Eq. (3), we obtain the complete solution of Eq. (1).

EXAMPLE

For an example of the above method, we consider the following fourth order nonlinear differential equation,

$$\frac{d^4x}{dt^4} + c_1\frac{d^3x}{dt^3} + c_2\frac{d^2x}{dt^2} + c_3\frac{dx}{dt} + c_4 x = -\varepsilon\ x^3 . \tag{11}$$

Here $f(x) = x^3$ and $x_0 = \frac{1}{2}a_1(e^{-\lambda_1 t} + e^{-\lambda_2 t}) + a_2\left(\frac{e^{-\lambda_1 t} - e^{-\lambda_2 t}}{\lambda_1 - \lambda_2}\right) + a_3 e^{-\lambda_3 t} + a_4 e^{-\lambda_4 t}$.

Thus, $f^{(0)} = \left\{\frac{1}{2}a_1(e^{-\lambda_1 t} + e^{-\lambda_2 t}) + a_2\left(\frac{e^{-\lambda_1 t} - e^{-\lambda_2 t}}{\lambda_1 - \lambda_2}\right) + a_3 e^{-\lambda_3 t} + a_4 e^{-\lambda_4 t}\right\}^3$,

$$F_0 = (a_3 e^{-\lambda_3 t} + a_4 e^{-\lambda_4 t})^3 ,\ F_1 = 3(a_3 e^{-\lambda_3 t} + a_4 e^{-\lambda_4 t})^2 . \tag{12}$$

According to the Eqs. (7) - (9), we obtain

$$\begin{aligned}
& e^{-\lambda_3 t}(D - \lambda_3 + \lambda_1)(D - \lambda_3 + \lambda_2)(D - \lambda_3 + \lambda_4)A_3 + e^{-\lambda_4 t}(D - \lambda_4 + \lambda_1)(D - \lambda_4 + \lambda_2)\times \\
& (D - \lambda_4 + \lambda_3)A_4 + \frac{1}{2}\{e^{-\lambda_1 t}(D - \lambda_1 + \lambda_2)(D - \lambda_1 + \lambda_3)(D - \lambda_1 + \lambda_4) \\
& + e^{-\lambda_2 t}(D - \lambda_2 + \lambda_1)(D - \lambda_2 + \lambda_3)(D - \lambda_2 + \lambda_3)\}A_1 + (D + \lambda_4)\{e^{-\lambda_1 t}(\lambda_1 - \lambda_3 - \frac{3}{2}D) \\
& + e^{-\lambda_2 t}(\lambda_2 - \lambda_3 - \frac{3}{2}D)\}A_2 - (\frac{(\lambda_1 e^{-\lambda_1 t} - \lambda_2 e^{-\lambda_2 t})}{(\lambda_1 - \lambda_2)})D(D + \lambda_3 - \frac{\lambda_1 + \lambda_2}{2})A_2 \\
& = -[(a_3 e^{-\lambda_3 t} + a_4 e^{-\lambda_4 t})^3 + 3(a_3 e^{-\lambda_3 t} + a_4 e^{-\lambda_4 t})^2 \frac{1}{2}a_1(e^{-\lambda_1 t} + e^{-\lambda_2 t})],
\end{aligned} \tag{13}$$

$$D(D + \lambda_4)(D + \lambda_3 - \frac{\lambda_1 + \lambda_2}{2})A_2 = -3a_2(a_3 e^{-\lambda_3 t} + a_4 e^{-\lambda_4 t})^2, \tag{14}$$

and

$$\begin{aligned}
& (D + \lambda_1)(D + \lambda_2)(D + \lambda_3)(D + \lambda_4)u_1 \\
& = -\sum_{r=2}^{3} F_r(a_3 e^{-\lambda_3 t}, a_4 e^{-\lambda_4 t})\left\{\frac{1}{2}a_1(e^{-\lambda_1 t} + e^{-\lambda_2 t}) + a_2\left(\frac{e^{-\lambda_1 t} - e^{-\lambda_2 t}}{\lambda_1 - \lambda_2}\right)\right\}^r .
\end{aligned} \tag{15}$$

