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ON THE EXACT SOLITON SOLUTIONS OF SOME NONLINEAR PDE BY TAN-COT METHOD

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Abstract**—** For many solitary wave applications, various approximate models have been proposed. Certainly, the most famous solitary wave equations are the K-dV, BBM and Boussinesq equations. The K-dV equation was originally derived to describe shallow water waves in a rectangular channel. Surprisingly, the equation also models ion-acoustic waves and magneto-hydrodynamic waves in plasmas, waves in elastic rods, equatorial planetary waves, acoustic waves on a crystal lattice, and more. If we describe all of the above situation, we must be needed a solution function of their governing equations. The Tan-cot method is applied to obtain exact travelling wave solutions to the generalized Korteweg-de Vries (gK-dV) equation and generalized Benjamin-Bona-Mahony (BBM) equation which are important equations to evaluate wide variety of physical applications. In this paper we described the soliton behavior of gK-dV and BBM equations by analytical system especially using Tan-cot method and shown in graphically.

Index Terms**—**Tan-cot function method; Travelling wave; BBM equation; gK-dV equation.

I. INTRODUCTION

IN many branches of physics and engineering nonlinear partial differential equations (PDEs) plays an important role. In terms of many physical phenomena where soliton waves can be found, it can be described by nonlinear PDEs. Solitons arise as the solutions of a widespread class of weakly nonlinear dispersive partial differential equations describing physical systems. Solitons are solitary waves with elastic scattering property. Due to dynamical balance between the nonlinear and dispersive effects these waves retain their shapes and speed to a stable waveform after colliding with each other.

One basic expression of a solitary wave solution is of the form [1]:

$$u(x,t) = f(x - ct),$$

where c is the speed of wave propagation. For $c > o$, the wave moves in the positive x direction, whereas the wave moves in the negative x direction for $c < 0$. Travelling waves, whether their solution expressions are in explicit or implicit forms are very interesting from the point of view of applications. Getting inspiration from many applications of gK-dV and BBM equations in real life problems, we attempt to obtain the solitary solutions of these equations. The term soliton is by Zabusky and Kruskal after their findings that waves like particles retained their shapes and velocities after interactions [2].

In recent years, quite a few methods for obtaining explicit traveling and solitary wave solutions of nonlinear evolution equations have been proposed. A variety of useful methods notably Algebraic method [3], Tanh function method [4], Variational method [5], Homotopy perturbation method [6], Hirota method [7], Backlund transformations method [8,9], Darboux transformations [10], Sine-cosine method [11], Tanh-sech method [12-14], Extended tanh method [15], Hyperbolic function method [16], Jacobi elliptic function expansion method [17], The First Integral method [18] and Tan-cot method [19] were applied to investigate the solutions of nonlinear partial differential equations. The Tan-cot method is a direct and effective algebraic method for handling many nonlinear evolution equations. It is a good tool to solve nonlinear partial differential equations with genuine non-linear dispersion where solitary patterns solutions are generated [20]. The goal of the present

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work is to get exact soliton solutions of gK-dV and BBM equations by using Tan-cot method.

The paper is organized in the following sequence. In Section II, we present the method of Tan-cot function. The Application of Tan-cot method in terms of gK-dV and BBM equation are derived in Section III. The results along with discussion discuss in section IV. Finally, this paper ends with conclusion in section V.

II. THE TAN-COT FUNCTION METHOD

Consider nonlinear partial differential equations in the form:

$$Q(u, u_x, u_t, u^n u_x, u_{xx}, u_{xt}, u_{xxt}, \dots) = 0, \quad (1)$$

where $u(x,t)$ is the soliton solution of (1). The term $u^n u_x$ shows non-linearity in (1). Since (1) admits soliton solution, we can use the transformation:

$$u(x,t) = u(\xi) \quad (2)$$

where $\xi = x - ct + d,\ c$ is the speed of traveling wave and d is the constant.