Solving Eq. (14), we obtain

$$A_2 = a_2[n_1 a_3^2 e^{-2\lambda_3 t} + n_2 a_3 a_4 e^{-(\lambda_3 + \lambda_4)t} + n_3 a_4^2 e^{-2\lambda_4 t}] , \tag{16}$$

where

$$\begin{aligned}
& n_1 = \frac{3}{\lambda_3(\lambda_1 + \lambda_2 + 2\lambda_3)(2\lambda_3 - \lambda_4)}, \quad n_2 = \frac{12}{\lambda_3(\lambda_3 + \lambda_4)(\lambda_1 + \lambda_2 + 2\lambda_4)}, \\
& n_3 = \frac{3}{\lambda_4^2(\lambda_1 + \lambda_2 - 2\lambda_3 + 4\lambda_4)}.
\end{aligned} \tag{17}$$

Substituting the value of A_2 from Eq. (16) into Eq. (13). In order to separate the Eq. (13) for determining the unknown functions A_1, A_3 and A_4, we use the conditions as discussed in the method (see also Akbar *et al*. 2007 and Alam 2002). It is to note that our solution approaches toward critically damped solution (see Alam 2002) if $\lambda_1 \to \lambda_2$. However, Eq. (13) has not an exact solution unless $\lambda_1 \to \lambda_2$. Now we consider $\lambda_4 \approx \lambda_3 + 2\lambda_1$ but $\lambda_4 \langle 2\lambda_3$. Under these imposed conditions and by equating like terms on both sides of the Eq. (13), we obtain

$$\begin{aligned} & e^{-\lambda_1 t}(D-\lambda_1+\lambda_2)(D-\lambda_1+\lambda_3)(D-\lambda_1+\lambda_4)A_1 \\ & = -a_2 a_3^2 n_1 \lambda_2 \lambda_3 (\lambda_1+\lambda_2+2\lambda_3) t e^{-(\lambda_1+2\lambda_3)t} - \frac{1}{2} a_2 a_3 a_4 n_2 \lambda_2 (2\lambda_4^2 + 2\lambda_3\lambda_4 + \lambda_1\lambda_3 + \lambda_2\lambda_3 \\ & \quad + \lambda_1\lambda_4 + \lambda_2\lambda_4) t e^{-(\lambda_1+\lambda_3+\lambda_4)t} - a_2 a_4^2 n_3 \lambda_2 \lambda_4 (\lambda_1+\lambda_2-2\lambda_3+4\lambda_4) t e^{-(\lambda_1+2\lambda_4)t}, \end{aligned} \tag{18}$$

$$\begin{aligned} & e^{-\lambda_3 t}(D-\lambda_3+\lambda_1)(D-\lambda_3+\lambda_2)(D-\lambda_3+\lambda_4)A_3 \\ & = [a_2 n_1 \{(\lambda_1+2\lambda_3)(\lambda_1+2\lambda_3-\lambda_4)+\lambda_3(\lambda_1+\lambda_2+2\lambda_3)\} - \frac{3}{2}a_1] a_3^2 e^{-(\lambda_1+2\lambda_3)t} \\ & \quad + [a_2 n_1 (\lambda_2+2\lambda_3)(\lambda_2+2\lambda_3-\lambda_4) - \frac{3}{2}a_1] a_3^2 e^{-(\lambda_2+2\lambda_3)t} - a_3^3 e^{-3\lambda_3 t} - 3a_3^2 a_4 e^{-(2\lambda_3+\lambda_4)t}, \end{aligned} \tag{19}$$

and

$$\begin{aligned} & e^{-\lambda_4 t}(D-\lambda_4+\lambda_1)(D-\lambda_4+\lambda_2)(D-\lambda_4+\lambda_3)A_4 \\ & = [\frac{1}{2} a_2 n_2 \{(\lambda_1+\lambda_3)(2\lambda_1+\lambda_3+3\lambda_4)+(2\lambda_4^2+2\lambda_3\lambda_4+\lambda_1\lambda_3+\lambda_2\lambda_3+\lambda_1\lambda_4+\lambda_2\lambda_4)\} \\ & \quad -3a_1] a_3 a_4 e^{-(\lambda_1+\lambda_3+\lambda_4)t} + [a_2 n_3 \{(\lambda_1+\lambda_4)(\lambda_1-\lambda_3+3\lambda_4)+\lambda_4(\lambda_1+\lambda_2-2\lambda_3+4\lambda_4)\} \\ & \quad -\frac{3}{2}a_1] a_4^2 e^{-(\lambda_1+2\lambda_4)t} + [\frac{1}{2} a_2 n_2 (\lambda_2+\lambda_3)(2\lambda_2+\lambda_3+3\lambda_4) - 3a_1] a_3 a_4 e^{-(\lambda_2+\lambda_3+\lambda_4)t} \\ & \quad + [a_2 n_3 (\lambda_2+\lambda_4)(\lambda_2-\lambda_3+3\lambda_4) - \frac{3}{2}a_1] a_4^2 e^{-(\lambda_2+2\lambda_4)t} - [3a_3 a_4^2 e^{-(\lambda_3+2\lambda_4)t} + a_4^3 e^{-3\lambda_4 t}]. \end{aligned} \tag{20}$$

The particular solutions of Eqs. (18)- (20) yield respectively

$$\begin{aligned} A_1 &= i_1 a_2 a_3^2 t e^{-(\lambda_1-\lambda_2+2\lambda_3)t} + i_2 a_2 a_3^2 t e^{-(\lambda_1-\lambda_2+2\lambda_3)t} + i_3 a_2 a_3 a_4 t e^{-(\lambda_1-\lambda_2+\lambda_3+\lambda_4)t} \\ & \quad + i_4 a_2 a_3 a_4 t e^{-(\lambda_1-\lambda_2+\lambda_3+\lambda_4)t} + i_5 a_2 a_4^2 t e^{-(\lambda_1-\lambda_2+2\lambda_4)t} + i_6 a_2 a_4^2 t e^{-(\lambda_1-\lambda_2+2\lambda_4)t}, \end{aligned} \tag{21}$$

$$\begin{aligned} A_3 &= (M_1 a_2 + M_2 a_1) a_3^2 e^{-(\lambda_1+\lambda_3)t} + (M_3 a_2 + M_4 a_1) a_3^2 e^{-(\lambda_2+\lambda_3)t} + M_5 a_3^3 e^{-2\lambda_3 t} \\ & \quad + M_6 a_3^2 a_4 e^{-(\lambda_3+\lambda_4)t}, \end{aligned} \tag{22}$$

and

$$\begin{aligned} A_4 &= (S_1 a_2 + S_2 a_1) a_3 a_4 e^{-(\lambda_1+\lambda_3)t} + (S_3 a_2 + S_4 a_1) a_4^2 e^{-(\lambda_1+\lambda_4)t} + (S_5 a_2 + S_6 a_1) a_3 a_4 e^{-(\lambda_2+\lambda_3)t} \\ & \quad + (S_7 a_2 + S_8 a_1) a_4^2 e^{-(\lambda_2+\lambda_4)t} + S_9 a_3 a_4^2 e^{-(\lambda_3+\lambda_4)t} + S_{10} a_4^3 e^{-2\lambda_4 t}, \end{aligned} \tag{23}$$

where

$$r_1 = -n_1\lambda_2\lambda_3(\lambda_1+\lambda_2+2\lambda_3),$$

$$r_2 = -\frac{1}{2}n_2\lambda_2(2\lambda_4^2+2\lambda_3\lambda_4+\lambda_1\lambda_3+\lambda_2\lambda_3+\lambda_1\lambda_4+\lambda_2\lambda_4),$$

$$r_3 = -n_3\lambda_2\lambda_4(\lambda_1+\lambda_2-2\lambda_3+4\lambda_4),$$

$$i_1 = -\frac{r_1}{2\lambda_3(\lambda_1+\lambda_3)(\lambda_1+2\lambda_3-\lambda_4)},$$

$$i_2 = -\frac{r_1}{2\lambda_3(\lambda_1+\lambda_3)(\lambda_1+2\lambda_3-\lambda_4)}\left(\frac{1}{2\lambda_3}+\frac{1}{(\lambda_1+\lambda_3)}+\frac{1}{(\lambda_1+2\lambda_3-\lambda_4)}\right),$$

$$i_3 = -\frac{r_2}{(\lambda_1+\lambda_3)(\lambda_1+\lambda_4)(\lambda_3+\lambda_4)},$$