If we use the above transformation, obviously we get the following changes:

$$\frac{\partial}{\partial t}(\,.\,) = -c\frac{d}{d\xi}(\,.\,),\ \frac{\partial}{\partial x}(\,.\,) = \frac{d}{d\xi}(\,.\,),$$

$$\frac{\partial^2}{\partial x^2}(\,.\,) = \frac{d^2}{d\xi^2}(\,.\,), \frac{\partial^2}{\partial u \partial t}(\,.\,) = -c\frac{d^2}{d\xi^2}(\,.\,) \quad (3)$$

Making use of (3) into (1) yields a nonlinear ordinary differential equation (ODE):

$$Q(u, u_\xi, u^n u_\xi, u_{\xi\xi}, u_{\xi\xi\xi} \dots \dots \dots) = 0 \quad (4)$$

The equation (4) is then integrated as long as all terms involve derivatives of u, where we ignore the constant of integration ($since\ u \to 0\ as\ \xi \to \pm\infty$). To get solution of (1) by applying Tan-cot method, the function $u(\xi)$ can be expressed either in the following forms:

$$u(\xi) = \alpha tan^\beta(\mu\xi)\ , \qquad |\mu\xi| \le \frac{\pi}{2} \quad (5)$$

or

$$u(\xi) = \alpha cot^\beta(\mu\xi), \qquad |\mu\xi| \le \frac{\pi}{2} \quad (6)$$

where $\alpha, \mu\ and\ \beta \neq 0$ are parameters that will be determined, $\mu\ and\ c$ are the wave number and the wave speed respectively. From (5),

$$u(\xi) = \alpha tan^\beta(\mu\xi)$$

$$u_\xi(\xi) = \alpha\beta\mu\left[tan^{\beta-1}(\mu\xi) + tan^{\beta+1}(\mu\xi)\right], \quad (7)$$

And

$$u_{\xi\xi} = \alpha\beta\mu^2\left[(\beta-1)\,tan^{\beta-2}(\mu\xi) + 2\beta tan^\beta(\mu\xi) + (\beta+1)\,tan^{\beta+2}(\mu\xi)\right], \quad (8)$$

Also,

$$u_{\xi\xi\xi} = \alpha\beta\mu^2\left[\mu(\beta-1)(\beta-2)tan^{\beta-3}(\mu\xi)\left(1 + tan^2(\mu\xi)\right) + 2\mu\beta^2 tan^{\beta-1}(\mu\xi)\left(1 + tan^2(\mu\xi)\right) + \mu(\beta+1)(\beta+2)tan^{\beta+1}(\mu\xi)\left(1 + tan^2(\mu\xi)\right)\right]$$

$$\Rightarrow u_{\xi\xi\xi} = \alpha\beta\mu^3\left[(\beta-1)(\beta-2)tan^{\beta-3}(\mu\xi) + (3\beta^2 - 3\beta + 2)tan^{\beta-1}(\mu\xi) + (3\beta^2 + 3\beta + 2)tan^{\beta+1}(\mu\xi) + (\beta+1)(\beta+2)tan^{\beta+3}(\mu\xi)\right], \quad (9)$$

We find all the solutions considering the form $\alpha tan^\beta(\mu\xi)$. Therefore derivatives are calculated intentionally only for this form. We can obtain an equation in different powers of tangent functions substituting the values of derivatives from (7)-(9) into (4).Then we collect the coefficients of each pair of tangent functions with same exponent from (4), where each term has to vanish. Consequently, one can obtain a system of algebraic equations in unknown parameters $c, \mu, \beta\ and\ \mu$. Solving this system, we can get the soliton solutions of partial differential equation (1) by substituting the values of these parameters in (5).