$$i_4 = -\frac{r_2}{(\lambda_1+\lambda_3)(\lambda_1+\lambda_4)(\lambda_3+\lambda_4)}\left(\frac{1}{(\lambda_1+\lambda_3)}+\frac{1}{(\lambda_1+\lambda_4)}+\frac{1}{(\lambda_3+\lambda_4)}\right),$$

$$i_5 = -\frac{r_3}{2\lambda_4(\lambda_1+\lambda_4)(\lambda_1+2\lambda_4-\lambda_3)},$$

$$i_6 = -\frac{r_3}{2\lambda_4(\lambda_1+\lambda_4)(\lambda_1+2\lambda_4-\lambda_3)}\left(\frac{1}{2\lambda_4}+\frac{1}{(\lambda_1+\lambda_4)}+\frac{1}{(\lambda_1+2\lambda_4-\lambda_3)}\right),$$

$$m_1 = n_1\{(\lambda_1+2\lambda_3)(\lambda_1+2\lambda_3-\lambda_4)+\lambda_3(\lambda_1+\lambda_2+2\lambda_3)\},\quad m_2 = -\frac{3}{2},$$

$$m_3 = n_1(2\lambda_3+\lambda_2)(2\lambda_3+\lambda_2-\lambda_4),\quad m_4 = -\frac{3}{2},$$

$$s_1 = \frac{1}{2}n_2\{(\lambda_1+\lambda_3)(2\lambda_1+\lambda_3+3\lambda_4)+(2\lambda_4^2+2\lambda_3\lambda_4+\lambda_1\lambda_3+\lambda_2\lambda_3+\lambda_1\lambda_4+\lambda_2\lambda_4)\},$$

$$s_2 = -3,\quad s_3 = n_3\{(\lambda_1+\lambda_4)(\lambda_1-\lambda_3+3\lambda_4)+\lambda_4(\lambda_1+\lambda_2-2\lambda_3+4\lambda_4)\},$$

$$s_4 = -\frac{3}{2},\quad s_5 = \frac{1}{2}n_2(\lambda_2+\lambda_3)(2\lambda_2+\lambda_3+3\lambda_4),\quad s_6 = -3,$$

$$s_7 = n_3(\lambda_2+\lambda_4)(\lambda_2-\lambda_3+3\lambda_4),\quad s_8 = -\frac{3}{2},$$

$$M_1 = -\frac{m_1}{2\lambda_3(\lambda_1+2\lambda_3-\lambda_2)(\lambda_1+2\lambda_3-\lambda_4)},\quad M_2 = -\frac{m_2}{2\lambda_3(\lambda_1+2\lambda_3-\lambda_2)(\lambda_1+2\lambda_3-\lambda_4)},$$

$$M_3 = -\frac{m_3}{2\lambda_3(\lambda_2+2\lambda_3-\lambda_1)(\lambda_2+2\lambda_3-\lambda_4)},\quad M_4 = -\frac{m_4}{2\lambda_3(\lambda_2+2\lambda_3-\lambda_1)(\lambda_2+2\lambda_3-\lambda_4)},$$

$$M_5 = \frac{1}{(\lambda_1-3\lambda_3)(\lambda_2-3\lambda_3)(3\lambda_3-\lambda_4)},\quad M_6 = \frac{3}{2\lambda_3(\lambda_1-2\lambda_3-\lambda_4)(\lambda_2-2\lambda_3-\lambda_4)},$$

$$S_1 = -\frac{s_1}{(\lambda_1+\lambda_4)(\lambda_3+\lambda_4)(-\lambda_1+\lambda_2-\lambda_3-\lambda_4)},\quad S_2 = -\frac{s_2}{(\lambda_1+\lambda_4)(\lambda_3+\lambda_4)(-\lambda_1+\lambda_2-\lambda_3-\lambda_4)},$$

$$S_3 = \frac{s_3}{2\lambda_4(\lambda_1-\lambda_2+2\lambda_4)(-\lambda_1+\lambda_3-2\lambda_4)},\quad S_4 = \frac{s_4}{2\lambda_4(\lambda_1-\lambda_2+2\lambda_4)(-\lambda_1+\lambda_3-2\lambda_4)},$$