III. APPLICATION

A. Generalized Korteweg-de Vries (gK-dV) equation

Consider the following generalized Korteweg-de Vries (gK-dV) equation:

$$u_t + (n+1)(n+2)u^n u_x + u_{xxx} = 0. \quad (10)$$

We now employ the Tan-cot method. Using the wave variable $\xi = x - ct + d$ transforms (10) into the ODE:

$$-cu_\xi + (n+1)(n+2)u^n u_\xi + u_{\xi\xi\xi} = 0, \quad (11)$$

Integrating (11) and considering constant of integration to be zero for simplicity, we get

$$-cu + (n+2)u^{n+1} + u_{\xi\xi} = 0. \quad (12)$$

Substituting values of (5) and (8) into (12) gives

$$-c\alpha tan^\beta(\mu\xi) + (n+2)\alpha^{n+1}tan^{\beta(n+1)}(\mu\xi) + \alpha\beta\mu^2\left[(\beta-1)tan^{\beta-2}(\mu\xi) + 2\beta tan^\beta(\mu\xi) + (\beta+1)tan^{\beta+2}(\mu\xi)\right] = 0$$

$$\Rightarrow (-c\alpha + 2\alpha\beta^2\mu^2)tan^\beta(\mu\xi) + (n+2)\alpha^{n+1}tan^{\beta(n+1)}(\mu\xi) + \alpha\beta\mu^2(\beta-1)tan^{\beta-2}(\mu\xi) + \alpha\beta\mu^2(\beta+1)tan^{\beta+2}(\mu\xi) = 0. \quad (13)$$

Equating the exponent and the coefficient of each pair of the tan function, we find the following system of algebraic equations:

$$-c\alpha + 2\alpha\beta^2\mu^2 = 0\,,$$
$$\beta(n+1) = \beta + 2\,, \quad (14)$$
$$(n+2)\alpha^{n+1} + \alpha\beta\mu^2(\beta+1) = 0.$$

Solving the system (14) by Mathematica yields

$$\beta = \frac{2}{n},$$
$$\mu = \frac{n}{2\sqrt{2}}\sqrt{c}\,, \quad (15)$$
$$\alpha = \left(-\frac{c}{4}\right)^{\frac{1}{n}}.$$

Therefore soliton solution of gK-dV equation (10) is obtain from (5) as:

$$\begin{aligned} \mathrm{u(x,t)} &= \alpha tan^{\beta}(\mu\xi) \\ &= \left(-\frac{c}{4}\right)^{\frac{1}{n}} tan^{\frac{2}{n}}\left(\frac{n}{2\sqrt{2}}\sqrt{c}\xi\right) \\ &= \left[-\frac{c}{4}tan^2\left(\frac{n}{2\sqrt{2}}\sqrt{c}\xi\right)\right]^{\frac{1}{n}} \\ &= \left[\frac{c}{4}\left(1 - sec^2\left(\frac{n}{2\sqrt{2}}\sqrt{c}(x-ct+d)\right)\right)\right]^{\frac{1}{n}} \quad (16) \end{aligned}$$

B. Benjamin-Bona-Mahony (BBM) equation

We consider the generalized Benjamin-Bona-Mahony (BBM) equation:

$$u_t + au_x + bu^2u_x + \delta u_{xxt} = 0, \quad (17)$$

where $a, b\ and\ \delta$ are non-zero arbitrary constants. Now we employ the Tan-cot method as we did in (3.1). Then (17) is transformed into the following ODE:

$$(a-c)\mathrm{u}_{\xi} + bu^2\mathrm{u}_{\xi} - c\delta\mathrm{u}_{\xi\xi\xi} = 0. \quad (18)$$

Integrating (18) and setting the constant of integration to zero for simplicity, we get

$$(a-c)u + \frac{b}{3}u^3 - c\delta\mathrm{u}_{\xi\xi} = 0. \quad (19)$$

Substituting (5) and (8) into (19) gives:

$$\left((a-c)\alpha - 2c\delta\alpha\beta^2\mu^2\right)tan^{\beta}(\mu\xi) + \frac{b}{3}\alpha^3 tan^{3\beta}(\mu\xi) - c\delta\alpha\beta\mu^2(\beta-1)\,tan^{\beta-2}(\mu\xi) - c\delta\alpha\beta\mu^2(\beta+1)\,tan^{\beta+2}(\mu\xi) = 0. \quad (20)$$