$$S_5 = -\frac{s_5}{(\lambda_2+\lambda_4)(\lambda_3+\lambda_4)(\lambda_1-\lambda_2-\lambda_3-\lambda_4)},\quad S_6 = -\frac{s_6}{(\lambda_2+\lambda_4)(\lambda_3+\lambda_4)(\lambda_1-\lambda_2-\lambda_3-\lambda_4)},$$

$$S_7 = -\frac{s_7}{2\lambda_4(\lambda_1-\lambda_2-2\lambda_4)(\lambda_2-\lambda_3+2\lambda_4)}, \qquad S_8 = -\frac{s_8}{2\lambda_4(\lambda_1-\lambda_2-2\lambda_4)(\lambda_2-\lambda_3+2\lambda_4)},$$
$$S_9 = \frac{3}{2\lambda_4(\lambda_1-\lambda_3-2\lambda_4)(\lambda_2-\lambda_3-2\lambda_4)}, \qquad S_{10} = -\frac{1}{(\lambda_1-3\lambda_4)(\lambda_2-3\lambda_4)(\lambda_3-3\lambda_4)}. \tag{24}$$

Here u_1 is a correction term and has also very small contribution in the solution. However it is laborious work to solve the Eq. (9) for u_1. So we neglect the calculation of u_1. Putting the values of A_1, A_2, A_3 and A_4 from Eqs. (21), (16), (22), (23) into Eq. (4) and integrating, we obtain

$$a_1(t) = a_{1,0} + \varepsilon\Big[a_{2,0}\, a_{3,0}^2 \times \left\{\begin{matrix} i_2\left(1-e^{(-\lambda_1+\lambda_2-2\lambda_3)t}\right) \\ -i_1\left(t\,e^{(-\lambda_1+\lambda_2-2\lambda_3)t} + \dfrac{e^{(-\lambda_1+\lambda_2-2\lambda_3)t}-1}{\lambda_1-\lambda_2+2\lambda_3}\right)\end{matrix}\right\}\Big/\left(\lambda_1-\lambda_2+2\lambda_3\right)$$
$$+ a_{2,0}\, a_{3,0}\, a_{4,0} \times \left\{\begin{matrix} i_4\left(1-e^{(-\lambda_1+\lambda_2-\lambda_3-\lambda_4)t}\right) \\ -i_3\left(t\,e^{(-\lambda_1+\lambda_2-\lambda_3-\lambda_4)t} + \dfrac{e^{(-\lambda_1+\lambda_2-\lambda_3-\lambda_4)t}-1}{\lambda_1-\lambda_2+\lambda_3+\lambda_4}\right)\end{matrix}\right\}\Big/\left(\lambda_1-\lambda_2+\lambda_3+\lambda_4\right)$$
$$+ a_{2,0}\, a_{4,0}^2 \times \left\{\begin{matrix} i_6\left(1-e^{(-\lambda_1+\lambda_2-2\lambda_4)t}\right) \\ -i_5\left(t\,e^{(-\lambda_1+\lambda_2-2\lambda_4)t} + \dfrac{e^{(-\lambda_1+\lambda_2-2\lambda_4)t}-1}{\lambda_1-\lambda_2+2\lambda_4}\right)\end{matrix}\right\}\Big/\left(\lambda_1-\lambda_2+2\lambda_4\right)\Big],$$

$$a_2(t) = a_{2,0} + \varepsilon\, a_{2,0}\left[n_1 a_{3,0}^2\left(\frac{1-e^{-2\lambda_3 t}}{2\lambda_3}\right) + n_2\, a_{3,0}\, a_{4,0}\left(\frac{1-e^{-(\lambda_3+\lambda_4)t}}{\lambda_3+\lambda_4}\right) + n_3 a_{4,0}^2\left(\frac{1-e^{-2\lambda_4 t}}{2\lambda_4}\right)\right],$$