Equating the exponents of second and third term of tangent functions in the equation (20), then collecting the coefficients of the terms involved in the tangent functions of the same exponent, where each term has to vanish, we obtain the following system of algebraic equations:

$$3\beta = \beta - 2,$$
$$\frac{b}{3}\alpha^3 - c\delta\alpha\beta\mu^2(\beta-1) = 0\,, \quad (21)$$
$$(a-c)\alpha - 2c\delta\alpha\beta^2\mu^2 = 0, a - c \neq 0\,.$$

Solving the system (21) using Mathematica yields

$$\beta = -1\,,$$
$$\alpha = -\sqrt{\frac{3(a-c)}{b}}, \quad (22)$$
$$\mu = \sqrt{\frac{a-c}{2c\delta}}.$$

It follows immediately, $\frac{a-c}{b} > 0\ and\ \frac{a-c}{c\delta} > 0.$

Thus soliton solution of BBM equation (17) is given by:

$$\begin{aligned} u(x,t) &= u(\xi) \\ &= \alpha tan^{\beta}(\mu\xi) \\ &= -\sqrt{\frac{3(a-c)}{b}}tan^{-1}\left(\sqrt{\frac{a-c}{2c\delta}}\ (x-ct+d)\right) \\ &= -\sqrt{\frac{3(a-c)}{b}}\cot\left(\sqrt{\frac{a-c}{2c\delta}}\ (x-ct+d)\right), \quad (23) \end{aligned}$$

IV. Result And Discussion

A. gK-dV equation

The equation (16) represents the soliton solution of the gK-dV equation. We analytically analyzed the solution for studying different types of solitons. For $n = 1$, we get the real soliton whereas for $n = 2$ or more, it becomes the complex soliton. The effect of c on the shape of the gK-dV soliton are discussed and represented in numerous figures in this section.

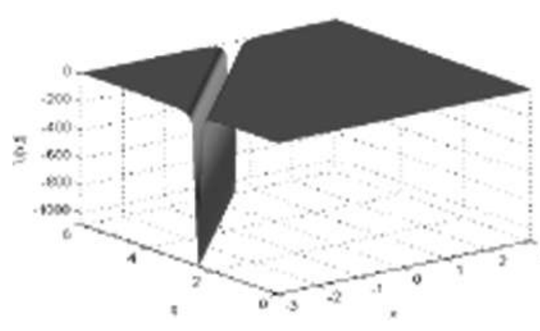

Fig. 1: *Soliton solution of $u(x,t)$ in (16) for $n = 1$, $d = 1$ and $c = 1$.*

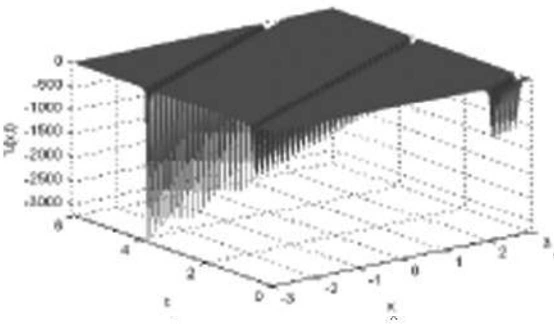

Fig. 2: *Soliton solution of* $u(x,t)$ *in (16) for* $n=1$, $d=1 and\ c=2$.

It is found that the width varies a little compare to the amplitude the gK-dV soliton which varies significantly with the variation of c. Fig. 1 represents the anti-modulus shape of (16) with wave speed $c=1$ within the interval $-3 \leq x \leq 3$, $0 \leq t \leq 6$.

The width decreases and soliton becomes more spiky and highly discrete, when the value of c increases i.e. $c=2\ to\ c=4$ (figs. 2 & 3). If we further increase the value of c, the width tend to zero and the soliton becomes flat shaped. This type of soliton profiles are represented in the figs. 4, & 5, respectively for the values of $c=8, \& \ 22$ and $d=4, \& \ 5$.

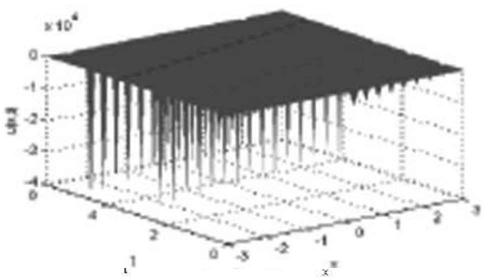

Fig. 3: *Soliton solution of* $u(x,t)$ *in (16) for* $n=1$, $d=2\ and\ c=4$.

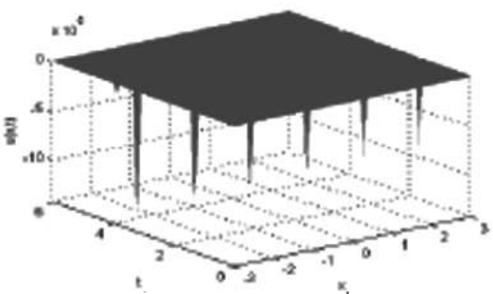

Fig. 4: *Soliton solution of* $u(x,t)$ *in (16) for* $n=1$, $d=4\ and\ c=8$.

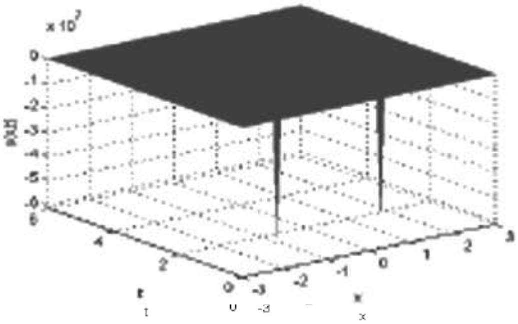

Fig. 5: *Soliton solution of* $u(x,t)$ *in (16) for* $n=1$, $d=5\ and\ c=22$.

B. BBM equation

In this section, we discuss the travelling wave solution of equation (23). Travelling wave solution of (23) is a solitary like solution which does not change its shape, and propagates at constant speed which oscillates between positive and negative direction for $c=1 \& 2$ (figs. 6 & 7). Figs. 6 & 7 represents that the soliton profile (23) is not continuous within the interval $-10 \leq x,t \leq 10$.

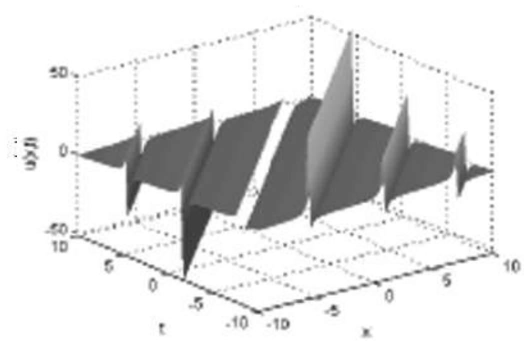

Fig. 6: *Soliton solution of* $u(x,t)$ *in (23) for* $a=\delta=d=2$, $b=4\ and\ c=1$.

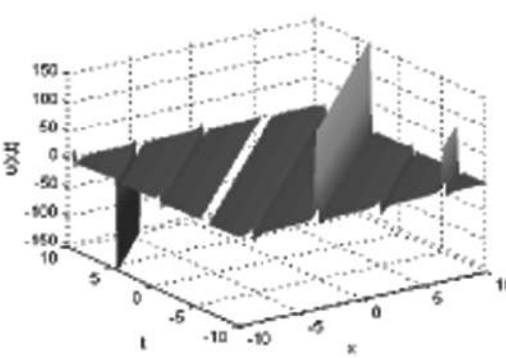

Fig. 7: *Soliton solution of* $u(x,t)$ *in (23) for* $a=\delta=d=4$, $b=9\ and\ c=1$.