$$a_3(t) = a_{3,0} + \varepsilon\Big[a_{3,0}^2\left\{M_1 a_{2,0} + M_2\, a_{1,0}\right\}\left(\frac{1-e^{-(\lambda_1+\lambda_3)t}}{\lambda_1+\lambda_3}\right) + a_{3,0}^2\left\{M_3 a_{2,0} + M_4 a_{1,0}\right\}\left(\frac{1-e^{-(\lambda_2+\lambda_3)t}}{\lambda_2+\lambda_3}\right)$$
$$+ a_{3,0}^3 M_5\left(\frac{1-e^{-2\lambda_3 t}}{2\lambda_3}\right) + a_{3,0}^2 a_4 M_6\left(\frac{1-e^{-(\lambda_3+\lambda_4)t}}{\lambda_3+\lambda_4}\right)\Big],$$

$$a_4(t) = a_{4,0} + \varepsilon\Big[a_{3,0}\, a_{4,0}\left\{S_1 a_{2,0} + S_2\, a_{1,0}\right\}\left(\frac{1-e^{-(\lambda_1+\lambda_3)t}}{\lambda_1+\lambda_3}\right) + a_{4,0}^2\left\{S_3 a_{2,0} + S_4 a_{1,0}\right\}\left(\frac{1-e^{-(\lambda_1+\lambda_4)t}}{\lambda_1+\lambda_4}\right)$$
$$+ a_{3,0}\, a_{4,0}\left\{S_5\, a_{2,0} + S_6\, a_{1,0}\right\}\left(\frac{1-e^{-(\lambda_2+\lambda_3)t}}{\lambda_2+\lambda_3}\right) + a_{4,0}^2\left\{S_7\, a_{2,0} + S_8\, a_{1,0}\right\}\left(\frac{1-e^{-(\lambda_2+\lambda_4)t}}{\lambda_2+\lambda_4}\right)$$
$$+ a_{3,0}\, a_{4,0}^2 S_9\left(\frac{1-e^{-(\lambda_3+\lambda_4)t}}{\lambda_3+\lambda_4}\right) + a_{4,0}^3 S_{10}\left(\frac{1-e^{-2\lambda_4 t}}{2\lambda_4}\right)\Big]. \tag{25}$$

Therefore, we obtain the first approximate solution of the Eq. (11) is given by

$$x(t,\varepsilon) = \frac{1}{2}a_1(e^{-\lambda_1 t} + e^{-\lambda_2 t}) + a_2\left(\frac{e^{-\lambda_1 t} - e^{-\lambda_2 t}}{\lambda_1-\lambda_2}\right) + a_3\, e^{-\lambda_3 t} + a_4\, e^{-\lambda_4 t} \tag{26}$$

where a_1, a_2, a_3, a_4 are given by the Eq. (25).

RESULTS AND DISCUSSION

To test the accuracy of the approximate analytical solutions obtained by the presented technique have been compared to the numerical solutions. Firstly, $x(t,\varepsilon)$ is calculated by the Eq. (26) by using the imposed conditions $\lambda_1 \to \lambda_2$, $\lambda_4 \approx \lambda_3 + 2\lambda_1$ but $\lambda_4 \langle 2\lambda_3$ in which a_1, a_2, a_3, a_4 are calculated by the Eq. (25). Moreover, if we replace $-\varepsilon f(x)$ by $\varepsilon f(x)$ with $\varepsilon << 1$, then the analytical approximate solutions will also be conversed to the corresponding numerical solutions. The corresponding numerical solutions of Eq. (11) are computed by fourth order Runge-Kutta method. The analytical approximate solutions and numerical solutions are plotted in Fig.1 and Fig. 2 for different initial conditions.