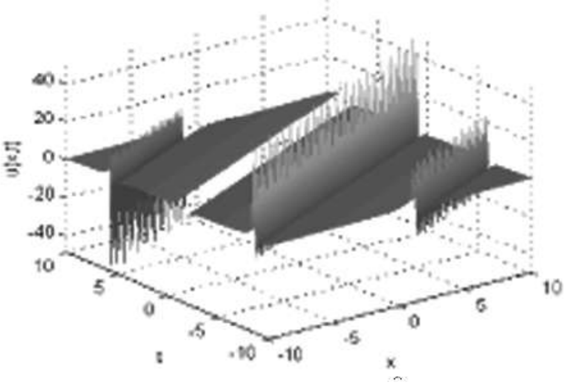

Fig. 8: *Soliton solution of* $u(x,t)$ *in (23) for* $a=\delta=d=4\,, b=9$ *and* $c=2$.

It is seen that as the value of c increases from $c=1$ to $c=2$, the wave width decreases and $u(x,t)$ profile becomes discrete (fig. 8). If we further increases the value of any others parameter including wave speed c, the width approaches to zero and the profile becomes more discrete and finally approaches to a flat position. This type of profile is represented in the figs. 9 & 10, respectively for the value of wave speed $c=5$ & $c=9$. It is obvious from following figure that physical behavior of solitons can be viewed.

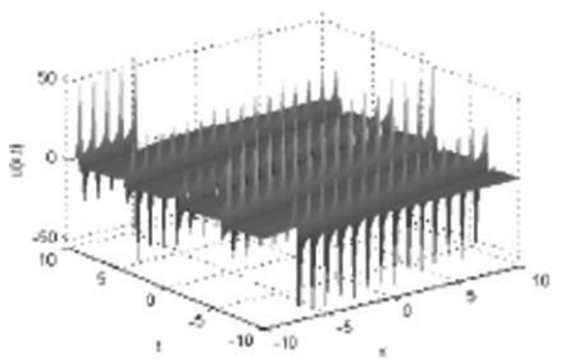

Fig. 9: *Soliton solution of* $u(x,t)$ *in (23) for* $a=\delta=d=6\,, b=9$ *and* $c=5$.

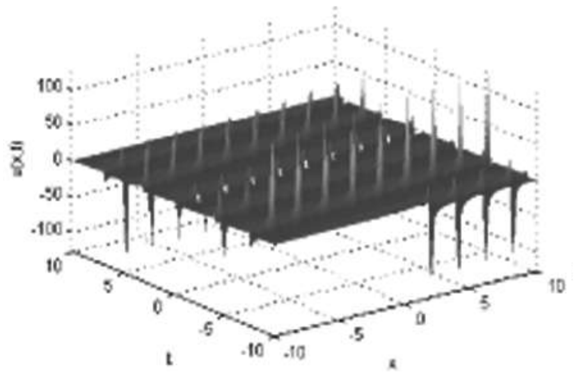

Fig. 10: *Soliton solution of* $u(x,t)$ *in (23) for* $a=10, \delta=d=6\,, b=9$ *and* $c=9$.

V. Conclusion

In this paper, the Tan-cot method has been successfully applied to find the solution of gK-dV and BBM equations and described many possible situation for different parameters. The method is used to find an exact and periodic solutions for the nonlinear travelling waves. It can be easily seen that the implemented method used in this paper is powerful and applicable to a large variety of nonlinear partial differential equations. The soliton behaviour of solutions with respect to time and space is shown and discussed graphically. Therefore we can say the method can be extended to solve the problems of nonlinear partial differential equations arising in the theory of solitons and other areas. In future we try to develop better analytical ways to solve gK-dV, BBM and others nonlinear differential equations. Standard methods from PDE appear inadequate to settle the problems, instead we probably need some deep new ideas in future.

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