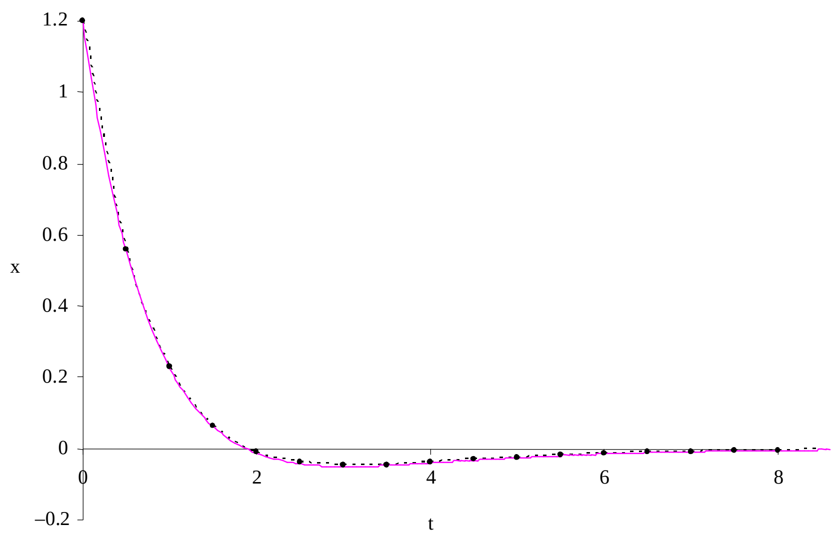

Fig.1. First approximate solution of Eq. (11) is denoted by $-\bullet-$ (dashed lines) by the presented method with the initial conditions $a_{1,0} = 0.6,\ a_{2,0} = 0.6,\ a_{3,0} = 0.6,\ a_{4,0} = 0.6$ or $[x(0) = 1.80000,\ \dot{x}(0) = -2.61561,\ \ddot{x}(0) = 3.33514,\ \dddot{x}(0) = -3.88353]$ when $\lambda_1 = 0.7$, $\lambda_2 = 0.95$, $\lambda_3 = 1.18$, $\lambda_4 = 1.35$, $\varepsilon = 0.1$ and $f = x^3$. Corresponding numerical solution is denoted by - (solid line).

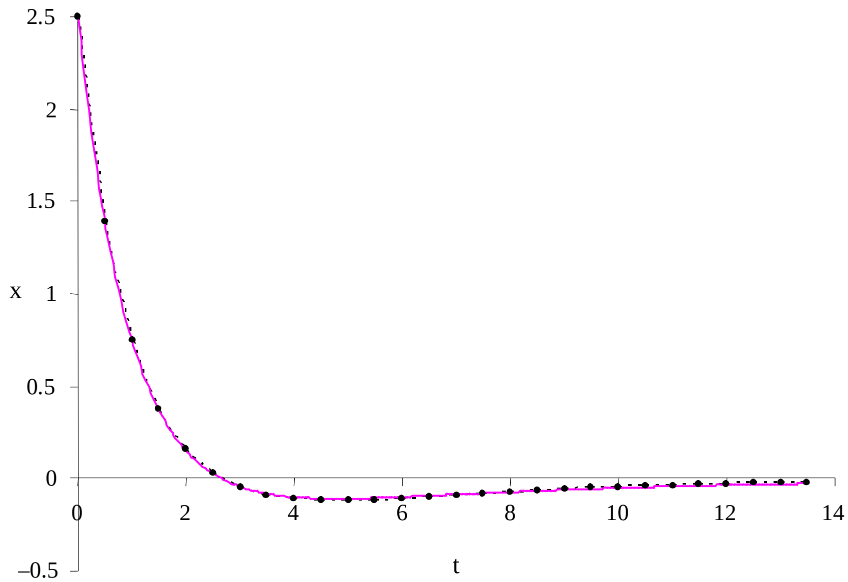

Fig. 2. First approximate solution of Eq. (11) is denoted by –•– (dashed lines) by the presented method with the initial conditions $a_{1,0} = 1.0,\ a_{2,0} = 0.5,\ a_{3,0} = 1.0,\ a_{4,0} = 0.5$ or $[x(0) = 2.50000,\ \dot{x}(0) = -2.88042,\ \ddot{x}(0) = 3.13488,\ \dddot{x}(0) = -3.07011]$ when $\lambda_1 = 0.25, \lambda_2 = 0.8,$ $\lambda_3 = 1.2,\ \lambda_4 = 1.43,\ \varepsilon = 0.1$ and $f = x^3$. Corresponding numerical solution is denoted by - (solid line).

CONCLUSION

The KBM method has been extended for solving the fourth order near critically damped nonlinear systems under some special conditions with small nonlinearities, when the four eigenvalues of the corresponding linear equation are real and negative numbers. It is also noted that the analytical approximate solutions will be conversed to the corresponding numerical solutions obtained by the fourth order Rangue-Kutta method whether the small parameter is positive or negative. From the Figs 1 - 2, it is noticed that the solutions obtained by the presented method show good agreement with those obtained by the numerical method.

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(Received revised manuscript on 8 May, 2012)